Hecke orbits in moduli spaces

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1. Hecke orbits. Consider $\mathcal{A}_g$, the moduli space of polarized abelian varieties. Let $[(A, \lambda)] = x \in \mathcal{A}_g$ be a moduli point. We define the **Hecke orbit** $\mathcal{H}(x)$ of this point:

Definition we say that $[(B, \mu)] = y \in \mathcal{H}(x)$ if there exist an isogeny $\varphi : A \to B$ and a positive integer $n \in \mathbb{Z}_{>0}$ such that

$$\varphi^*(\mu) = n{\cdot}\lambda,$$

i.e. "A and B are isogenous taking into account their polarizations".

Remark. For $x \in \mathcal{A}_g(\mathbb{C})$ the Hecke orbit $\mathcal{H}(x)$ is *dense* in $\mathcal{A}_g(\mathbb{C})$.

From now on: *all base fields considered have characteristic* $p > 0$.

2. Newton polygons.

Dieudonné - Manin:

$$\{X\}/\sim_k \quad \stackrel{\sim}{\longrightarrow} \quad \{\mathrm{NP}\}:$$

for every p-divisible group X over any field we define its Newton polygon $\mathcal{N}(X)$,

$$X \sim Y \quad \Longleftrightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y),$$

and for every Newton polygon β there exists X with $\mathcal{N}(X) = \beta$;

here k is an *algebraically closed field.*

Newton polygon:
  lower convex, integral break points.

We take the "Newton polygon of the Frobenius".

We have a partial ordering:

$$\beta \succ \gamma \quad \text{iff} \quad \text{“}\gamma \text{ is above } \beta\text{”}.$$

Write $\mathcal{A}_g = \mathcal{A}_g \otimes \mathbb{F}_p$.

$$\mathcal{W}_\xi(\mathcal{A}_g) :=$$
$$:= \{[(B,\mu)] = y \in \mathcal{A}_g \mid \mathcal{N}(B) \prec \xi\},$$

$$\mathcal{W}^0_\xi(\mathcal{A}_g) :=$$
$$:= \{[(B,\mu)] = y \in \mathcal{A}_g \mid \mathcal{N}(B) = \xi\},$$

Theorem (Grothendieck and Katz). *$\mathcal{W}_\xi(\mathcal{A}_g) \subset \mathcal{A}_g$ is closed.*

Obvious. Let $[(A,\lambda)] = x$ and $\xi = \mathcal{N}(A)$; then

$$\boxed{\mathcal{H}(x) \subset \mathcal{W}^0_\xi(\mathcal{A}_g)}.$$

Conjecture (FO, 1993).

$$\boxed{(\mathcal{H}(x))^{\mathrm{Zar}} = \mathcal{W}_\xi(\mathcal{A}_g)}.$$

This conjecture will be indicated by **(HO)**, the *Hecke orbit conjecture*.

3. The Hecke orbit conjecture.

theorem (Ching-Li Chai & Frans Oort, using a result by Chia-Fu Yu).
The conjecture **(HO)** *is true:*
$[(A,\lambda)] = x$ *and* $\xi = \mathcal{N}(A)$
$\boxed{\Longrightarrow \mathcal{H}(x) \text{ is dense in } \mathcal{W}_\xi(\mathcal{A}_g).}$

For the proof of this theorem several methods were used/developed:

- Newton polygon strata, EO strata,
- Foliations,
- slope filtrations on p-divisible groups,
- generalized Serre-Tate canonical coordinates,
- rigidity results for p-divisible groups,
- "splitting at supersingular primes",
- hypersymmetric abelian varieties,
- Hilbert modular varieties.

Just to mention some names of mathematicians involved in developing these methods:

Newton polygon strata, EO strata
(Dieudonné, Manin, FO, Ekedahl),

Foliations (FO),

slope filtrations on p-divisible groups
(Zink, FO),

generalized Serre-Tate canonical coordinates
(Chai),

rigidity results for p-divisible groups (Chai),

"splitting at supersingular primes" (Chai),

hypersymmetric abelian varieties (Chai, FO),

Hilbert modular varieties (Rapoport, Deligne & Pappas, Bachmet & Goren, FO, Andreatta, Chia-Fu Yu, Chai).

In this talk I will not discuss all methods.
I will sketch

the main lines of the proof of **(HO)**.

Then I will restrict myself to a discussion of two basic, new techniques:

Foliations

and

generalized Serre-Tate canonical coordinates.

I hope in this way the audience will obtain a feeling for this beautiful topic.

We see a general phenomenon in mathematics: a challenging problem generates new methods, which are even more interesting than the final solution of the problem....

A survey: Ching-Li Chai, "Hecke orbits on Siegel modular varieties". Progr.Math. 235, Birkhäuser 2004, 71-107. – More material on:
http://www.math.upenn.edu/ chai/
http://www.math.uu.nl/people/oort/

The Hecke orbit conjecture was proved by Ching-Li Chai, Invent. Math. **121** (1995), for *ordinary abelian varieties*. His fundamental work was important for developing further methods/ideas.

The Hecke orbit conjecture for moduli points in *Hilbert modular varieties* has been studied and proved by Ching-Li Chai and Chia-Fu Yu; that result is used in the proof of the Hecke orbit conjecture for Siegel moduli varieties.

The Hecke orbit conjecture is easily proved for supersingular abelian varieties; from now on that case will be excluded from the considerations (but in the proof of the general case degeneration techniques to supersingular points will be used).

In order to prove the Hecke orbit conjecture $(\mathcal{H}(x))^{\mathrm{Zar}} = \mathcal{W}_{\xi}(\mathcal{A}_g)$, it suffices to show this in case $x \in \mathcal{A}_g(\overline{\mathbb{F}_p})$.

4. Foliations. See JAMS **17** (2004).

Definition. Let (A,λ) be a polarized abelian variety, and $(X,\lambda) := (A,\lambda)[p^\infty]$ its quasi-polarized p-divisible group. We write
$\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g) := \{[(B,\mu)] = y \in \mathcal{A}_g \mid$
$(B,\mu)[p^\infty] \otimes \Omega \cong (X,\lambda) \otimes \Omega\}$.

Theorem.
$C(x) := \mathcal{C}_{(X,\lambda)}(\mathcal{A}_g) \subset \mathcal{W}^0_{\mathcal{N}(A)}(\mathcal{A}_g)$
is a closed subset.

We will say that $C(x)$ is the (*central*) *leaf* passing through $[(A,\lambda)] = x \in \mathcal{A}_g$; we write C_x for the irreducible component of $C(x)$ containing x.

Using Hecke correspondences only involving isogenies with kernel successive extensions of α_p starting from $x \in \mathcal{A}_g$ defines a subset of $\mathcal{A}_g$ (which, in general, is not closed). The union of all irreducible components of this set passing through x is denoted by $I(x)$.

Theorem. $I(x) \subset \mathcal{A}_g$ *is closed.*

An irreducible component of $I(x)$ will be called an *isogeny leaf*. Related notion: Rapoport-Zink spaces.

Product structure on NP strata:

Theorem. *Work over k, an algebraically closed field. Choose a symmetric Newton polygon ξ, choose an irreducible component $W \subset \mathcal{W}^0(\mathcal{A}_g)$, choose an isogeny leaf $I \subset W$, and an irreducible component $C \subset W$ of a central leaf. There exist integral schemes T, J of finite type over k, finite morphisms $T \twoheadrightarrow C$ and $J \twoheadrightarrow I$ and a finite surjective morphism*

$$\Phi : T \times J \twoheadrightarrow W$$

such that

$$\forall u \in J(k), \quad \Phi(T \times \{u\}))$$

is a central leaf in W, and

$$\forall t \in T(k), \quad \Phi(\{t\} \times J))$$

is an isogeny leaf in W.

Isogeny correspondences. Let $[(A,\lambda)] = x$ and $[(B,\mu)] = y \in \mathcal{H}(x)$. Then there exist finite, surjective morphisms:

$$C_x \leftarrow T \twoheadrightarrow C_y.$$

Comment. This is remarkable. In general isogeny correspondences in positive characteristic blow up and down in a rather unpredictable way. The result says that, restricted to central leaves, Hecke correspondences are finite-to-finite. It looks as: central leaves have "the same properties" as moduli spaces in characteristic zero.

We will see that central leaves are exactly the loci where finer structures are true, and group-theoretic methods can be applied.

Note that an isogeny of degree prime to p gives an isomorhism on the p-divisible group. Hence such isogenies "move" a moduli point in a (central) leaf.

We sketch the proof that
$C(x) \subset \mathcal{W}^0_{\mathcal{N}(A)}(\mathcal{A}_g)$ is **closed**.
Here $[(A,\lambda)] = x$, and $(A,\lambda)[p^\infty] = (X,\lambda)$. These arguments can be found in my paper on Foliations; I profited a lot from the cooperation with and ideas of Thomas Zink.

Step 1. For given h there exists $n_0(h)$ such that for $n \geq n_0(h)$, and p-divisible groups X, Y, of height h, the existence of an isomorphism $X[p^n] \cong Y[p^n]$ implies the existence of an isomorphism $X \cong Y$.

Step 2. For (A,λ) over k fixed we construct $S \subset \mathcal{A}_g$ as "the set of $s \in S$ such that there exists an isomorphism $(A,\lambda)[p^n] \cong (\mathcal{U},\mu)_s[p^n]$". We see that $S \twoheadrightarrow C(x)$ is surjective; conclusion:

$$C(x) \text{ is constructible.}$$

Let D be the Zariski closure of $C(x)$ inside $\mathcal{W}^0_{\mathcal{N}(A)}(\mathcal{A}_g)$ (and we want to show $C(x) = D$). We replace D by an irreducible component of its normalization, after taking sufficiently level structure.

Step 3. The universal family $(\mathcal{U}, \lambda) \to D$ has constant Newton polygon. By a theorem by Thomas Zink and FO there exist a polarized abelian scheme $(\mathcal{P}, \nu) \to D$ which is gfc (geometrically fiberwise constant), and an isogeny $\varphi : (\mathcal{P}, \nu) \to (\mathcal{U}, \lambda)$. Choose an integer q (a power of p) annihilating $\mathrm{Ker}(\varphi)$. Say all fibers of $(\mathcal{P}, \mu) \to D$ are geometrically isomorphic with (Z, ν).

Step 4. There exists a finite group scheme L over k such that after replacing D by a finite cover, there exists an isomorphism $L \times D \cong \mathcal{P}[q]$.

Step 5. The kernel $\mathcal{M} = \mathrm{Ker}(\varphi)$ defines $s : D \to G$, a section in the Grassmannian G of all free subgroup schemes over D of this rank inside $L \times D \cong \mathcal{P}[q]$.

Step 6. Lemma. *Given p-divisible groups Z and X, and an integer r. Then:*

$$\#(\{M \subset Z \mid \mathrm{rk}(M) = r,\ Z/M \cong X\}) < \infty.$$

Step 7. We see that the section s is constant on the inverse image of $C(x)$ inside D. As $C(x)$ is constructible, it is dense in its cloure D. Hence for every $d \in D$ we have $(\mathcal{P}, \nu)_d/s(d) \cong (X, \lambda)$. Hence $C(x) = D$.

Q.E.D.

5. Logic of the proof of (HO).

Notation. Let $\mathcal{H}_\ell(x)$ be the Hecke orbit, where $\ell \neq p$ is a prime, and all degrees involved are a power of ℓ. From now on this prime will be fixed. We formulate the ℓ-Hecke orbit conjecture:

(HO)$_\ell$: $\mathcal{H}_\ell(x) \cap C(x)$ is dense in $C(x)$. (?)

Corollary of the product structure.

$$\textbf{(HO)}_\ell \Rightarrow \textbf{(HO)}.$$

Hence it suffice to prove **(HO)**$_\ell$. This conjecture we split up into two parts:

(HO)$_{\mathrm{ct}}$ (ct stands for "continuous"):

$$\boxed{(\mathcal{H}_\ell(x))^{Zar} \textit{ contains } C_x.} \text{ (?)}$$

(HO)$_{\mathrm{dc}}$ (dc stands for "discrete"):
Hecke-prime-to-p correspondences act transitively on the set $\Pi_0(C(x))$ *of geometrically irreducible components of* $C(x)$.
(?)

It is not difficult to see:

$$\mathbf{(HO)} \quad \Longleftrightarrow \quad \mathbf{(HO)}_{\mathrm{ct}} + \mathbf{(HO)}_{\mathrm{dc}}.$$

Note that $\mathbf{(HO)}_{\mathrm{dc}}$ is known for the supersingular situation.

We start with the discrete part.

We write $\mathcal{A}_{g,1}$ for the moduli space in characteristic p of principally polarized abelian varieties. We write $W_\xi := \mathcal{W}_\xi(\mathcal{A}_{g,1})$.

6. Newton polygon strata are irreducible.

Theorem. *For ξ not supersingular, the locus W_ξ is geometrically irreducible.*

This was conjectured more than 10 years ago (FO). The proof of this theorem uses: the Cayley-Hamilton method, details in the proof of a conjecture by Grothendieck (FO), the Raynayd trick for EO-strata, purity of the stratification by Newton polygons (A.J. de Jong and FO), and a result by Chai on ℓ-adic monodromy.

7 Central leaves are irreducible.

Theorem (Chai and FO). *Let* $[(A, \lambda)] = x$, *and* $\xi = \mathcal{N}(A)$. *For* ξ *not supersingular, the leaf* $C(x)$ *is geometrically irreducible.* I.e. $C(x) = C_x$.

Clearly this proves $\mathbf{(HO)}_{\mathrm{dc}}$ for the non-supersingular situation.

Q.E.D discrete part of **(HO)**

The irreducibility of non-supersingular leaves uses: the irreducibility of non-supersingular NP strata, the notion of hypersymmetric abelian varieties (see below), and a result by Chai on prime-to-p-adic monodromy.

8. We come to the continuous part.

Definition. An abelian variety A over a field $L \supset \mathbb{F}_p$ is called *hypersymmetric* if for $L \subset k = \overline{k}$:

$$\mathrm{End}(A \otimes k) \otimes \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(A[p^\infty] \otimes k)$$

is an isomorphism.

This implies (by a theorem of Grothendieck) that A is defined over a finite field K, and (by a theorem of Tate) that the Frobenius operates in a diagonal way on $\mathrm{End}^0(A_K[p^\infty])$. We show that for any Newton polygon there exists a hypersymmetric abelian variety having that Newton polygon.

Here is a *very rough sketch* of the (quite involved) proof of $\mathbf{(HO)}_{\mathrm{ct}}$, the continuous part of the Hecke orbit conjecture for the Siegel moduli space: $\boxed{(\mathcal{H}_\ell(x))^{Zar} \text{ contains } C_x.}$

Step 1.(Chai and Chia-Fu Yu). *The Hecke orbit conjecture holds for Hilbert Modular Varieties.*
Comments: The most difficult part is the discrete part. For p inert (Goren & FO), and for p totally ramified (Andreatta & Goren) this situation was studied; however in the proof of $\mathbf{(HO)}_{\mathrm{ct}}$ we have no a priori information on the behaviour in the totally real field used for the HMV situation. The difficult, final step was done by Chia-Fu Yu.
Step 2. Using HO for HMV:
for every x there is a split hypersymmetric
point in $\overline{\mathcal{H}_\ell} \cap C(x)$.
Here several ingredients are used. "Degeneration to supersingular points" is one of them. For canonical coordinates: see below.

Step 3. Using Step 2, analogously to "Larsen's example" in the 1995 Hecke orbit paper by Chai $\mathbf{(HO)}_{\mathrm{ct}}$ follows.

Q.E.D. **(HO)**

Note the curious “twist” in the proof:

the *discrete part* of **(HO)** basically is a geometric result on irreducibility of Newton polygon strata and of leaves;

the *continuous part* of **(HO)** uses the discrete part of HO for HMV (and many other ingredients).

In several parts of the proof canonical coordinates play a crucial role. In characteristic zero “group theory” can be used to prove density of Hecke orbits in a moduli space. In positive characteristic the situation is not completly analogous, but here the “infinitesimal group theorey” on leaves gives a firm grasp on the situation; that is precisely what the new technique *”generalized Serre-Tate canonical coordinates”* does for us.

9 Generalized Serre-Tate canonical coordinates (Chai).
For an ordinary $[(A, \lambda)] = x$ there is an isomorphism:

$$(\mathcal{A}_{g,1})^{/x} \quad \cong \quad (\mathbb{G}_m[p^\infty])^{g(g+1)/2},$$

canonical up to $\mathbb{Z}_p$-linear transformations. This was constructed by Serre and Tate (Woods Hole, 1964). It works even in mixed characteristics.

There have been many attempts to generalize the existence of such coordinates to a neighbourhood of a non-ordinary point.

Aanalysis of Serre-Tate coordinates. Let A be an ordinary abelian variety over k; write $Y = A[p^\infty]_{\mathrm{loc}}$ and $X = A[p^\infty]_{\mathrm{et}}$. An infinitesimal deformation of A results in a slope divisible extension

$$\mathcal{Y} \subset \mathcal{Z} \twoheadrightarrow \mathcal{Z}/\mathcal{Y} = \mathcal{X}.$$

As $\mathcal{X} \cong (\mathbb{G}_m[p^\infty])^g$ and $\mathcal{Y} \cong (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g$ the deformation is determined by an element of $\mathrm{Ext}(\mathcal{X}, \mathcal{Y}) \cong (\mathrm{Ext}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mathbb{G}_m[p^\infty]))^{g^2}$.

How can this be generalized to a non-ordinary situation? Obstacles:
- a deformation in general does not allow a slope filtration.
- the p-divisible group of an ordinary abelian variety over a perfect field splits into the divisible parts, but the extension of isoclinic parts does not split for non-ordinary abelian varieties in general.

Solution by Chai of this problem:
- work over a central leaf (where there is a slope filtration), and
- consider only that part of the extension data giving a geometrically split extension.

Here is a description of the result in the simplest situation.

We assume that Z is a p-divisible group whose Newton polygon only has two slopes, or A is an abelian variety, and $A[p^\infty] = Z$ having two slopes. Asume moreover that $Z \cong X \oplus Y$, a direct sum of isoclinic parts, with $\nu_Y := \mathrm{sl}(Y) > \mathrm{sl}(X) =: \nu_X$.

Study the local deformation space of (A,λ), or the deformation space of Z. Through $[(A,\lambda)] = x$ we have the leaf $C(x)$. Over a leaf (Zink & FO) the universal deformation $\mathcal{Z} \to C(x)$ admits a slope filtration

$$\mathcal{Y} \subset \mathcal{Z} \twoheadrightarrow \mathcal{Z}/\mathcal{Y} = \mathcal{X}.$$

This defines an extension class. Write h_X, resp. h_Y for the height of X, resp. Y.
Claim: the p-divisible part of this extension-class-functor is canonically isomorphic with the formal scheme $C(x)^{/x}$:

Theorem (Chai). *The central leaf $\mathcal{C}_Z(\mathrm{Def}(Z))$ has the structure of a p-divisible group, isoclinie of slope $\nu_Y - \nu_X$, of height $h_X{\cdot}h_Y$ and of dimension $h_X{\cdot}h_Y(\nu_Y - \nu_X)$.*
The formal completion

$$C(x)^{/x} \subset \mathrm{Def}(A,\lambda) = (\mathcal{A}_{g,1})^{/x}$$

has the structure of a p-divisible group, isoclinie of slope $\nu_Y - \nu_X$, of
height: $h_X{\cdot}(h_X+1)/2$ and
dimension: $h_X{\cdot}(h_X+1)(\nu_Y - \nu_X)/2$.