Irreducibility of strata and leaves in the moduli space of abelian varieties

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Introduction

In the moduli space $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of principally polarized abelian varieties in characteristic p several strata and leaves are defined. These are nature-given locally closed subsets (which depend on the structure of the p-divisible groups of the fibers). In these talks

we give a survey of what is known about dimensions
and (ir)reducibility of these subspaces.

Let us start with an example (which is very easy now, but which was hard, say 25 years ago).

(0.1) Example. *Supersingular abelian surfaces.* We choose a prime number p and we write $V_0 \subset \mathcal{A}_{2,1} \otimes \mathbb{F}_p$ for the moduli space of principally polarized abelian surfaces with p-rank equal to zero. Note that $g = 2$ and p-rank is zero implies A is supersingular. It is known that V_0 has pure dimension one, see [24] or [35]. Families of supersingular abelian surfaces were studied in [38]. Moreover in [34] we see how to give such a family a principal polarization. It follows that V_0 is a union of rational curves.
Question. *Is V_0 irreducible?*
A partial answer. If $p > 1500$, the space V_0 is geometrically reducible.

Let us show an easy proof of this statement (which was a starting point of a lot of research). We know that the number h of isomorphism classes of supersingular j-values is approximately $p/12$ (Deuring, Igusa); a more precise result can be given, e.g. see [19], page 118; for any p we have $h \geq (p-1)/12$

By [34] we know that in every irreducible Moret-Baily family the number of geometric fibers with reducible theta-divisor equals $5p - 5$, see [33], Prop. 2.5; such a divisor is the union of two supersingular elliptic curves; the number of isomorphism classes of reducible supersingular genus 2 curves is $h(h+1)/2$.

Suppose V_0 is geometrically irreducible. By the previous results we see that $h(h+1)/2 \leq 5p - 5$. This implies $(p-1)(p+11) \leq 5(p-1){\cdot}2{\cdot}12^2$; hence $p \leq 1440 - 11$. If $p > 1500$ we conclude that V_0 is geometrically reducible.) $\square$

Later we derived a more precise result. In [19] we find:

$$V_0 \text{ is geometrically irreducible} \quad \Longleftrightarrow \quad p \leq 11.$$

(0.2) Remark. In these lectures we consider the case of polarized abelian varieties and quasi-polarized p-divisible groups. The situation of unpolarized p-divisible groups is also interesting (and usually much easier); however we leave that out of our considerations.

Also, interesting things can be said about the non-principally polarized case; however we will mainly describe strata and leaves inside $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$.

For an abelian variety A, or an abelian scheme, and a prime number p we write $A[p^\infty] = \cup_i A[p^i] = X$ for its p-divisible group.

For a p-divisible group in characteristic p one can define its Newton Polygon; its slopes are basically given by the "p-adic values of Frobenius"; for a precise definition see [27], or [7]; we write $\mathcal{N}(X)$ for the Newton polygon of X. We write $\mathcal{N}(A)$ instead of $\mathcal{N}(A[p^\infty])$ for the Newton polygon of an abelian variety A.

(0.3) NP - EO - Fol. Strata in $\mathcal{A}_g \otimes \mathbb{F}_p$ will be given by properties of the p-divisible group: $A \mapsto X = A[p^\infty]$, and $(A,\mu) \mapsto (X,\mu) = (A,\mu)[p^\infty]$. Here is a short survey of strata and foliations, to be explained and used below. For an abelian variety A, with a polarization (sometimes supposed to be principal) we can study the following objects:

NP $\quad A \quad \mapsto \quad A[p^\infty] \quad \mapsto \quad A[p^\infty]/\sim_k$ over an algebraically closed field:
the isogeny class of its p-divisible group; by the Dieudonné - Manin theorem,
we can identify this isogeny class of p-divisible groups with the Newton polygon of A. We obtain the Newton polygon strata, see (1.5).

EO $\quad (A,\lambda) \quad \mapsto \quad (A,\lambda)[p] \quad \mapsto \quad (A,\lambda)[p]/\cong_k$ over an algebraically closed field:
we obtain EO-strata; see [42]. Important feature (Kraft, Oort): the number of geometric isomorphism classes of group schemes of a given rank annihilated by p is *finite*.

Fol $\quad (A,\mu) \quad \mapsto \quad (A,\mu)[p^\infty] \mapsto (A,\mu)[p^\infty]/\cong_k$ over an algebraically closed field:
we obtain a foliation of an open Newton polygon stratum; see [44]. Note that for $f < g-1$ the number of central leaves is *infinite*.

Note: $X \cong Y \quad \Rightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y)$; conclusion: every central leaf in **Fol** is contained in exactly one Newton polygon stratum in **NP**.
Note: $X \cong Y \quad \Rightarrow \quad X[p] = Y[p]$; conclusion: every central leaf in **Fol** is contained in exactly EO-stratum in **EO**.

However, a NP-stratum may contain many EO-strata, an EO-stratum may intersect several NP-strata, see (9.2). Whether an EO-stratum equals a central leaf is studied and answered in the theory of minimal p-divisible groups, see Section 7.

Isogeny correspondences are finite-to-finite above central leaves, but may blow up and down subsets of isogeny leaves; see (1.7).

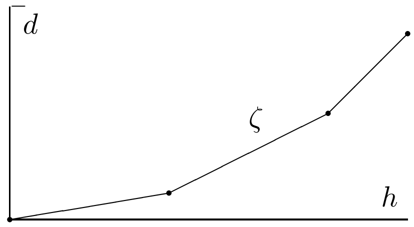
Notation. We fix a prime number p. All base fields and base schemes will be in characteristic p. We write k and Ω for an algebraically closed field of characteristic p. We write A for an abelian variety and X for a p-divisible group. From now on we usually write $\mathcal{A}_g$ instead of $\mathcal{A}_g \otimes \mathbb{F}_p$. For an algebraic scheme T over a field K we write $\Pi_0(T)$ for the set of irreducible components of T_k, where k is an algebraically closed field containing K.

1 NP-strata

$(A, \mu) \mapsto (A, \mu)[p^\infty] \mod \sim_k$.

(1.1) Newton polygons. Suppose given integers $h, d \in \mathbb{Z}_{\geq 0}$; here h = "height", d = "dimension". In case of abelian varieties we will choose $h = 2g$, and $d = g$. A Newton polygon γ (related to h and d) is a polygon $\gamma \subset \mathbb{Q} \times \mathbb{Q}$ (or, if you wish in $\mathbb{R} \times \mathbb{R}$), such that:

- γ starts at $(0,0)$ and ends at (h, d);
- γ is lower convex;
- any slope β of γ has the property $0 \leq \beta \leq 1$;
- the breakpoints of γ are in $\mathbb{Z} \times \mathbb{Z}$; hence $\beta \in \mathbb{Q}$.



Note that a Newton polygon determines (and is determined by)

$$\beta_1, \cdots, \beta_h \in \mathbb{Q} \text{ with } 0 \leq \beta_1 \leq \cdots \leq \beta_h \leq 1 \quad \leftrightarrow \quad \zeta.$$

Sometimes we will give a Newton polygon by data $\sum_i (m_i, n_i)$; here $m_i, n_i \in \mathbb{Z}_{\geq 0}$, with $\gcd(m_i, n_i) = 1$, and $m_i/(m_i + n_i) \leq m_j/(m_j + n_j)$ for $i \leq j$, and $h = \sum_i (m_i + n_i)$, $d = \sum_i m_i$. From these data we construct the related Newton polygon by choosing the slopes $m_i/(m_i + n_i)$ with multiplicities $h_i = m_i + n_i$. Conversely clearly any Newton polygon can be encoded in a unique way in such a form.

(1.2) Theorem (Dieudonné and Manin), see [27], "Classification theorem " on page 35,

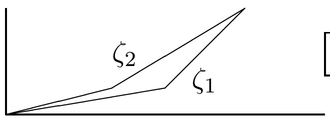
$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}.$$

This means: for every p-divisible group X over a field we define its Newton polygon $\mathcal{N}(X)$; over en algebraically closed field, every Newton polygon comes from a p-divisible group and

$$X \quad \sim_k \quad Y \quad \Longleftrightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y).$$

(1.3) We introduce a partial ordering in the set of Newton polygons having the same end-

We write $\zeta_1 \succ \zeta_2$ if ζ_1 is "below" ζ_2,
i.e. if no point of ζ_1 is strictly above ζ_2.
point.



$\boxed{\zeta_1 \succ \zeta_2}$

This might seem strange. However, explanation if ζ_1 is "below" ζ_2, the locus defined by ζ_1 is "bigger" than the locus defined by ζ_2.

(1.4) If S is a base scheme, and $\mathcal{X} \to S$ is a p-divisible group over S we write

$$\mathcal{W}_\zeta(S) = \{s \in S \mid \mathcal{N}(\mathcal{X}_s) \prec \zeta\} \subset S$$

and

$$\mathcal{W}^0_\zeta(S) = \{s \in S \mid \mathcal{N}(\mathcal{X}_s) = \zeta\} \subset S.$$

(1.5) Grothendieck - Katz. A theorem by Grothendieck and Katz, see [22], Th. 2.3.1 on page 143 says that for any $\mathcal{X} \to S$ and for any Newton polygon ζ

$$\mathcal{W}(S) \subset S \text{ is a } \textit{closed} \text{ set.}$$

(1.6) An abelian variety admits a polarization. This implies that the Newton polygon of an abelian variety is symmetric, use the duality theorem for abelian schemes, see [37], 19.1. For a symmetric Newton polygon ξ we write

$$W_\xi = \mathcal{W}_\xi \left(\mathcal{A}_{g,1} \otimes \mathbb{F}_p\right).$$

(1.7) Remark. Isogeny correspondences in characteristic p in general blow up and down in a rather wild pattern. The dimension of Newton polygon strata and of EO-strata in general depends very much on the degree of the polarization in consideration. However for p-rank strata the dimension in the whole of $\mathcal{A}_g \otimes \mathbb{F}_p$ solely depends on the p-rank, see [35], Theorem 4.1. It seems a miracle that the dimension on central leaves does not depend on the degree of a polarization. See [51].

In [44] we also find the definition of *isogeny leaves*, and we see that any irreducible component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ up to a finite morphism is isomorphic with the product of a central leaf and an isogeny leaf, see [44], 5.3. Note that all central leaves with the same Newton polygon have the same dimension, see (3.4) and (3.6). However for Newton polygon strata, and hence also for isogeny leaves, the dimension in general depends very much on the degree of the polarization; for more information see Section 3 of [51].

2 EO-strata

Basic reference [42]. $\quad (A, \mu) \mapsto (A, \mu)[p] \mod \cong_k$.

(2.1) For a polarized abelian variety (A, μ) we write S_φ for the set of points which give geometrically the same isomorphism class $\varphi = ((A, \mu)[p]/\cong)$. For the moduli point $x = [(A, \mu)]$ this subset is denoted by S_φ, the Ekedahl-Oort stratum defined by φ.

$$S_\varphi \quad = \quad \{(A, \lambda) \mid (A, \lambda)[p] \otimes \Omega \cong (G, \lambda)\}; \quad \varphi = (A, \lambda)[p] \otimes \Omega \mod \cong .$$

Usually we consider these strata only for principal polarizations. Care should be take what exactly is $(A, \lambda)[p]$, especially in characteristic $p = 2$, see [42], Section 9. The number of EO-strata is finite in every component of the moduli space; every EO-stratum is locally closed in $\mathcal{A}_g$. These strata form a stratification of the moduli space,

$$\mathcal{A}_g \quad = \quad \bigsqcup S_\varphi.$$

Note that every stratum S_φ is quasi-affine. We know the dimension of S_φ if we work in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$. The smallest stratum is the superspecial locus, which is zero-dimensional. The unique one-dimensional stratum in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is connected.

3 Foliations

Basic reference [44]. $\quad (A,\mu) \mapsto (A,\mu)[p^\infty] \quad \bmod \cong_k$.

(3.1) For a polarized abelian variety (A,μ) with moduli point $x = [(A,\mu)]$ we write $C(x)$ for the set of points which geometrically have the same p-divisible group; this is called the *central leaf* passing through x:

$$C(x) \quad = \quad \{(A,\lambda) \mid (A,\lambda)[p^\infty] \otimes \Omega \cong (X,\mu)\}; \quad \varphi = (A,\mu)[p^\infty] \otimes \Omega \quad \bmod \cong .$$

This we consider for all kind of polarizations. This set is closed in its open NP-stratum

$$C(x) \quad \subset \quad \mathcal{W}^0_\xi(\mathcal{A}_g).$$

(3.2) One can study Hecke-α-correspondences, by which we mean those defined by quasi-isogenies which are geometrically successive quotients by a subgroup isomorphic with α_p. Such spaces appear in the mod p-aspect of the theory of Rapoport-Zink spaces, see [53]. For a moduli point $x = [(A,\mu)]$ we define $I(x)$ as the union of irreducible components of Hecke-α orbits passing through x, called the *isogeny leaf* passing through x. Every isogeny leaf is closed in $\mathcal{A}_g$, and it is contained in one NP-stratum.

(3.3) Isogeny correspondences, unpolarized case. Let $\psi : X \to Y$ be an isogeny between p-divisible groups. Then there exist finite surjective morphisms

$$\mathcal{C}_X(\mathrm{Def}(X)) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_Y(\mathrm{Def}(Y)).$$

(3.4) Corollary. *The dimension of* $\mathcal{C}_X(\mathrm{Def}(X))$ *only depends on the isogeny class of* X.

(3.5) Isogeny correspondences, polarized case. Let $\psi : A \to B$ be an isogeny, and let λ respectively μ be a polarization on A, respective on B, and suppose there exists an integer $n \in \mathbb{Z}_{>0}$ such that $\psi^*(\mu) = n{\cdot}\lambda$. Then there exist finite surjective morphisms

$$\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [44], 3.16.

(3.6) Corollary. The dimension of $\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$ only depends on the isogeny class of (X,λ).

Remark / Notation. In fact, this dimension only depends on the isogeny class of X. We write

$$c(\xi) := \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)\right), \quad X = A[p^\infty], \quad \xi := \mathcal{N}(X);$$

this is well-defined: all irreducible components have the same dimension.

A proof of all previous results on isogeny correspondences, and the independence of the dimension of the leaf in an isogeny class can be given using [52]; see [44], 2.7 and 3.13.

(3.7) Product structure. See [44], Th. 5.3. Suppose given a moduli point $x = [(A,\mu)]$, work over an algebraically closed field k, and let W be an irreducible component of the open NP-stratum containing x. Let C be the irreducible component of the central leaf passing through x, and let I be an irreducible component of $I(x)$. Then there exist

finite covers $T \to C$, *and* $J \to I$ *and a finite surjective morphism* $T \times J \to W$.

(3.8) Remark. Central leaves get strictly smaller if you go to higher Newton polygons; the ordinary locus is one leaf; the locus defined by p-rank $f = g - 1$ is one leaf. For $f < g - 1$ a leaf is smaller than the Newton polygon stratum; in such a case the number of central leaves is infinite. Inside the supersingular locus W_σ every central leaf is finite. The dimension of a central leaf only depends on the Newton polygon, not on the polarization.

Going to a higher Newton polygon, dimensions of isogeny leaves either stay the same or go up. The dimension of an isogeny leaf depends in an essential way, not only on the Newton polygon, but also on the type of the polarization. For more information on this see [51], Section 3.

Note that the dimension of a central leaf depends on the Newton polygon, and not on the degree of the polarization. The dimension of an isogeny leaf very much depends on the polarization.

(3.9) An example. We discuss isogeny leaves. We will see that some isogeny leaves are irreducible, and some are reducible. *There seems to be no uniform statement describing (ir)reducibility of isogeny leaves.*

Choose $g = 3$, choose the Newton polygon $\xi = (2,1) + (1,2)$. Consider $\mathcal{A}_{3,1,n} \otimes \mathbb{F}_p$, and write $W = \mathcal{W}^0_\xi(\mathcal{A}_{3,1,n} \otimes \mathbb{F}_p)$. For a point

$$x = [(A,\lambda)] \in W, \quad \text{either} \quad a(A) = 1 \quad \text{or} \quad a(A) = 2.$$

Suppose x is defined over an algebraically closed field k. Write $(A,\lambda)[p^\infty] = (X,\lambda)$. We see:

$$a(A) = 2 \Longrightarrow X \cong G_{2,1} + G_{1,2}; \quad a(A) = 1 \Longrightarrow X \cong (G_{2,1} + G_{1,2})/\alpha_p.$$

If $a(A) = 1$ we see that $I(x)$ is an *irreducible* rational curve; it is obtained by constructing a polarization μ on Y with $Y[p^\infty] = G_{2,1} + G_{1,2}$ such that $\mathrm{Ker}(\mu) = \alpha_p + \alpha_p$, and by performing a Moret-Baily type of construction.
If $a(A) = 2$, we see that X is minimal, $x \in \mathcal{Z}_\xi$. If A is defined over a finite field, the full H_α-orbit of x inside W has a finite number of irreducible components. However, if A has the property that $\mathrm{End}(A_k) = \mathbb{Z}$, where k is an algebraically closed field (and hence A cannot bed defined over a finite field), then the full H_α-orbit of x inside W has an infinite number of irreducible components; note cases with $\mathrm{End}(A_k) = \mathbb{Z}$ do occur.

(3.10) If $x \in \mathcal{A}_{g,1}$ is a point which is either ordinary, or of p-rank $g - 1$, or supersingular the H_α-orbit of x in $\mathcal{A}_{g,1}$ is finite. It could be interesting to describe all H_α-orbits in all other cases.

4 Dimension and (ir)reducibility of an NP-strata

Basic references: [25], [41], [43], [45], [5]. $\quad (A,\mu) \mapsto (A,\mu)[p^\infty] \mod \sim_k$.

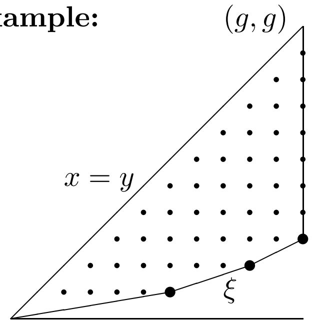
(4.1) The dimension of Newton polygon strata, polarized case. We fix an integer g. For every *symmetric* NP ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g, \ \ (x,y) \ \text{ on or above } \ \xi\},$$

and we write

$$\boxed{\mathrm{sdim}(\xi) := \#(\triangle(\xi)).}$$

Example:



$$\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)) = \#(\triangle(\xi))$$
$$\xi = (5,1) + (2,1) + 2{\cdot}(1,1) + (1,2) + (1,5),$$
$$\text{g=11; slopes: } \{6 \times \tfrac{5}{6}, 3 \times \tfrac{2}{3}, 4 \times \tfrac{1}{2}, 3 \times \tfrac{1}{3}, 6 \times \tfrac{1}{6}\}.$$
$$\text{This case: } \dim(\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)) = \mathrm{sdim}(\xi) = 48.$$

(4.2) Theorem, see [41], Th. 3.4 and [43], Th. 4.1. *Let ξ be a symmetric Newton polygon. Then:*

$$\dim\left(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)\right) = \mathrm{sdim}(\xi).$$

Note that this applies to *principally polarized abelian varieties.* For Newton polygon strata in the non-principally polarized case different dimensions can appear. For more details see [51].

(4.3) Example. Note that the supersingular locus has "many components" for fixed g and large p; see [25], 4.9:

$$\#(\Pi_0(W_\sigma)) = H_g(p,1) \quad \text{if} \quad g \quad \text{is odd,}$$
$$\#(\Pi_0(W_\sigma)) = H_g(1,p) \quad \text{if} \quad g \quad \text{is even.}$$

Note that for g fixed, and $p \to \infty$, indeed $\#(\Pi_0(W_\sigma)) \to \infty$.

Some of these class numbers have been computed; e.g. see [15]

One might wonder what the number of irreducible components is for an arbitrary Newton polygon stratum; it seems quite a job to compute the number of components of all Newton polygon strata. However this turns out to be "easy".

(4.4) Theorem. *Let $\xi \neq \sigma$ be a* non-supersingular *symmetric Newton polygon. The locus*

$$W_\xi \quad := \quad \mathcal{W}_\xi\left(\mathcal{A}_{g,1} \otimes \mathbb{F}_p\right)$$

is geometrically irreducible, i.e. $\#(\Pi_0(W_\xi)) = 1$.
See [45], [5].

(4.5) Note that the previous result uses *principal polarizations.* To compute dimension and number of components of Newton polygon strata for arbitrary polarizations is a difficult question which has a not uniform answer. Here is an example: in [25], 12.5, we compute the number of components of strata of supersingular abelian 4-folds for some degrees of the polarization.

5 Dimension and (ir)reducibility of an EO-strata

Basic references: [42], [10], [5].

The dimension of an EO-stratum in the principally polarized case was determined in [42], Theorem 1.2. Later results, more general results, and different proofs can be found in [32], [55].

In [42], 14.1 we find the *conjecture* that EO-strata in $\mathcal{A}_{g,1}$ not contained in the supersingular locus are *irreducible*. That has been proved now. Here are the details.

(5.1) For a principally polarized abelian variety (A,λ) the isomorphism class of $(A,\lambda)[p]$ is given by an elementary sequence, see [42], 5.6 and Section 9 of that paper. That is a map $\varphi : \{1,\cdots,g\} \to \mathbb{Z}$ satisfying certain properties: write $\varphi(0)=0$, and the condition is $\varphi(i) \leq \varphi(i+1) \leq \varphi(i)+1$ for $0 \leq i < g$.
Lemma. *Let $g = 2r-1$ or $g = 2r$.*

$$S_\varphi \subset W_\sigma \quad \Longleftrightarrow \quad \varphi(r) = 0.$$

I proved this, did not publish up to now, but see [13].
Sketch of a proof of the lemma.

Let φ' be the elementary sequence $\{0,\cdots,0,\varphi'(r-1)=1,\cdots,\varphi'(g)=1.\}$. We see that this belongs to the minimal p-divisble group of the Newton polygon $(r,r+1)+(r+1,r)$ respectively $(r-1,r)+(1,1)+(r,r-1)$. In particular $S_{\varphi'} \cap W_\sigma = \emptyset$. This shows $\Rightarrow$.

Suppose $\varphi(r)=0$. Consider a final filtration $G_1 \subset \cdots \subset G_i \subset \cdots \subset G$. Note that G_{g+r}/G_r is annihilated by V and by F. This shows that X/G_r is superspecial. Hence X supersingular in this case. This proves $\Leftarrow$.

(5.2) **Theorem** (T. Ekedahl & G. van der Geer). *Let Let φ be an elementary sequence. Suppose $\varphi(r) \neq 0$.* (Equivalently: $S_\varphi \not\subset W_\sigma$. *The* EO-*stratum $S_\varphi \subset \mathcal{A}_{g,1}$ is geometrically irreducible.*
See [10], Th. 11.5.

(5.3) Some time ago I made the conjecture that for $S_\varphi \subset W_\sigma$ and p large the stratum S_φ is geometrically reducible. This seems to be correct, see [13].

6 Dimension and (ir)reducibility of a central leaf

Basic references: [51], [5].

(6.1) Let ζ be a Newton polygon. Suppose that the slopes of ζ are $1 \geq \beta_1 \geq \cdots \geq \beta_h \geq 0$; this polygon has slopes $\beta_h,\cdots,\beta_1$ (non-decreasing order), and it is *lower convex*. We write ζ^* for the polygon starting at $(0,0)$ constructed using the slopes $\beta_1,\cdots,\beta_h$ (non-increasing order); note that ζ^* is *upper convex*, and that the beginning and end point of ζ and of ζ^* coincide.

For a symmetric Newton polygon ξ we write:

$$\triangle(\xi;\xi^*) := \{(x,y) \in \mathbb{Z} \mid (x,y) \prec \xi, \quad (x,y) \gneqq \xi^*, \quad x \leq g\}, \qquad \boxed{\mathrm{cdp}(\xi) := \#\left(\triangle(\xi;\zeta^*)\right)};$$

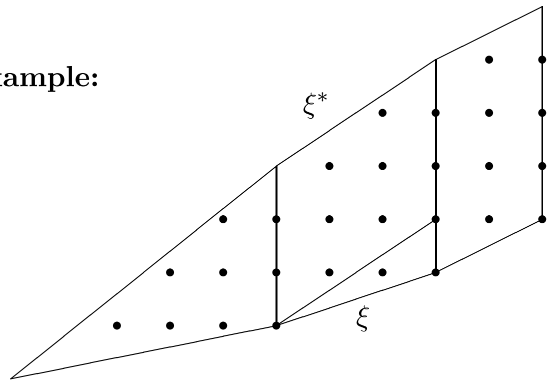
"cdp" = dimension of central leaf, polarized case.

(6.2) **Theorem** (Dimension formula, the polarized case.) *Let (A,λ) be a polarized abelian variety. Let $(X,\lambda)=(A,\lambda)[p^\infty]$; write $\xi = \mathcal{N}(A)$; then*

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) \quad = \quad \mathrm{cdp}(\xi).$$

See [51], Th. 2.4.

Example:



$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \sum_{0<i\leq g} (\xi^*(i) - \xi(i))$.
slopes $1/5, 4/5, h = 5$: $\quad \frac{1}{2}4{\cdot}5 - \frac{1}{2}1{\cdot}2 = 9$,
slopes $1/3, 2/3, h = 3$: $\quad \frac{1}{2}2{\cdot}3 - \frac{1}{2}1{\cdot}2 = 2$,
$(d_1 - c_1)h_2 = 3{\cdot}3 = 9$,
$(d_1 + d_2 - c_1 - c_2)r = 4{\cdot}2 = 8$,
$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \#\left(\triangle(\zeta;\zeta^*)\right) = 28$.

(6.3) Corollary. *Let* $\xi' \precneqq \xi$. *Then* $\mathrm{cdp}(\xi') < \mathrm{cdp}(\xi)$. That is: a central leaf in the stratum of a higher Newton polygon has a strictly smaller dimension.

(6.4) Question. I do not know a proof of this fact without using the previous theorem. *Can we give a more direct proof?* It seems plausible that one can give a direct proof, but I was not able to find such a proof. In particular I do not know how to describe the closure of a central leaf.

(6.5) Theorem (C.-L. Chai & F. Oort). *Let* $[(B, \mu)] = y \in \mathcal{A}_{g,d,n}$, *where* $d, n \in \mathbb{Z}_{>0}$ *and* n *relatively prime to* pd. *Suppose that* B *is not supersingular. Write* $(Y, \mu) = (B, \mu)[p^\infty]$. *Then the central leaf* $\mathcal{C}_{(Y,\mu)}(\mathcal{A}_{g,d,n})$ *is geometrically irreducible.*
See [5], Corollary 5.6. One of the tools in proving this theorem is the fact that p-adic monodromy on a central stream in W_ξ is maximal, use Section 8 and see [5], Th. 5.2.

7 Minimal p-divisible groups

Basic reference: [48]. Also see [49].

(7.1) Starting from a p-divisible group X we obtain a BT_1 group scheme:

$$[p] : \{X \mid \text{a } \ p-\text{divisible group}\}/\cong_k \quad \longrightarrow \{G \mid \text{a } \ \mathrm{BT}_1\}/\cong_k; \quad X \mapsto G := X[p].$$

This map is known to be surjective; see [17], 1.7, see [42], 9.10; it is not difficult to construct a section for this map, e.g. see [49], 2.5. It is the main theorem of [48] that *the fiber of this map over* (G *up to* $\cong_k$) *is precisely one p-divisible group* X *if* G *is minimal.* Also see [49].

(7.2) For a pair (m, n) of coprime non-negative integers we define $H_{m,n}$ to be a p-divisible group of dimension m, of height $m+n$, of slope $m/(m+n)$ such that over k the endomorphism ring $\mathrm{End}(H_{m,n})$ is the *maximal order* in $\mathrm{End}^0(H_{m,n}) := \mathrm{End}(H_{m,n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; this defines this group uniquely up to isomorphism over an algebraically closed field; the group can be constructed over $\mathbb{F}_p$. For $\zeta = \sum_i \ (m_i, n_i)$ we write $H(\zeta) = \prod_i \ H_{m_i,n_i}$. This p-divisible group is defined over $\mathbb{F}_p$, and if no confusion can arise we consider $H(\zeta)$ over a field K, meaning $H(\zeta) \otimes_{\mathbb{F}_p} K$.

Definition. We say a p-divisible group X is *minimal* if there exists an isomorphism $X_\Omega \cong H(\zeta)$.

(7.3) Theorem (Oort), see [48], Th. 1.2. *Work over an algebraically closed field k. Let X be a p-divisible group, and let ζ be some Newton polygon. Suppose there exists an isomorphism $X[p] \cong H(\zeta)[p]$. Then there exists an isomorphism $X \cong H(\zeta)$.*

Suppose $[(A, \lambda)] = x$ where $A[p^\infty]$ is *minimal* and λ is a *principal* polarization. In this case the central leaf $C(x)$ is denoted by $\mathcal{Z}_\xi = C(x)$, where $\xi = \mathcal{N}(A[p^\infty])$, and it is called the *central stream* inside W^0_ξ. Note that a principal quasi-polarization on the minimal p-divisible $A[p^\infty]$ is unique, see [44], 3.7.

(7.4) Let $[(B, \mu)] = y$, with $\xi = \mathcal{N}(B)$ and $\varphi = ((Y, \mu)[p]/\cong)$. Then $C(y) \subset \mathcal{W}_\xi(\mathcal{A}_g)$ and $C(y) \subset S_\varphi$. These inclusions are not equality in most cases. However:

(7.5) Corollary. *Suppose moreover $B[p^\infty]$ is minimal. Then $C(y) = S_\varphi$.*
This central leaf, which is also an EO-stratum is called the central stream defined by $\xi := \mathcal{N}(B)$, notation $\mathcal{Z}_\xi$

8 Monodromy

Basic reference: [2]. The result in this section is completely due to Ching-Li Chai.

(8.1) Let $d \in \mathbb{Z}_{>0}$, choose an integer $n \in \mathbb{Z}_{\geq 3}$ relatively prime with pd and let $\mathcal{B} := \mathcal{A}_{g,d,n} \otimes \mathbb{F}_p$. Let $Z \subset \mathcal{B}$ be a locally closed subset. Let Z' be an irreducible component of Z. Let $\eta \in Z'$ be the generic point. Let $A \to Z$ be the universal abelian scheme restricted to Z. Let

$$\rho_{A,\ell} : \pi_1(Z', \overline{\eta}) \quad \longrightarrow \quad \mathrm{Sp}(T_\ell, <, >_\ell)$$

be the ℓ-adic monodromy representation in the Tate-ℓ-group of A_η. Identify the Tate-ℓ-group of A_η over $\overline{\eta}$ with $\mathbb{Z}_\ell^{2g}$ with the standard pairing.

(8.2) Theorem (C.-L. Chai). *Choose notation as above. Let $Z \subset \mathcal{B}$ be a locally closed subscheme, smooth over* $\mathrm{Spec}(k)$*, such that:*

Z is Hecke-ℓ-stable, and

the Hecke-ℓ-action on the set $\Pi_0(Z)$ is transitive, and

$Z \not\subset W_{N,\sigma}$ (equivalently: Z contains a non-supersingular point).

Then:

$$\rho_{A,\ell} : \pi_1(Z', \overline{\eta}) \longrightarrow \mathrm{Sp}(T_\ell, <, >_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$$

is surjective, *and*

$$Z \textit{ is irreducible, i.e. } Z = Z'.$$

See [2], 4.4.

9 Remarks and questions

(9.1) Strata and leaves above are defined by specifying properties of the points in that stratum. Then, one has to do some (or sometimes a lot of) work to prove that the stratum exists as a locally closed set. Often it is not clear what a canonical scheme structure could be. It seems desirable to have better definitions and better constructions. In the case of EO strata this can be done via [10].

(9.2) Suppose given a Newton polygon ξ. How can one decide which EO-stratum have a non-empty intersection with W^0_ξ ? If φ belongs to an abelian variety with $a = 1$, then φ appears on every NP-stratum with that p-rank. If φ belong to a minimal abelian variety, then φ appears on precisely one NP-stratum. For the general case here a possible partial answer.

Let φ_ξ be the elementary sequence given by the minimal p-divisible group $H(\xi)$.

Conjecture.

$$S_\varphi \cap W_\xi \neq \emptyset \quad \Longrightarrow \quad \varphi \supset \varphi_\xi.$$

Note that the opposite implication certainly does not hold. Even if we prove the conjecture we do not have a necessary and sufficient condition.

(9.3) *What is the boundary of a central leaf?* Suppose $\xi' \subsetneqq \xi$ and let $C \subset W_\xi$ be a central leaf. Try to describe $C^{\mathrm{Zar}} \cap W_{\xi'}$.

Conjecture. *With notation as above, I conjecture that C^{Zar} contains the central stream $\mathcal{Z}_{\xi'}$.*

Even if the conjecture is proved, this does not fully answer the question. One could even ask whether $C^{\mathrm{Zar}} \cap W_{\xi'}$ is independent of the central leaf C chosen; this could be possible, but I do not see a proof or evidence. For a partial answer see [14].

(9.4) Question. Note that a generic supersingular abelian variety has a large endomorphism ring; the number of components of $\mathcal{W}_\sigma(\mathcal{A}_{g,1})$ can be computed using this ring. A generic non-supersingular abelian variety has endomorphism ring equal to $\mathbb{Z}$. Can we prove (4.4) using this fact? This was one of my dreams for some time, but I did not succeed along these lines.

(9.5) Problem. Determine for a given p and g which Newton polygons appear on the moduli space of curves of genus g in charactersitc p. I have some ideas, but no explicit conjecture to offer. It is not clear to me whether the answer for a given g depends on p.

The list of references below is perhaps more than we need. However, it can be of service to the reader to record more relevant papers.

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