Hecke orbits
and stratifications in moduli spaces of abelian varieties.

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Some results in this talk are joint work with Ching-Li Chai; the influence of his ideas will be clear in several aspects. I received further inspiration from joint work with Tadao Oda, Ke-Zheng Li, Torsten Ekedahl, Eyal Goren, Johan de Jong, Thomas Zink and Chai-Fu Yu.

1 Introduction

(1.1) Suppose given a field K, a polarized abelian variety (A, λ) over K. We define the *Hecke orbit* of the moduli point $x := [(A, \lambda)]$ to be the set of points $y = [(B, \mu)]$ over a field L such that there exist a field Ω containing K and L, an integer $n \in \mathbb{Z}_{>0}$ and an isogeny

$$\varphi : A_\Omega \to B_\Omega \quad \textit{such that} \quad \varphi^*(\mu) = n\cdot \lambda.$$

Notation:

$$[(B, \mu)] = y \in \mathcal{H}(x).$$

We write $y \in \mathcal{H}_\ell(x)$ when ℓ is a prime number, and in the notation above, n and $\deg(\varphi)$ are powers of ℓ.
Question. *What is the closure of* $\mathcal{H}(x)$ *and of* $\mathcal{H}_\ell(x)$?

(1.2) If the characteristic of $K = \mathbb{C}$, then:

$$\mathcal{H}(x) \textit{ is dense in } \mathcal{A}_g(\mathbb{C}),$$

in the sense of the "classical topology", and (then, of course)

$$\mathcal{H}(x) \textit{ is dense in } \mathcal{A}_g,$$

in the sense of the Zariski topology; the same statements hold for $\mathcal{H}_\ell(x) \subset \mathcal{A}_g(\mathbb{C})$ and $\mathcal{H}_\ell(x) \subset \mathcal{A}_g$. *The Hecke Orbit problem seems to be clear over* $\mathbb{C}$, leaving aside interesting arithmetic questions in characteristic zero.

(1.3) Example. Consider $K \supset \mathbb{F}_p$, let $g = 1$, and consider an elliptic curve E such that $E[p](k) = 0$, i.e. the elliptic curve has no geometric points of order exactly p; such an elliptic curve is called *supersingular*. Let $x = [(E, \lambda)]$, where λ is the unique principal polarization on E. Note that if $\varphi : E \to E'$ is an isogeny, then also E' is supersingular. Note that the set of supersingular j-values is finite. Conclusion: $\mathcal{H}(x) \cap \mathcal{A}_{1,1} \otimes k$ is *finite*. We see that $\mathcal{H}(x)$ is *not Zariski-dense in* $\mathcal{A}_{1,1} \otimes \mathbb{F}_p$.

Try to predict what the Zariski closure is of $\mathcal{H}(x)$ for a moduli point $x \in \mathcal{A}_g \otimes \mathbb{F}_p$. The analogous question: what is the Zariski closure of $\mathcal{H}_\ell(x) \subset \mathcal{A}_g \otimes \mathbb{F}_p$?

As points in a Hecke orbit correspond with isogenous abelian varieties, it seems logical to look for (isogeny-) invariants attached to an abelian variety invariant under an isogeny (or invariant under an isogeny of degree prime to p).

From now on in this note, we fix a prime number p, all base fields considered are of characteristic p, and all schemes are over $\mathrm{Spec}(\mathbb{F}_p)$.

Some notations and abbreviations we are going to use.

$k = \overline{k} \supset \mathbb{F}_p$, an algebraically closed field.
$m = \overline{\mathbb{F}_p}$

NP = Newton polygon
EO = Ekedahl-Oort
Fol = Foliation(s)
HO = Hecke orbit

$\infty \sim$ $\quad A[p^\infty]$ up to isogeny
$1 \cong$ $\quad A[p]$ up to isomorphism
$\infty \cong$ $\quad A[p^\infty]$ up to isomorphism

A, B, ... abelian varieties
X, Y, ... p-divisible groups
$\mathcal{A} = \mathcal{A}_{g,1} \otimes k, \quad \mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$
here $N \in \mathbb{Z}_{>0}$, not divisible by p, usually $N \geq 3$

$W_\xi = \mathcal{W}_\xi(\mathcal{A}), \quad W_{\xi,N} = \mathcal{W}_\xi(\mathcal{B})$

$\mathcal{N}(X)$ = the Newton polygon of X
$\mathcal{N}(A)$ = the Newton polygon of $A[p^\infty]$

ss = supersingular, i.e.
E, an elliptic curve in characteristic p, is supersingular if $E[p](k) = 0$,
an abelian variety A is supersingular if Ak is isogenous with E^g,
where E is a supersingular elliptic curve
Fact: A is ss iff all slopes in $\mathcal{N}(A)$ are equal to $\frac{1}{2}$;
this NP is denoted by σ, the ss NP

Let W be a scheme algebraic over a field K. We write $\Pi_0(W)$ for the set of irreducible components of W_k, where k is an algebraic closure of K.

We write ℓ for a prime number not dividing $p{\cdot}N$

2 p-adic invariants

To an abelian A over a field $K \supset \mathbb{F}_p$ we attach its p-divisible group

$$A \quad \mapsto \quad X := A[p^\infty].$$

To a p-divisible group we can attach various "invariants":

$X[p^\infty]$ up to $\sim$	ξ	NP	W_ξ
$X[p^1]$ up to $\cong$	φ	EO	S_φ
$(X[p^\infty], \lambda)$ up to $\cong$	X	Fol	$C(x)$

We explain these notions and notations below.

3 Newton polygon strata

(3.1) NP: $X[p^\infty]$ up to $\sim$.
Dieudonné - Manin:

$$\{X\}/\sim \quad \xrightarrow{\sim} \quad \{\mathrm{NP}\} \qquad X \mapsto \mathcal{N}(X):$$

Over an algebraically closed field the set of isogeny classes of p-divisible groups is the same as the set of Newton polygons.

Newton polygons are partially ordered; we write $\gamma \prec \beta$ if no point of γ is below β:

$$\gamma \prec \beta \quad \Leftrightarrow \quad \gamma \text{ is "above" } \beta.$$

An abelian variety A is isogenous with its dual A^t; using the duality theorem, see [10], 19.1 we conclude that $X \sim X^t$; hence $\mathcal{N}(A) =: \xi$ is symmetric: if the slope λ appears in ξ with multiplicity n_λ, then $1 - \lambda$ also appears with that multiplicity: $n_\lambda = 1 - n_{1-\lambda}$.

For a symmetric Newton polygon ξ we write:

$$\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) = \{[A, \lambda] \mid \mathcal{N}(A) \prec \xi\},$$

$$\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) = \{[A, \lambda] \mid \mathcal{N}(A) = \xi\}.$$

Grothendieck - Katz:

$$\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \textit{ is closed.}$$

Hence

$$\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \textit{ is locally closed.}$$

See [5]. These are called the **Newton polygon strata.**

4 EO - strata

(4.1) EO: $X[p^1]$ up to $\cong$.
In [15] we find: Let G be a "1-truncated Barsotti-Tate group". This means that there exists a p-divisible group X such that $X[p] \cong G$. Denote its isomorphism class by φ. *For a fixed $n \in \mathbb{Z}_{>0}$, the set of isomorphism classes of finite group schemes over k of rank p^n is finite* (proved by Kraft, and rediscovered by FO). For an abelian scheme $A \to T$ we define

$$\mathcal{S}_\varphi(T) = \{s \in T \mid \exists \Omega : A_s[p]_\Omega \cong G_\Omega\} \subset T.$$

In [15] we find:

$$S_\varphi := \mathcal{S}_\varphi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \quad \textit{is locally closed.}$$

Also we find a discussion of the isomorphism class of $(A, \lambda)[p]$ in case λ is a principal polarization, and we find a description of the strata S_φ (their dimensions, etc.).
Important point: *Every S_φ is quasi-affine.* This generalizes and was inspired by a result by Raynaud, who proved this for $(\mathcal{A}_g \otimes \mathbb{F}_p)^{\mathrm{ord}}$ (published by Szpiro and by Moret-Bailly).
The EO-strata give a (finite) stratification of every component of $\mathcal{A}_g \otimes \mathbb{F}_p$.

5 Foliations

(5.1) Fol: $X[p^\infty]$ up to $\cong$.
Choose $x \in \mathcal{A}_g \otimes \mathbb{F}_p$; write $x = [(A, \lambda)]$, over some field, say over k, and write $(X, \lambda) = (A, \lambda)[p^\infty]$. Define:

$$C(x) = \mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes k) = \{[(B, \mu)] \mid \exists \Omega : (B, \mu)_\Omega \cong (X, \lambda)_\Omega\}.$$

In [18] we find:
Theorem. *$\mathcal{C}(x) \subset \mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ is a* closed *subset.*
This uses [20]. An irreducible component of $\mathcal{C}(x)$ is called a *central leaf.* One also defines an isogeny leaf. We prove that any component of $\mathcal{W}^0_\xi(\mathcal{A}_g \otimes k)$, up to a finite morphism, is the product of a central and an isogeny leaf. For all this, see [18].

If x and y are two moduli-points in the same open Newton polygon stratum, then any component of $C(x)$ has the same dimension as a component of $C(y)$. Isogeny correspondences are finite-to-finite on central leaves. We could say: "abelian varieties in one central leaf behave as if they live in characteristic zero".

Note:

∞ $\sim$	**NP**	$\mathcal{N}(A)$ is invariant under isogenies.
1 $\cong$	**EO**	φ is invariant under isogenies of degree prime to p.
∞ $\cong$	**Fol**	$(A, \lambda)[p^\infty]$ invariant under isogenies of degree prime to p.

6 The Hecke orbit conjecture

(6.1)$_?$ The Hecke orbit conjecture. *For any point $x := [(A,\lambda)] \in \mathcal{A}_g \otimes \mathbb{F}_p$ its Hecke orbit*

$$\mathcal{H}(x) \text{ is Zariski-dense in its Newton polygon stratum } \mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [14].

This conjecture was proved in 1995 by Chai for *ordinary* abelian varieties, see [1].

This conjecture can be "split up" into:

(6.2)$_?$ The Hecke orbit conjecture, continuous form. *For any point $x := [(A,\lambda)] \in \mathcal{A}_{g,1} \otimes k$ and C_x the irreducible component of $C(x)$ containing X, and for any prime number ℓ different from p*

$$\text{the intersection } \mathcal{H}_\ell \cap C_x \text{ is dense in } C_x.$$

(6.3)$_?$ The Hecke orbit conjecture, discrete form. *For every symmetric Newton polygon,*

$$\text{the action of } \mathcal{H}_\ell \text{ on } \Pi_0(W_\xi) \text{ is transitive.}$$

Note that

$$(6.2) + (6.3) + [18] \quad \Rightarrow \quad (6.1).$$

We show that (6.3) is correct: (7.1).

An ingenuous proof, mainly by Ching-Li Chai, reduces (6.2) to an analogous discrete Hecke orbit conjecture for Hilbert modular varieties. This problem for Hilbert modular varieties has now been solved by Chai-Fu Yu. A combination of all these building blocks is expected to give, once written down in full detail, a proof of (6.1).

7 Some results

Note that the supersingular locus has "many components"; see [7], 4.9:

$$\#(\Pi_0(W_\sigma)) = H_g(p,1) \quad \text{if} \quad g \quad \text{is odd,}$$

$$\#(\Pi_0(W_\sigma)) = H_g(1,p) \quad \text{if} \quad g \quad \text{is even.}$$

We want to compute the number of components of all Newton polygon strata. As it will turn out: *any non-supersingular NP stratum W_ξ is irreducible,* see (7.4) below.

(7.1) Theorem. *For every g, for every prime number $\ell \neq p$, for every N as above, and for every symmetric Newton polygon ξ, the action of $\mathcal{H}_\ell$ on the set of irreducible components $\Pi_0(W_{N,\xi})$ of the Newton polygon stratum $W_{N,\xi} \subset \mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$ is* transitive.

(7.2) Let $Z \subset \mathcal{B}$ be a locally closed subset. Let Z' be an irreducible component of Z. Let $\eta \in Z'$ be the generic point. Let $A \to Z$ be the universal abelian scheme restricted to Z. Let

$$\rho_{A,\ell} : \pi_1(Z', \overline{\eta}) \quad \longrightarrow \quad \mathrm{Sp}(T_\ell, <, >_\ell)$$

be the ℓ-adic monodromy representation in the Tate-ℓ-group of A_η. Identify the Tate-ℓ-group of A_η over $\overline{\eta}$ with $\mathbb{Z}_\ell^{2g}$ with the standard pairing.

(7.3) **Theorem** (C.-L. Chai). *Choose notation as above. Let $Z \subset \mathcal{B}$ be a locally closed subscheme, smooth over* $\mathrm{Spec}(k)$, *such that:*

Z is Hecke-ℓ-stable, and

the Hecke-ℓ-action on the set $\Pi_0(Z)$ is transitive, and

$Z \not\subset W_{N,\sigma}$ (equivalently: Z contains a non-supersingular point).

Then:

$$\rho_{A,\ell} : \pi_1(Z', \overline{\eta}) \longrightarrow \mathrm{Sp}(T_\ell, <, >_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$$

is surjective, *and*

$$Z \textit{ is irreducible, i.e. } Z = Z'.$$

See [2], 4.4.

(7.4) **Corollary.** *For every $\xi \supsetneqq \sigma$ the locus $W_{N,\xi} \subset \mathcal{B}$ is* irreducible; i.e.

$$\xi \neq \sigma \quad \Rightarrow \quad \#(\Pi_0(W_{N,\xi})) = 1.$$

This was conjectured in [14]. This answers [21], 3.8.

(7.5) **Irreducibility of EO-strata.** T. Ekedahl and G. van der Geer have proved (unpublished) a result proving the irreducibility of certain EO-strata. A description (FO) which EO-strata contain non-supersingular points then results in:

Let φ be such that $S_\varphi \not\subset W_\sigma$. Then S_φ is geometrically irreducible.

This was conjectured in [15].
However, one can expect that for any φ such that $S_\varphi \subset W_\sigma$ we have $\Pi_0(S_\varphi) > 1$ for $p \gg 0$.

(7.6) **Definition** (C.-L. Chai) *An abelian variety A is called* hypersymmetric *if it is defined over a finite field*, hence A is defined over $m = \overline{\mathbb{F}_p}$, *and*

$$\mathrm{End}(A_m) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{End}(A_m[p^\infty])$$

is an isomorphism.
Note that being hypersymmetric is invariant under an isogeny; we could have formulated the definition by saying that the isogeny class is defined over a finite field, and then proceed as before.

(7.7) Irreducibility of leaves. Let $[(A,\lambda,\beta)] = x \in \mathcal{B}$ be the moduli point of a *non-supersingular* polarized abelian variety A with level structure; C.–L. Chai proves:

if $x \notin W_\sigma$ *then* $C(x)$ *is geometrically irreducible.*

In the proof of this result (work over k):

- it is shown that for any symmetric NP there exists a hypersymmetric abelian variety; fix a principally polarized hypersymmetric (B,μ) with $\mathcal{N}(B) = \xi$;
- irreducibility of a NP-stratum is used, see (7.4);
- use the foliation structure by isogeny leaves and by central leaves, see [18], and conclude that for any $C_i \in \Pi_0(C(x))$, and an isogeny leaf I passing through (B,μ) we have $C_i \cap I \neq \emptyset$; choose $[(B_i,\mu_i)] \in C_i \cap I$;
- use weak approximation in order to conclude that for every i and j we have that (B_i,μ_i) is in the prime-to-p Hecke orbit of (B_j,μ_j);
- use a variant of (7.3) in order to conclude that $C(x)$ is irreducible. □

8 A sketch of a proof of (7.1)

(8.1) Step 1.
Claim. *For every symmetric Newton polygon* ξ *and every irreducible component* W *of* W_ξ *we have* $W \cap W_\sigma \neq \emptyset$, *i.e.* W *contains a supersingular point.*

A proof of Step 1 uses:

T1 EO-strata, especially that every S_φ is *quasi-affine* (a variant of a theorem by Raynaud), and it uses:

T2 Finiteness of Hecke orbits (Chai).
Proposition. *A Hecke-*ℓ *orbit* $\mathcal{H}_\ell(x)$ *is finite for* $x = [(A_x,\lambda)]$ *iff* A_x *is supersingular.*
See [1], Prop. 1 on page 448.

(8.2) Step 2.
Claim. (a) *For every symmetric Newton polygon* ξ *and every component,* $W \in \Pi_0(W_\xi)$ *there is a component* $T \in \Pi_0(W_\sigma)$ *such that* $T \subset W$.
(b) *For every* $\zeta \prec \xi$,
every component of W_ζ *is contained in a unique component of* W_ξ *and*
every component of W_ξ *contains at least one of* W_ζ*;*
by inclusion of components we obtain a well-defined map

$$i : \Pi_0(W_{N,\zeta}) \to \Pi_0(W_{N,\xi}), \quad W' \mapsto i(W') = W \quad \text{if} \quad W' \subset W,$$

which moreover is surjective.
(c) *This map* i *is Hecke-*ℓ *equivariant.*

A proof of Step 2 uses:

T3 Purity. See [4], 4.1: *If the Newton polygon jumps in a family of p-divisible groups, it already jumps in codimension one.*

T4 Deformations with constant Newton polygon. *For any* principally *polarized abelian variety (A, λ) there exists a deformation with generic fiber (A', λ') with $\mathcal{N}(A) = \mathcal{N}(A')$ and $a(A') \leq 1$.*
See [4], (5.12), and [17], (3.11) and (4.1).

T5 Cayley-Hamilton. See [16], 3.5 and use T4. *In particular this shows:*
(a) *Around any point $x \in \mathcal{A}$, with $\mathcal{N}(A_x) = \zeta$ and $a(A_x) = 1$ the Newton strata $\{W_\xi^{/x} \mid \xi \succ \zeta\}$ are formally smooth and nested as given by the graph of all Newton polygons below ζ.*
(b) *The dimension of $W_\xi \subset \mathcal{A}$ equals* $\mathrm{sdim}(\xi)$.
(c) *The generic point of any component of W_ξ has a-number at most one* (i.e. equal to zero in the ordinary case, and equal to one otherwise).

Remark. T4 and T5 give a proof of a conjecture by Grothendieck, see [16], [17]. Instead of using results in [16], one can use this conjecture by Grothendieck (now a theorem) and Purity, see T3, in order to compute dimensions of Newton polygon strata in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$.

(8.3) Step 3. *The action of $\mathcal{H}_\ell$ on $\Pi_0(W_{N,\sigma})$ is* transitive.

A proof of Step 3 uses:

Notation. For $g \in \mathbb{Z}_{>1}$ and $j \in \mathbb{Z}_{\geq 0}$ we write

$$\Lambda_{N,g,j} := \{(B, \mu, \beta)\}/ \cong$$

for the set of such triples over k up to isomorphism, where $B \cong E^g$ is a superspecial abelian variety over k, and μ a polarization on B with $\mathrm{Ker}(\mu) = B[F^j]$ and $\beta : B[N] \xrightarrow{\sim} (\mathbb{Z}/N)^{2g}$ is a symplectic isomorphism.

T6 *There is a canonical, Hecke-ℓ-equivariant, bijective map*

$$\Pi_0(W_{N,\sigma}) \quad \xrightarrow{\sim} \quad \Lambda_{N,g,g-1}.$$

See [7], 3.6 and 4.2; this uses [9], 2.2 and 3.1.

T7 *The action of $\mathcal{H}_\ell$ on $\Lambda_{N,g,g-1}$ is* transitive.

In Step 3 we have seen that $\mathcal{H}_\ell$ acts transitively on $\Pi_0(W_{N,\sigma})$. By the first and the second Step we see that this implies that $\mathcal{H}_\ell$ acts transitively on $\Pi_0(W_{N,\xi})$. This proves (7.1). By (7.3) this shows (7.4). $\square$

Note that Step 2 finishes the proof of all claims in [13]. At that time (1991), I did not expect (7.4).

9 Some topics and questions not treated in my talk.

(9.1) Let $x = [(A, \lambda)] \in W_\xi^0$ and let $\varphi = \mathrm{ES}(A[p])$. Then $C(x) \subset W_\xi^0$ and $C(x) \subset S_\varphi$. we have not discussed:

When is $C(x) = W_\xi^0$? Iff $f(A) \geq g - 1$.

When is $C(x) \subset S_\varphi$? See [19].

For which φ and ξ is $W_\xi^0 \cap S_\varphi \neq \emptyset$?

(9.2) *Let $x = [(A, \lambda)] \in W_\xi^0$ and let $\zeta \prec \xi$. What is the "boundary" $\overline{C(x)} \cap W_\zeta^0$?*

There are examples of this situation where $y \in W_\zeta^0$ and $y \notin C(x)$ for every $x = [(A, \lambda)] \in W_\xi^0$. But there is a "guess" what the boundary of $C(x)$ in W_ζ^0 could be.

(9.3) See [15], 14.3. On the set of isomorphism classes of $(A, \lambda)[p]$ there are two partial orderings "$\prec$" and "$\subset$". It seems unknown how to give an (easy, combinatorial) algorithm which decides for φ and φ' whether $\varphi' \subset \varphi$.

(9.4) We did not discuss structures needed for proving Conjecture (6.2), and details of that proof.

(9.5) We did not discuss the story of beautiful stratifications on Hilbert modular varieties (Rapoport, Deligne & Pappas, Bachmat & Goren, Goren & Oort, Andreatta & Goren, Yu, Harashita), and on other modular Shimura varieties (Moonen, Wedhorn).

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