Stratifications and foliations of moduli spaces

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We study the moduli spaces $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of principally polarized abelian varieties in characteristic p, and also $\mathcal{A}_g \otimes \mathbb{F}_p$. The p-structure on the abelian varieties in consideration provides us with several naturally defined subsets. In this talk we discuss the definitions and constructions of these sets, we indicate interrelations, and we sketch proofs, applications, open problems and conjectures.

Much of this research originated in the seminal paper:
[13] Yu. I. Manin –
–*The theory of commutative formal groups over fields of finite characteristic.* (1963).
The fascinating structures, studied by Manin forty years ago, reveal more and more their beautiful features to us..... after tenacious research.

We study:

NP *The stratification defined by $X = A[p^\infty]$ up to isogeny over k.*
For every abelian variety A, or p-divisible group X, we consider the related Newton polygon $\mathcal{N}(A)$, respectively $\mathcal{N}(X)$. This is an isogeny invariant. By Grothendieck-Katz we obtain closed subsets of $\mathcal{A}$.

EO *The stratification defined by $A[p]$ up to isomorphism over k.*
In [22] we have defined a natural *stratification* of the moduli space of polarized abelian varieties in positive characteristic: moduli points are in the same stratum iff the corresponding p-kernels are geometrically isomorphic. Such strata are called Ekedahl-Oort-strata.

Fol *Subschemes defined by $(X = A[p^\infty], \lambda)$ up to isomorphism over k.*
In [25] we define in $\mathcal{A}_g \otimes \mathbb{F}_p$ a *foliations* : moduli points are in the same leaf iff the corresponding p-divisible groups are geometrically isomorphic; in this way we obtain a foliation of every open Newton polygon stratum.

Fol $\subset$ EO The observation $X \cong Y \Rightarrow X[p] \cong Y[p]$ shows that any leaf in the last sense is contained in precisely one NP-stratum (in the first sense); the main result of [26], "*X is minimal iff $X[p]$ is minimal*", shows that *a stratum* (in the first sense) *and a leaf* (in the second sense) *are equal* iff we are in the minimal situation, see Section 4.

All base fields, and base schemes will be of characteristic p. We denote by k an algebraically closed field. We consider mainly moduli of polarized abelian varieties, although many theorems can be formulated also for (and many proofs are via) p-divisible groups.

1 Newton polygon strata

See [13], [29], [5], [8], [20], [23], [7], [24].

Grothendieck showed that "Newton polygons go up under specialization". This was made more precise in:

(1.1) Theorem (Grothendieck-Katz). *Let $X \to S$ be a p-divisible group over a scheme in characteristic p. Let β be a NP. The set*

$$\mathcal{W}_\beta(S) := \{s \in S \mid \mathcal{N}(X_s) \prec \beta\} \quad \subset \quad S$$

is a closed subset. See: [8].

(1.2) Definition. For a symmetric NP ξ, ending at $(2g, g)$ we define

$$\mathrm{sdim}(\xi) = \#\left(\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g, \quad (x,y) \prec \xi\}\right),$$

where $(x, y) \prec \xi$ means "(x, y) is on or above ξ", see [23], 3.3.

We write $W_\xi = \mathcal{W}_\xi(\mathcal{A})$.

(1.3) Theorem. *For a symmetric NP ξ, ending at $(2g, g)$:*

$$\dim(W_\xi) \quad = \quad \mathrm{sdim}(\xi).$$

A purely combinatorial formula computes these dimensions. Note that we essentially use the fact that we consider *principally* polarized abelian varieties.

(1.4) Corollary (the Manin conjecture, see [13], page 76). *For any prime number p, for any symmetric Newton polygon ξ there exists an abelian variety A defined over $\overline{\mathbb{F}_p}$ with $\mathcal{N}(A) = \xi$.*

A proof was given in the Honda-Serre-Tate theory, see [29]. Another proof was given via deformation theory in characteristic p, see [24]: for every symmetric Newton polygon ξ the locus $W^0_\xi \neq \emptyset$, and the conjecture by Manin follows. Also see [23], Section 5, for a shorter proof.

Much more can be said about the structure of NP-strata in $\mathcal{A}_g \otimes \mathbb{F}_p$.

(1.5) Theorem (conjectured by Grothendieck, Montreal 1970). *Let G_0 be a p-divisible group, and $\mathcal{N}(G_0) =: \gamma \prec \beta$. There exists a deformation $\mathcal{G}_\eta$ of G_0 such that $\mathcal{N}(G_\eta) = \beta$.*
I.e. any Newton polygon below $\mathcal{N}(G_0)$ appears on $\mathrm{Def}(G_0)$.

We sketch a proof of (1.3) and (1.5).

Defo to $a = 1$. Using [7], joint work with Johan de Jong, we see that every isosimple p-divisible group can be deformed, keeping the same NP, to a p-divisible group with $a = 1$; see [7], 5.12.

From this we deduce that any p-divisible group, or any principally polarized abelian variety can be deformed, keeping the same NP, to $a = 1$, see [24], 2.8 and 3.10.

Cayley-Hamilton. A non-commutative variant of the CH-theorem from linear algebra can be applied to the matrix of Frobenius of a p-divisible group with $a = 1$, see [23]. This proves the Grothendieck conjecture (1.5), and it allows us to read off the dimension of NP strata.

Computations are made possible via the theory of "displays" as initiated by Mumford, see [15], [16], [30].

(1.6) Remark. Note the rather indirect way to prove (1.5) and (1.3). I do not know a direct approach. However deformation theory is manageable in the case $a = 1$ (these points are non-singular on the NP stratum considered !), and the Purity-result allows us to "move away" from the points with $a > 1$ (which can be singular points).

2 Ekedahl-Oort strata

This is joint work Torsten Ekedahl - Frans Oort. See: [9], [10], [22]; also see [28].

(2.1) Basic idea: *over an algebraically closed field k, for a given rank, all commutative group schemes over k, of that rank, annihilated by p give only finitely many isomorphism classes.* See [9], [10].

Let us denote by Φ_g the set of isomorphism classes of BT_1 group schemes with an alternating, non-degenerate bilinear form over k. Remark: for $p > 2$ this is literally what is written; for $p = 2$ we have to consider such a form on its Dieudonné module, see [22], Section 9 for a discussion, and for theorems. We write $\mathrm{ES}(A, \lambda) \in \Phi_g$ for the isomorphism class, over k, of $(X = A[p^\infty], \lambda)[p]$.

(2.2) Property/definition. *Let $\varphi \in \Phi_g$, and consider*

$$S_\varphi := \{[(A, \lambda)] \mid \mathrm{ES}(X, \lambda) = \varphi\} \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p.$$

This subset is locally closed in $\mathcal{A}$. It is called the EO*-stratum defined by φ.*

(2.3) Theorem. *For every $\varphi \in \Phi_g$ the stratum $S_\varphi \subset \mathcal{A}$ is quasi-affine* (i.e. an open set in an affine scheme). *The boundary of every $S_\varphi \subset \mathcal{A}$ is the union of lower dimensional strata.*

In [22], 1.2, and in [28], 6.10 we find a computation of the dimension of these strata.

3 Foliations

See: [25], [31], [27], [14].

(3.1) Let (X,λ) be a quasi-polarized p-divisible group. Let $X \to S$ be a quasi-polarized p-divisible group defined over a field L. We define

$$\mathcal{C}_{(X,\lambda)}(S) = \{s \mid \kappa(s) \subset k, \quad L \subset k \quad (X,\lambda) \otimes_L k \cong (X_s,\lambda_s) \otimes_{\kappa(s)} k\}.$$

(3.2) Theorem. *The subset $\mathcal{C}_{(X,\lambda)}(S) \subset S$ is locally closed, and $\mathcal{C}_{(X,\lambda)}(S) \subset \mathcal{W}^0_{\mathcal{N}(X)}(S)$ is closed.*

See [25]. The proof uses the notion "completely slope divisible p-divisible groups" as introduced by Zink, and it uses the main result of [27].

(3.3) Definition. Let $x = (A,\lambda) \in \mathcal{A}_g \otimes \mathbb{F}_p$ and $(X,\lambda) := (A_0,\lambda_0)[p^\infty]$. We write $\mathcal{C}(x) = \mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$; an irreducible component of $\mathcal{C}(x)$ will be called a *central leaf.*

(3.4) Theorem. (i) *For central leaves C', C'' contained in the same NP-stratum there is a correspondence $C' \leftarrow \Gamma \to C''$ which in both directions is finite surjective.*
(ii) *For every $d \in \mathbb{Z}_{>0}$, a perfect field K and every $x \in \mathcal{A}_{g,d}(K)$, the scheme (with induced, reduced structure) $\mathcal{C}(x)$ is smooth over K; there is a number $c = c(\xi)$ depending only on ξ such that every irreducible component of $\mathcal{C}(x)$, with $x \in \mathcal{W}^0_\xi(\mathcal{A}_{g,d}(K)$, has dimension equal to $c(\xi)$.*

(3.5) Remark. The EO stratification is given by isomorphism classes of BT_1 group schemes; It is easy to see that the EO-strata are locally closed. In contrast however, my proof of the theorem above is not so easy.

(3.6) Let $[(A,\lambda)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$. Consider the "Hecke-$\alpha$-orbit" of x, i.e. all moduli points obtained by isogenies with local-local kernels (iterated α_p-isogenies). An geometrically irreducible component of a Hecke-α-orbit is called an *isogeny leaf.*

It is not difficult to see that an isogeny leaf $I \subset \mathcal{A}_g \otimes \mathbb{F}_p$ is a closed subset, it is complete as a (reduced) scheme, and it is contained in precisely one open NP stratum.

The structure given by central leaves and isogeny leaves is of great beauty:

(3.7) Theorem (the product structure defined by central and isogeny leaves). **(i)** *Let C and I be a central leaf, respectively an isogeny leaf through $x \in \mathcal{A}_g \otimes k$. Every irreducible component of $C \cap I$ has dimension equal to zero.*
(ii) *Let ξ be a symmetric NP, and let W be an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g \otimes k)$. Let C be a central leaf in W and I an isogeny leaf in W. There exists an integral schemes T of finite type over k, a finite morphism $T \to C$ and a finite surjective morphism*

$$\Phi : \quad T \times I \quad \twoheadrightarrow \quad W \subset \mathcal{A}_{g,d} \otimes k$$

such that

$$\forall u \in I(k), \quad \Phi(T \times \{u\}) \quad \textit{is a central leaf in} \quad W,$$

every central leaf in W can be obtained in this way,

$$\forall t \in T(k), \quad \Phi(\{t\} \times I) \quad \textit{is an isogeny leaf in} \quad W,$$

and every isogeny leaf in W can be obtained in this way.
See [25], 5.3; for an application of this product structure in mixed characteristics, see [14].

(3.8) Remark. Any two central leaves associated with the same NP have the same dimension (independently of the degree of the polarizations in consideration). However the dimensions of NP-strata and of isogeny leaves in general do depend on the degree of the polarizations.

Motivation / explanation. In the moduli space of abelian varieties in characteristic p we consider Hecke orbits related to isogenies of degree prime to p, and Hecke orbits related to iterated α_p-isogenies. The first "moves" points in a central leaf: under such isogenies the geometric p-divisible group does not change; the second moves points in an isogeny leaf. We can expect that these two natural foliations describe these two "transversal" actions: see (5.2) and (5.3).

(3.9) As illustration we record for $g = 4$ the various data considered:

NP	ξ	f	$sdim(\xi)$	$c(\xi)$	$i(\xi)$	ES($H(\xi)$)
ρ	$(4,0)+(0,4)$	4	10	10	0	$(1,2,3,4)$
$f=3$	$(3,0)+(1,1)+(0,3)$	3	9	9	0	$(1,2,3,3)$
$f=2$	$(2,0)+(2,2)+(0,2)$	2	8	7	1	$(1,2,2,2)$
β	$(1,0)+(2,1)+(1,2)+(0,1)$	1	7	6	1	$(1,1,2,2)$
γ	$(1,0)+(3,3)+(0,1)$	1	6	4	2	$(1,1,1,1)$
δ	$(3,1)+(1,3)$	0	6	5	1	$(0,1,2,2)$
ν	$(2,1)+(1,1)+(1,2)$	0	5	3	2	$(0,1,1,1)$
σ	$(4,4)$	0	4	0	4	$(0,0,0,0)$

Here $\rho \succ (f=3) \succ (f=2) \succ \beta \succ \gamma \succ \nu \succ \sigma$ and $\beta \succ \delta \succ \nu$. The notation ES, encoding the isomorphism type of a BT_1 group scheme, is as in [22]; the number f indicates the p-rank; the number $i(\xi)$ denotes the dimension of isogeny leaves in $\mathcal{W}^0_\xi(\mathcal{A})$.

(3.10) Note that the dimensions of all central leaves in $\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ are equal; however the dimensions of the components of this NP-stratum and the dimensions of the isogeny eaves depends on the degree of the polarization considered.

4 Minimal p-divisible groups

Clearly every central leaf is contained in a unique EO-stratum. Does it ever happen that an EO-stratum is equal to a central leaf? There is a complete and simple answer to this.

We study the following:
Question. *Suppose X and Y defined over k have the property that $X[p] \cong Y[p]$; does this imply that $X \cong Y$?* Clearly in general the answer is negative.

For every isogeny class of p-divisible groups we define a unique member, which we call a "minimal p-divisible group": if X is isosimple over k, it is moreover minimal iff $\mathrm{End}(X)$ is the maximal order in its full ring of fractions $\mathrm{End}^0(X)$; a p-divisible group is minimal iff it is a direct sum of isosimple minimal ones. Every p-divisible group is isogenous to a unique minimal one (unique up to isomorphism, everything over k).

(4.1) Theorem. *Work over k. Let H be a minimal p-divisible group. Let X be a p-divisible group such that $X[p] \cong H[p]$. Then $X \cong H$.* See [26].
Remark. We have no a priori condition on the Newton polygon of X.

In fact, this theorem is "optimal": if X is not minimal, there exists infinitely many mutually non-isomorphic Y with $Y[p] \cong X[p]$.

(4.2) Remark. Here are two special cases of the theorem:
ordinary Suppose that $X[p]$ is isomorphic with a direct sum of copies of μ_p and $\underline{\mathbb{Z}/p}$: this is called the ordinary case; for this the claim of the theorem is clear.
superspecial Suppose that X has "no étale part" and that $a(X) = \dim(X)$; then X is isomorphic with a product of supersingular one-dimensional formal groups (the "superspecial case", see [19], Th. 2).
The theorem is a generalization of these two cases to all possible Newton polygons.

5 Some questions and conjectures.

(5.1) Strata and subsets defined above are obtained via "set-theoretical" definitions; after having proved we obtain a (locally) closed set, over a perfect field we give the induced, reduced scheme structure. It would be much better to define these objects via a functorial approach (and to have possible non-reduced scheme structures ?). I have not been able to work via these lines. Please tell me if you have an idea, a result, or anything else in this direction.

(5.2) Conjecture. Consider a point $[(A, \lambda)] = x \in \mathcal{A} = \mathcal{A}_g \otimes \mathbb{F}_p$, and consider its Hecke orbit $\mathcal{H}(x) \subset \mathcal{A}$. *We expect this Hecke orbit to be dense in its Newton polygon stratum in the moduli space, i.e. the Zariski closure is expected to be:*

$$\overline{\mathcal{H}(x)} \quad \stackrel{?}{=} \quad \mathcal{W}_{\mathcal{N}(A)}(\mathcal{A}).$$

Notation: $\mathcal{H}(x)$ is the set of all $y = [(B, \mu)]$ such that there exists a field L and isogenies $(A, \lambda)_L \leftarrow (M, \zeta) \rightarrow (B, \mu)_L$. We write $y \in \mathcal{H}_\ell(x)$ if moreover ℓ is a prime number and the degrees of the isogenies considered are powers of ℓ.

Note that Hecke-ℓ-actions "move" in a central leaf: under ℓ-degree-isogenies, with $\ell \neq p$ the p-divisible groups are unchanged. Note that Hecke-α-actions (isogenies with local-local kernel) "move" in an isogeny leaf. Conjecture (5.2) follows, in case we can prove:

(5.3) Conjecture. *For every prime number ℓ, different from p, the Hecke-ℓ-orbit $\mathcal{H}_\ell(x)$ in $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is Zariski-dense in $\mathcal{C}(x)$, a union of central leaves.*

(5.4) Canonical coordinates. In [2] we give a generalization of Serre-Tate canonical coordinates on the ordinary locus to "canonical coordinates" locally at a point of an arbitrary central leaf.

(5.5) Note that isogeny correspondences blow up and down between NP strata in characteristic p. However *isogeny correspondences are finite-to-finite between central leaves.* Central leaves in positive characteristic seem very much analogous to behavior in characteristic zero.

Components of a NP-stratum can have different dimensions if all degrees of polarizations are considered. In contrast with Th. (1.3) we expect:

(5.6) Conjecture. *Let ξ be a symmetric Newton polygon, with p-rank equal to f, i.e. ξ has exactly f slopes equal to zero. We expect that $\mathcal{W}_\xi(\mathcal{A}_g)$ has a component of dimension precisely* $(g(g-1)/2) + f$. Note that it is clear that every such component has at most this dimension.

(5.7) Let us choose a number $i \in \mathbb{Z}_{>0}$. For any point $[(X, \lambda)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$ we can consider $\varphi := [(X, \lambda)[p^i]]$, and we can study $S_\varphi^{(i)}(\mathcal{A}_g \otimes \mathbb{F}_p)$, the set of points $y = [(Y, \mu)]$ such that there exist: an algebraically closed field k over which y is defined, and an isomorphism $(X, \lambda)[p^i] \otimes k \cong (Y, \mu)[p^i] \otimes k$; probably this is a locally closed set in $\mathcal{A}_g \otimes k$.

Choosing $i = 1$ we obtain $S_\varphi^{(1)}(\mathcal{A}_g \otimes \mathbb{F}_p) = S_\varphi$, the EO-strata as defined in [22].
Note that the leaves defined by $S^{(i+1)}$ are contained in leaves defined by $S^{(i)}$: for $\varphi_1 = , \varphi_2, \cdots, \varphi_i = [(X, \lambda)[p^i]], \cdots$ all coming from the same (X, λ) we obtain $S_{\varphi_1}^{(1)}(-) \supset S_{\varphi_2}^{(2)}(-) \supset \cdots$; this descending chain stabilizes after a finite number of steps.
For $i >> 1$ we obtain central leaves: given g, there exists N such that for every x we have $S_{\varphi_N}^{(N)}(-) = \mathcal{C}_x(-)$.

We studied $S^{(1)}$ in [22], and we consider $S^{(N)} = S^{(\infty)}$ in [25]; one could also study the "intermediate" cases $S^{(i)}$.

We know that the supersingular locus has "many" components (if p is large). However we expect:

(5.8) Conjecture. *Let ξ be a symmetric Newton polygon, of height $2g$, not equal to the supersingular one: $\xi \gneq \sigma$. We expect that $\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$ is* geometrically irreducible.

(5.9) Let W be a component of a NP-stratum, and let η be its generic point, and A the corresponding abelian variety over $\overline{\kappa(\eta)}$. If A is not supersingular then $\mathrm{End}(A) = \mathbb{Z}$; this can be proved, using [11]. Can this be used to prove the conjecture above? Note that different components of the supersingular locus are given by "different polarizations", see [12].

(5.10) *We try to study the closure of central leaves in lower Newton polygon strata.* Consider two symmetric Newton polygons $\xi' \prec \xi$. We expect that in general a central leaf in $\mathcal{W}^0_{\xi'}(\mathcal{A})$

need not be contained in the closure of a central leaf in $\mathcal{W}^0_\xi(\mathcal{A})$. I am trying to formulate at least an expectation describing a relation between the central foliations of different NP-strata.

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