Hecke orbits in moduli spaces of abelian varieties and foliations.

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Most results in this talk are joint work with Ching-Li Chai; the influence of his ideas will be clear in several aspects. I received further inspiration from joint work with Tadao Oda, Ke-Zheng Li, Torsten Ekedahl, Eyal Goren, Johan de Jong, Thomas Zink and Chai-Fu Yu.

1 Introduction

(1.1) Notation. *We write $\mathcal{A}_g \to \mathrm{Spec}(\mathbb{Z})$ for the moduli space of polarized abelian varieties.* We will discuss characteristic zero briefly, and then turn to abelian varieties and their moduli spaces in **positive characteristic**.

(1.2) Suppose given a field K, a polarized abelian variety (A,λ) over K. We define the *Hecke orbit* of the moduli point $x := [(A,\lambda)]$ to be the set of points $y = [(B,\mu)]$ over some field L such that there exist a field Ω containing K and L, an integer $n \in \mathbb{Z}_{>0}$ and an isogeny

$$\varphi : A_\Omega \to B_\Omega \quad \textit{such that} \quad \varphi^*(\mu) = n\cdot \lambda,$$

that is:

A and B are geometrically isogenous,
and the $\mathbb{Q}$-classes of λ and μ are equal;

Notation:

$$[(B,\mu)] = y \in \mathcal{H}(x).$$

We write $y \in \mathcal{H}_\ell(x)$ when ℓ is a prime number, and in the notation above, n and $\deg(\varphi)$ are powers of ℓ.

$\boxed{\textbf{Question. } \textit{What is the closure of } \mathcal{H}(x) \textit{ and of } \mathcal{H}_\ell(x)\textit{ ?}}$

(1.3) If $K = \mathbb{C}$, then:

$$\mathcal{H}(x) \textit{ is dense in } \mathcal{A}_g(\mathbb{C}),$$

in the sense of the "classical topology", and (then, of course)

$$\mathcal{H}(x) \textit{ is dense in } \mathcal{A}_g,$$

in the sense of the Zariski topology; the same statements hold for $\mathcal{H}_\ell(x) \subset \mathcal{A}_g(\mathbb{C})$ and $\mathcal{H}_\ell(x) \subset \mathcal{A}_g$. *The Hecke Orbit problem seems to be clear over* $\mathbb{C}$, leaving aside interesting arithmetic questions in characteristic zero.

(1.4) Example. Consider $K \supset \mathbb{F}_p$, let $g = 1$, and consider an *elliptic curve* E such that $E[p](k) = 0$, i.e. the elliptic curve has no geometric points of order exactly p; such an elliptic curve is called *supersingular*. Let $x = [(E, \lambda)]$, where λ is the unique principal polarization on E. Note that if $\varphi : E \to E'$ is an isogeny, then also E' is supersingular. Note that the set of supersingular j-values is finite.

Conclusion: $\mathcal{H}(x) \cap \mathcal{A}_{1,1} \otimes k$ is *finite*.
We see that $\mathcal{H}(x)$ is *not Zariski-dense in* $\mathcal{A}_{1,1} \otimes \mathbb{F}_p$.

Note that if $x = [(E, \lambda)]$, where E is an *ordinary* elliptic curve over $K \supset \mathbb{F}_p$, then $\mathcal{H}(x)$ is *Zariski-dense in* $\mathcal{A}_{1,1} \otimes \mathbb{F}_p$. This is not difficult to show. For arbitrary g:
Theorem (Chai) *If* $[(A, \lambda)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$ *is the moduli point of an* ordinary *abelian variety in positive characteristic, then* $\mathcal{H}_\ell(x)$ *is Zariski-dense in* $\mathcal{A}_g \otimes \mathbb{F}_p$.
See [1].

Try to predict what the Zariski closure is of $\mathcal{H}(x)$ *for a moduli point* $x \in \mathcal{A}_g \otimes \mathbb{F}_p$.
The analogous question: what is the Zariski closure of $\mathcal{H}_\ell(x) \subset \mathcal{A}_g \otimes \mathbb{F}_p$?

As points in a Hecke orbit correspond with isogenous abelian varieties, it seems logical to look for (isogeny-) invariants attached to an abelian variety invariant under an isogeny (or invariant under an isogeny of degree prime to p).

From now on in this note, we fix a prime number p*, all base fields considered are of characteristic* p*, and all schemes are over* $\mathrm{Spec}(\mathbb{F}_p)$.

2 The Hecke orbit conjecture

(2.1)$_?$ **The Hecke orbit conjecture;** see [15].

For any point $x := [(A, \lambda)] \in \mathcal{A}_g \otimes \mathbb{F}_p$ *its Hecke orbit*
$\mathcal{H}(x)$ *is Zariski-dense in its Newton polygon stratum* $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$.

Notions used in this statement will be explained below.

3 Results

From now on we choose an algebraically closed field $k \supset \mathbb{F}_p$. We write

$$\mathcal{A} = \mathcal{A}_{g,1} \otimes k$$

for the moduli space of principally polarized abelian varieties defined over fields containing k.

(3.1) p-adic invariants. To an abelian A over a field $k = \overline{k} \supset \mathbb{F}_p$ we attach its p-divisible group

$$A \quad \mapsto \quad X := A[p^\infty].$$

To a p-divisible group we can attach various "invariants":

$X[p^\infty]$ up to $\sim$	ξ	NP	W_ξ
$X[p^1]$ up to $\cong$	φ	EO	S_φ
$(X[p^\infty], \lambda)$ up to $\cong$	X	Fol	$C(x)$

We explain these notions and notations below.

(3.2) Newton polygon strata. For every symmetric Newton polygon ξ we define a closed subset $W_\xi \subset \mathcal{A}$. From the definitions it follows that for $x \in W_\xi$ we have:

$$\mathcal{H}(x) \subset W_\xi.$$

Foliation. For every $x = [(A, \lambda)] \in \mathcal{A}$, with $\mathcal{N}(A) = \xi$ we define a locally closed set $\mathcal{C}(x) \subset \mathcal{A}$, in fact $\mathcal{C}(x) \subset W^0_\xi$ is closed. From the definitions it follows that

$$\mathcal{H}_\ell(x) \subset \mathcal{C}(x).$$

(3.3) Theorem. (FO and Ching-Li Chai).
(1) *For every ξ, a Newton polygon which is* not supersingular, $\xi \neq \sigma$, *the Newton polygon stratum*

$$W_\xi \subset \mathcal{A} \quad \textit{is} \text{ irreducible.}$$

(2) *For every $x = [(A, \lambda)] \in \mathcal{A}$ such that A is* not supersingular, $x \notin W_\sigma$, *the leaf*

$$\mathcal{C}(x) \subset \mathcal{A} \quad \textit{is} \text{ irreducible.}$$

[This is called the discrete form of the Hecke orbit conjecture.]
For this theorem see [2], [22], [23], [21].
Part (1) will be discussed in Section 7. Part (2) is included for completeness. Details in Section 8 will not be discussed in my talk.

(3.4) Theorem. (Ching-Li Chai, FO, using a result by Chai-Fu Yu)
For every $x = [(A, \lambda)] \in \mathcal{A}$ which is not supersingular, *and for every prime number ℓ different from p*

$$\mathcal{H}_\ell(x) \quad \text{is dense in} \quad \mathcal{C}(x).$$

[This is called the continuous form of the Hecke orbit conjecture.]
An ingenuous proof, mainly by Ching-Li Chai (one of the arguments is a generalization of Serre-Tate local coordinates) reduces (3.4) to an analogous discrete Hecke orbit conjecture for Hilbert modular varieties. This problem for Hilbert modular varieties has now been solved by Chai-Fu Yu (August 2003, not yet published). Further details will be published in [3].

In this talk I will explain these notions and I will show, see Section 9 that these theorems imply the full Hecke orbit conjecture:

$$\boxed{(3.3) + (3.4) + [19] \quad \Longrightarrow \quad (2.1) \text{ is true,}}$$

i.e.

$$\boxed{x \in W^0_\xi \quad \Longrightarrow \quad \mathcal{H}(x) \quad \text{is dense in} \quad W_\xi.}$$

4 Newton polygon strata

(4.1) NP: $X[p^\infty]$ up to $\sim$.
Dieudonné - Manin:

$$\{X\}/\sim \quad \xrightarrow{\sim} \quad \{\mathrm{NP}\} \qquad X \mapsto \mathcal{N}(X):$$

Over an algebraically closed field the set of isogeny classes of p-divisible groups is the same as the set of Newton polygons.

Newton polygons are partially ordered; we write $\gamma \prec \beta$ if no point of γ is below β:

$$\gamma \prec \beta \quad \Leftrightarrow \quad \gamma \text{ is "above" } \beta.$$

An abelian variety A is isogenous with its dual A^t; using the duality theorem, see [11], 19.1 we conclude that $X \sim X^t$; hence $\mathcal{N}(A) =: \xi$ is symmetric: if the slope λ appears in ξ with multiplicity n_λ, then $1-\lambda$ also appears with that multiplicity: $n_\lambda = 1 - n_{1-\lambda}$.

For a symmetric Newton polygon ξ we write:

$$\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) = \{[(A,\lambda)] \mid \mathcal{N}(A) \prec \xi\},$$

$$\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) = \{[(A,\lambda)] \mid \mathcal{N}(A) = \xi\}.$$

Grothendieck - Katz:

$$\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \textit{ is closed.}$$

See [6]; hence

$$\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \textit{ is locally closed.}$$

These are called the **Newton polygon strata.** We write

$$W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p), \quad W^0_\xi = \mathcal{W}^0_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p).$$

We write:

$$\mathcal{A} = \mathcal{A}_g \otimes \mathbb{F}_p, \quad W_\xi = \mathcal{W}_\xi(\mathcal{A}), \quad W^0_\xi = W^0_\xi(\mathcal{A}).$$

(4.2) Isogeny correspondences on $\mathcal{A}_g(\mathbb{C})$, also called Hecke correspondences, *are finite-to-finite.* However (for $g \geq 2$, and for isogeny degrees divisible by p), isogeny correspondences on $\mathcal{A}_g \otimes \mathbb{F}_p$ *blow up and down.*

There is a nice and clean formula for the dimension of $W_\xi \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$. However different components of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ can have different dimensions. These phenomena make the study of Newton polygon strata so hard, and also very interesting.

5 EO - strata

This section is included for completeness sake, but the contents of this section will not be discussed

(5.1) EO: $X[p^1]$ up to $\cong$.
In [16] we find: Let G be a "1-truncated Barsotti-Tate group". This means that there exists a p-divisible group X such that $X[p] \cong G$. Denote its isomorphism class by φ. *For a fixed $n \in \mathbb{Z}_{>0}$, the set of isomorphism classes of finite group schemes over k of rank p^n is finite* (proved by Kraft, and rediscovered by FO). For an abelian scheme $A \to T$ we define

$$\mathcal{S}_\varphi(T) = \{s \in T \mid \exists\Omega : A_s[p]_\Omega \cong G_\Omega\} \subset T.$$

In [16] we find:

$$S_\varphi := \mathcal{S}_\varphi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \quad \textit{is locally closed.}$$

Also we find a discussion of the isomorphism class of $(A, \lambda)[p]$ in case λ is a principal polarization, and we find a description of the strata S_φ (their dimensions, etc.).
Important point: *Every S_φ is quasi-affine.* This generalizes and was inspired by a result by Raynaud, who proved this for $(\mathcal{A}_g \otimes \mathbb{F}_p)^{\mathrm{ord}}$ (published by Szpiro and by Moret-Bailly).
The EO-strata give a (finite) stratification of every component of $\mathcal{A}_g \otimes \mathbb{F}_p$.

EO is an abbreviation for T. Ekedahkl and F. Oort, describing our joint work, see [16].

The set of isomorphism classes of group schemes of given rank annihilated by p is finite; this was discovered by H. Kraft, see [7], later rediscovered by FO. This fact shows that the stratification by EO-strata is a finite stratification.

6 Foliations

(6.1) Fol: $X[p^\infty]$ up to $\cong$.
Choose $x \in \mathcal{A}_g \otimes \mathbb{F}_p$; write $x = [(A, \lambda)]$, over some field, say over k, and write $(X, \lambda) = (A, \lambda)[p^\infty]$. Define:

$$C(x) = \mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes k) = \{[(B, \mu)] \mid \exists\Omega : (B, \mu)_\Omega \cong (X, \lambda)_\Omega\}.$$

In [19] we find:
Theorem. *$\mathcal{C}(x) \subset \mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ is a* closed *subset.*
This uses [24]. An irreducible component of $\mathcal{C}(x)$ is called a *central leaf.*
Remark. For x supersingular, $\mathcal{C}(x)$ has "many" components (for $p >> 0$). For x not supersingular we will see that $\mathcal{C}(x)$ is irreducible.

If x and y are two moduli-points in the same open Newton polygon stratum (with maybe different degrees of polarization), then any component of $C(x)$ has the same dimension as a component of $C(y)$. *Isogeny correspondences (Hecke correspondences) are finite-to-finite on central leaves.* We could say: "abelian varieties in one central leaf behave as if they live in characteristic zero".

(6.2) Define Hecke-α_p-orbits in the following way. We write $\alpha_p := \mathbb{G}_a[F]$ for the finite group scheme which is the kernel of F on the linear group $\mathbb{G}_a$ in characteristic p.
Notation: the Hecke-α-orbit. *If (A, λ) is a polarized abelian variety, we write*

$$[(B, \mu)] = y \in \mathcal{H}_\alpha(x)$$

if there exists a field Ω and isogenies of polarized abelian varieties

$$(B, \mu)_\Omega \longleftarrow (Z, \zeta) \longrightarrow (A, \lambda)_\Omega$$

such that the kernels of these isogenies are successive extensions of α_p.
An irreducible component of a Hecke-α-orbit is called an *isogeny leaf.*
Instead of "isogeny leaf" the terminology of "Rapoport-Zink space" is used. Warning: in general $\mathcal{H}_\alpha(x) \subset W_\xi$ is not closed (it can have an infinite number of irreducible components); however every irreducible component of $\mathcal{H}_\alpha(x)$ is proper over the base field (gives rise to a complete variety).

(6.3) A product structure on an open Newton polygon stratum.
See [19], Theorem (5.3):
For every symmetric Newton polygon ξ, and every irreducible component W of W_ξ there exist varieties T and J and a finite surjective morphism

$$\Phi : T \times J \longrightarrow W$$

such that:

$$\forall u \in J(k), \quad \Phi(T \times \{u\}) \quad \textit{is a central leaf in} \quad W,$$

every central leaf in W can be obtained in this way,

$$\forall t \in T(k), \quad \Phi(\{t\} \times J) \quad \textit{is an isogeny leaf in} \quad W,$$

and every isogeny leaf in W can be obtained in this way.

(6.4) Remark. For the supersingular locus $\xi = \sigma$, every irreducible component $W \subset W_\sigma$ is an isogeny leaf. For $\xi \neq \sigma$ we will show that $W = W_\xi$ is irreducible.

(6.5) Write $\dim(\xi)$ for the dimension of $W_\xi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, and $c(\xi)$ for the dimension of a central leaf in W_ξ and $i(\xi)$ for the dimension of a isogeny leaf in W_ξ. Clearly $\dim(\xi) = c(\xi) + i(\xi)$. There are formulas to compute these invariants; see [18], see [3]. Here are some examples: take $g = 3$, write f for the p-rank; for $\dim(\xi) = c(\xi) + i(\xi)$ we obtain:
$f = 3, \quad \xi = \rho = 3(1,0) + 3(0,1), \quad 6 = 6 + 0;$
$f = 2, \quad \xi = 2(1,0) + (1,1) + 2(0,1), \quad 5 = 5 + 0;$
$f = 1, \quad \xi = (1,0) + 2(1,1) + (0,1), \quad 4 = 3 + 1;$
$f = 1$ and $\xi = (2,1) + (1,2), \quad 3 = 2+1;$
$f = 1$ and $\xi = \sigma = 3(1,1)), \quad 2 = 0 + 2$.
The example $g = 4$ is presented in [19].

(6.6) Remark. For all polarization degrees, the dimension of all central leaves in a NP stratum are equal; however the NP stratum in general has components of different dimensions, and hence the dimension of isogeny leaves can vary inside one NP stratum.

(6.7) Note:

∞ $\sim$	**NP**	$\mathcal{N}(A)$ is invariant under isogenies.
1 $\cong$	**EO**	φ is invariant under isogenies of degree prime to p.
∞ $\cong$	**Fol**	$(A, \lambda)[p^\infty]$ invariant under isogenies of degree prime to p.

7 Irreducibility of NP strata

(7.1) Example; $g = 1$ For $g = 1$, the case of elliptic curves, there are just two possible Newton polygons:

- either $\rho = (1, 0) + (0, 1)$ the *ordinary case*,
- or $\sigma = (1, 1)$ the *supersingular case.*

We know, even for arbitrary g, that $W_\sigma = \mathcal{A}$; this moduli space is irreducible (Chai, Faltings).

What can be said about the number of components of the supersingular stratum $W_\sigma \subset \mathcal{A}$?

For the case of elliptic curves in characteristic p this is classical. By Hasse, Deuring, Igusa we know:

$$\sum_{j(E) \text{ is ss}} \frac{1}{\#(\mathrm{Aut}(E))} = \frac{p-1}{24};$$

this implies that the number of supersingular j-invariants is asymptotically $p/12$ for $p \to \infty$.

(7.2) Example. Note that the supersingular locus has "many components" for fixed g and large p; see [8], 4.9:

$$\#(\Pi_0(W_\sigma)) = H_g(p, 1) \quad \text{if} \quad g \quad \text{is odd,}$$
$$\#(\Pi_0(W_\sigma)) = H_g(1, p) \quad \text{if} \quad g \quad \text{is even.}$$

Note that for g fixed, and $p \to \infty$, indeed $\#(\Pi_0(W_\sigma)) \to \infty$.

One might wonder what the number of irreducible components is for an arbitrary Newton polygon stratum; it seems quite a job to compute the number of components of all Newton polygon strata. However, as it will turn out: *any non-supersingular NP stratum W_ξ is irreducible,* see (7.6) below.

(7.3) Theorem. *For every g, for every prime number $\ell \neq p$, for every N as above, and for every symmetric Newton polygon ξ, the action of $\mathcal{H}_\ell$ on the set of irreducible components $\Pi_0(W_{N,\xi})$ of the Newton polygon stratum $W_{N,\xi} \subset \mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$ is* transitive.
See [23].

(7.4) Let $Z \subset \mathcal{B}$ be a locally closed subset. Let Z' be an irreducible component of Z. Let $\eta \in Z'$ be the generic point. Let $A \to Z$ be the universal abelian scheme restricted to Z. Let

$$\rho_{A,\ell} : \pi_1(Z', \overline{\eta}) \quad \longrightarrow \quad \mathrm{Sp}(T_\ell, <, >_\ell)$$

be the ℓ-adic monodromy representation in the Tate-ℓ-group of A_η. Identify the Tate-ℓ-group of A_η over $\overline{\eta}$ with $\mathbb{Z}_\ell^{2g}$ with the standard pairing.

(7.5) Theorem (C.-L. Chai). *Choose notation as above. Let $Z \subset \mathcal{B}$ be a locally closed subscheme, smooth over* $\mathrm{Spec}(k)$*, such that:*
Z is Hecke-ℓ-stable, and
the Hecke-ℓ-action on the set $\Pi_0(Z)$ *is transitive, and*
$Z \not\subset W_{N,\sigma}$ (equivalently: Z contains a non-supersingular point).
Then:

$$\rho_{A,\ell} : \pi_1(Z', \overline{\eta}) \longrightarrow \mathrm{Sp}(T_\ell, <, >_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$$

is surjective, *and*

$$Z \text{ is irreducible, i.e. } Z = Z'.$$

See [2], 4.4.

(7.6) Corollary. *For every* $\xi \supsetneq \sigma$ *the locus* $W_{N,\xi} \subset \mathcal{B}$ *is* irreducible; i.e.

$$\xi \neq \sigma \quad \Rightarrow \quad \#(\Pi_0(W_{N,\xi})) = 1.$$

This was conjectured in [15]. This answers [26], 3.8. This implies Theorem (3.3), (1).

(7.7) A sketch of a proof of (7.6) **and hence of** (3.3).
In the proof of this result we work over $k = \overline{k}$. We study NP-strata $W_\xi \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes k$.

- *Every* EO-*strata is quasi-affine.* This a variant of a theorem by Raynaud; see [16].
- *Finiteness of Hecke orbits* (Chai):
  Proposition. *A Hecke-ℓ orbit $\mathcal{H}_\ell(x)$ is finite for $x = [(A_x, \lambda)]$ iff A_x is supersingular.*
  See [1], Prop. 1 on page 448.
- *For every symmetric Newton polygon ξ the (closed) locus W_ξ contains a supersingular point.* This follows using the previous two results.
- **Purity.** See [5], 4.1: *If the Newton polygon jumps in a family of p-divisible groups, it already jumps in codimension one.*
- **Claim.** (a) *For every symmetric Newton polygon ξ and every component, $W \in \Pi_0(W_\xi)$ there is a component $T \in \Pi_0(W_\sigma)$ such that $T \subset W$.*
  (b) *For every $\zeta \prec \xi$,*
  every component of W_ζ is contained in a unique component of W_ξ and
  every component of W_ξ contains at least one of W_ζ;
  by inclusion of components we obtain a well-defined map
  $$i : \Pi_0(W_{N,\zeta}) \to \Pi_0(W_{N,\xi}), \quad W' \mapsto i(W') = W \quad \text{if} \quad W' \subset W,$$
  which moreover is surjective.
  (c) *This map i is Hecke-ℓ equivariant.*
  This follows using the fact that W_ξ meets W_σ, using [17] and using Purity, and using my proof of a conjecture by Grothendieck, see [18].
- *The action of $\mathcal{H}_\ell$ on $\Pi_0(W_{N,\sigma})$ is* transitive.
  This uses a precise description of the set $\Pi_0(W_{N,\sigma})$ (here N indicates a level-structure), given in [10], 2.2 and 3.1 and [8], 3.6 and 4.2, and using the strong approximation theorem, see [25], Theorem 7.12 on page 427.
- **Conclusion.** This proves (7.3). By (7.5) this proves (7.6); hence (3.3), (1), follows.
- Details will appear in [23]

(7.8) Irreducibility of EO-strata. T. Ekedahl and G. van der Geer have proved (unpublished) a result proving the irreducibility of certain EO-strata. A description (FO) precisely which EO-strata are contained in the supersingular locus then results in:

Let φ be such that $S_\varphi \not\subset W_\sigma$ in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$. Then S_φ is geometrically irreducible.

This was conjectured in [16].
However, one can expect that for any φ such that $S_\varphi \subset W_\sigma$ we have $\Pi_0(S_\varphi) > 1$ for g fixed and $p \gg 0$. It seems that this has been proved now by S. Harashita.

8 Irreducibility of leaves

(8.1) Definition (C.-L. Chai) *An abelian variety A is called* hypersymmetric *if it is defined over a finite field*, hence A is defined over $m = \overline{\mathbb{F}_p}$, *and*

$$\mathrm{End}(A_m) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{End}(A_m[p^\infty])$$

is an isomorphism.
Note that being hypersymmetric is invariant under an isogeny; we could have formulated the definition by saying that the isogeny class is defined over a finite field, and then proceed as before.

(8.2) Irreducibility of leaves. Let $[(A, \lambda, \beta)] = x \in \mathcal{B}$ be the moduli point of a *non-supersingular* principally polarized abelian variety A with level structure; C.–L. Chai proves Theorem (3.3), (2):

if $x \notin W_\sigma$ then $C(x)$ is geometrically irreducible.

In the proof of this result (work over k):

- it is shown that for any symmetric NP there exists a hypersymmetric abelian variety; fix a principally polarized hypersymmetric (B, μ) with $\mathcal{N}(B) = \xi$;
- irreducibility of a NP-stratum is used, see (7.6), see (3.3), (1);
- use the foliation structure by isogeny leaves and by central leaves, see [19], see (6.3), and conclude that for any $C_i \in \Pi_0(C(x))$, and an isogeny leaf I passing through (B, μ) we have $C_i \cap I \neq \emptyset$; choose $[(B_i, \mu_i)] \in C_i \cap I$;
- use weak approximation in order to conclude that for every i and j we have that (B_i, μ_i) is in the prime-to-p Hecke orbit of (B_j, μ_j);
- use a variant of (7.5) in order to conclude that $C(x)$ is irreducible. $\square$

details will appear in [3].

9 The Hecke Orbit Conjecture via the product structure

Assume (3.3) **and** (3.4); **we show** (2.1). Note that $x \in W_\sigma$ We have $\mathcal{H}(x) = W_\sigma$. Let $x = [(A, \lambda)] \in \mathcal{A}$ such that A is not supersingular. We know by the continuous Hecke orbit conjecture, see (3.4), that for any prime number ℓ the Hecke-ℓ-orbit $\mathcal{H}_\ell(x)$ is dense in $C(x)$, which is irreducible by (3.3). Use (3.3) and conclude $W = W_\xi$ is irreducible. Use (6.3); let $H' \subset T$ be the inverse image under $\Phi : T \times \{u\} \to C(x)$ of $\mathcal{H}_\ell(x) \subset C(x)$. We see on the one hand that

$$H' \times J \quad \text{is dense in} \quad T \times J.$$

On the other hand

$$\Phi(H' \times J) \subset \mathcal{H}(x).$$

Hence

$$\mathcal{H}(x) \quad \text{is dense} \quad W_\xi.$$

This proves:

$$\text{(3.3) and (3.4)} \quad \Longrightarrow \quad \text{the Hecke orbit conjecture holds.}$$

10 Some topics and questions not treated in my talk.

(10.1) Let $x = [(A, \lambda)] \in W^0_\xi$ and let $\varphi = \mathrm{ES}(A[p])$. Then $C(x) \subset W^0_\xi$ and $C(x) \subset S_\varphi$. we have not discussed:

When is $C(x) = W^0_\xi$? Iff $f(A) \geq g - 1$.
When is $C(x) \subset S_\varphi$? See [20].
For which φ and ξ is $W^0_\xi \cap S_\varphi \neq \emptyset$?

(10.2) *Let $x = [(A, \lambda)] \in W^0_\xi$ and let $\zeta \prec \xi$. What is the "boundary" $\overline{C(x)} \cap W^0_\zeta$?*

There are examples of this situation where $y \in W^0_\zeta$ and $y \notin C(x)$ for every $x = [(A, \lambda)] \in W^0_\xi$. But there is a "guess" what the boundary of $C(x)$ in W^0_ζ could be.

(10.3) See [16], 14.3. On the set of isomorphism classes of $(A, \lambda)[p]$ there are two partial orderings "$\prec$" and "$\subset$". It seems unknown how to give an (easy, combinatorial) algorithm which decides for φ and φ' whether $\varphi' \subset \varphi$.

(10.4) We did not discuss structures needed for proving Conjecture (3.4), and details of that proof.

(10.5) We did not discuss the story of beautiful stratifications on Hilbert modular varieties (Rapoport, Deligne & Pappas, Bachmat & Goren, Goren & Oort, Andreatta & Goren, Yu, Harashita), and on other modular Shimura varieties (Moonen, Wedhorn).

11 Appendix: Some notations and abbreviations we are using.

$k = \overline{k} \supset \mathbb{F}_p$, an algebraically closed field.
$m = \overline{\mathbb{F}_p}$

NP = Newton polygon
EO = Ekedahl-Oort
Fol = Foliation(s)
HO = Hecke orbit

$\infty \sim$ $A[p^\infty]$ up to isogeny
$1 \cong$ $A[p]$ up to isomorphism
$\infty \cong$ $A[p^\infty]$ up to isomorphism

A, B, ... abelian varieties
X, Y, ... p-divisible groups
$\mathcal{A} = \mathcal{A}_{g,1} \otimes k, \quad \mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$
here $N \in \mathbb{Z}_{>0}$, not divisible by p, usually $N \geq 3$

$W_\xi = \mathcal{W}_\xi(\mathcal{A}), \quad W_{\xi,N} = \mathcal{W}_\xi(\mathcal{B})$

$\mathcal{N}(X)$ = the Newton polygon of X
$\mathcal{N}(A)$ = the Newton polygon of $A[p^\infty]$

ss = supersingular, i.e.

E, an elliptic curve in characteristic p, is supersingular if $E[p](k) = 0$,
an abelian variety A is supersingular if Ak is isogenous with E^g,
where E is a supersingular elliptic curve

Fact: A is ss iff all slopes in $\mathcal{N}(A)$ are equal to $\frac{1}{2}$;
this NP is denoted by σ, the ss NP

Let W be a scheme algebraic over a field K. We write $\Pi_0(W)$ for the set of irreducible components of W_k, where k is an algebraic closure of K.

We write ℓ for a prime number not dividing $p{\cdot}N$

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