Stratifications and foliations of moduli spaces of abelian varieties

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Introduction

In this survey we discuss *stratifications* and *foliations* of our basic hero $\mathcal{A}_g \otimes \mathbb{F}_p$: the moduli spaces of polarized abelian varieties of dimension g in positive characteristic p.

In characteristic zero we have strong tools at our disposal: besides algebraic-geometric theories we can use analytic and topological methods. It seems that we are at lost in positive characteristic. However the opposite is true. Phenomena, only occurring in positive characteristic provide us with strong tools to study moduli spaces. And, as it turns out again and again, several results in characteristic zero can be derived using reduction modulo p.

But we start with a warning. In algebraic geometry constructing varieties, schemes, spaces, a good method is to define a priori a functor, and then prove the functor is representable, as Grothendieck taught us. This functorial approach has proved to be very successful. However, strata and leaves defined and constructed below all are given by characterizing the points contained in that space. This is not a very elegant way of doing algebraic geometry, I know. However in the cases considered below, I cannot do better. And we pay a price. Usually the definitions are easy. However the fact that these spaces do exist often is cumbersome. We construct several subvarieties of $\mathcal{A}_g \otimes \mathbb{F}_p$, giving stratifications and foliations. We see, that the properties indeed define the spaces constructed, and we reap the fruits, because of the many applications. But in general the scheme structure we choose, the reduced structure on the (locally) closed set, need not be the most natural one.

Base fields, and base schemes will be in characteristic p, unless otherwise specified. We write $\mathcal{A}_g \otimes \mathbb{F}_p$ for the moduli scheme of polarized abelian schemes in characteristic p. We write $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ for the moduli scheme of *principally* polarized abelian schemes in characteristic p. Sometimes a level-n-structure will be assumed, with n prime to p.

The terminology "stratification" and "foliation" will be used in a loose sense:
A *stratification* will be a way of writing a space as a finite disjoint union of subspaces of that space; in some cases we will actually check whether the boundary of one stratum is the union of "lower strata".
A *foliation* will be a way of writing a space as a disjoint union of subspaces of that space.

Here are some motivating questions and problems:

- What is the Hecke orbit of a point in the moduli space of polarized abelian varieties? Over $\mathbb{C}$: such an orbit is dense in the moduli space $\mathcal{A}(\mathbb{C})$. What can we say about this question in positive characteristic?
- What is the maximal dimension of a complete subvariety of $\mathcal{A}_g(\mathbb{C})$?
- A conjecture by Grothendieck: which Newton polygons occur in the local deformation space of a given p-divisible group, or a given polarized abelian variety?
- Describe NP strata in the moduli space of abelian varieties in characteristic p.
- What strata are given by fixing the p-kernel of the abelian varieties studied.
- What kind of leaves are given by fixing the p-divisible group of the abelian varieties studied.

It will turn out that various stratifications and foliations of $A_g \otimes \mathbb{F}_p$, and a description of these structures give access to some of these questions.

Hecke correspondences have the property that they may blow up and down subsets of the moduli space (if we consider "α-Hecke-orbits"). For some of our results we have to restrict to principally polarized abelian varieties, in order to obtain nice, coherent statements. In some cases, by some miracle, statements holds more generally for all degrees of polarizations (e.g. the dimensions of the p-rank-strata). In some cases the condition that the polarization is principal is essential, e.g. the analog of a conjecture of Grothendieck translated to polarized p-divisible groups, e.g. the question whether $(a = 1)$-locus is dense in a NP stratum. In some of our considerations we restrict to the principally polarized case.

Here is a survey of the strata and leaves we are going to construct. For an abelian variety A over a field, a positive integer m and a prime number p we define the group scheme:

$$A[m] := \mathrm{Ker}(\times m : A \longrightarrow A),$$

and we define the p-divisible group of A by:

$$A[p^\infty] = \cup_{1 \leq i < \infty} \;\; A[p^i].$$

NP $\quad A \mapsto A[p^\infty] \mapsto A[p^\infty]/\sim$ over an algebraically closed field: the isogeny class of its p-divisible group; by the Dieudonné - Manin theorem we can identify this isogeny class of p-divisible groups with the Newton polygon of A. We obtain the Newton polygon strata.

EO $\quad (A, \lambda) \mapsto (A, \lambda)[p] \mapsto (A, \lambda)[p]/\cong$ over an algebraically closed field: we obtain EO-strata; see [52]. Important feature (Kraft, Oort): the number of geometric isomorphism classes of group schemes of a given rank annihilated by p is *finite.*

Fol $\quad (A, \lambda) \mapsto (A, \lambda)[p^\infty] \mapsto (A, \lambda)[p^\infty]/\cong$ over an algebraically closed field: we obtain a foliation of an open Newton polygon stratum; see [54]. Note that for $f < g - 1$ the number of leaves is *infinite.*

Note: $X \cong Y \Rightarrow \mathcal{N}(X) = \mathcal{N}(Y)$; conclusion: every central leaf in **Fol** is contained in exactly one Newton polygon stratum in **NP**.

However, a NP-stratum may contain many EO-strata, an EO-stratum may intersect several NP-strata. If the p-rank is smaller than $g-1$ a NP-stratum contains infinitely many central leaves. Whether an EO-stratum equals a central leaf is studied and answered in Section 11.
To a p-divisible group we can attach various "invariants":

$X[p^\infty]$ up to $\sim$	ξ	NP	W_ξ
$X[p^1]$ up to $\cong$	φ	EO	S_φ
$(A[p^\infty],\lambda)$ up to $\cong$	(X,λ)	Fol	$C(x)$

We explain these notions and notations below.

For all these concepts we show that we obtain (locally) closed subsets of the moduli space $\mathcal{A}_g \otimes K$, where $K \supset \mathbb{F}_p$ is a field. We will see that these spaces do exit. We determine their dimension. We decide which ones are (ir)reducible. And we show various applications of the concepts thus defined.

We will write k and Ω for an algebraically closed field. We will write $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ for the moduli space of principally polarized abelian varieties in characteristic p.

Questions studied below can be stated for the unpolarized case and in the case of quasi-polarized p-divisible groups or polarized abelian varieties. This last case is for us the most interesting, and often we will not state the analogous (usually easier) questions and theorems in the unpolarized case.

We have seen many times that supersingular abelian varieties on the one hand and non-supersingular abelian varieties on the other hands in general behave very differently. What has been proved now:

supersingular NP-*strata,* EO-*strata and central leaves in general*
(we mean $p \gg 0$), *are* **reducible** (Katsura-Oort, Li-Oort, Harashita)

but

non-supersingular NP-*strata*, EO-*strata and central leaves*
in the principally polarized case are **irreducible**
(Oort, Ekedahl-Van der Geer, Chai-Oort).

1 Newton polygons: The Manin conjecture

NP $\boxed{\text{Newton polygon strata are given by } A[p^\infty]/\sim .}$

(1.1) Our story starts with Manin's thesis [31]. For an abelian variety A we define its p-divisible group X as the union (inductive limit) of all subgroup schemes $A[p^i]$.

In Chapter II Manin describes isogeny classes of p-divisible groups over a perfect field and over an algebraically closed field. Here is the main result: *for a pair of coprime positive integers $m, n \in \mathbb{Z}$ there exists an isosimple formal group $G_{m,n}$ of dimension m, with* $\dim((G_{m,n})^t) = n$. *Any formal p-divisible group X over k is up to isogeny a direct sum of such groups:*

$$X \quad \sim \quad \sum_i G_{m_i,n_i};$$

see [31], Classification Theorem on page 35. Note that for any abelian variety (for example defined over a field of large transcendence degree over $\mathbb{F}_p$), its p-divisible group is isogenous with a p-divisible group defined over the prime field.

Note that this shows there is a bijection between the set of k-isogeny classes of p-divisible groups over k and the set of Newton polygons:

(1.2) Theorem (Dieudonné and Manin), see [31], "Classification theorem " on page 35 .

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}$$

A Newton polygon is a *lower convex polygon* in $\mathbb{Q} \times \mathbb{Q}$, starting at $(0,0)$, with *integral break points*, and ending at (h,d) for certain integers h and d; for abelian varieties of dimension g we have $h = 2g$ and $d = g$.

We write $\mathcal{N}(X)$ for the Newton polygon of X. For an abelian variety A over a field of characteristic p we write $\mathcal{N}(A) = \mathcal{N}(A[p^\infty])$. To $G_{m,n}$ one assigns $m+n$ slopes equal to $m/(m+n)$; for $X \sim \sum_i G_{m_i,n_i}$ the Newton polygon $\mathcal{N}(X)$ is formed by taking all these slopes and ordering them in non-decreasing order. This is the theory of the "F-slopes". In some papers the different (but equivalent) notion of "V-slopes" is used: in that notation $\mathcal{N}(G_{m,n})$ is isoclinic of slope equal to $n/(m+n)$.

(1.3) In Chapter IV Manin discusses Newton polygons coming from an abelian variety. The "formal structure of an abelian variety A", now called the Newton polygon of A, is discussed, written as $\mathcal{N}(A)$:

$$A \quad \mapsto \quad A[p^\infty] =: X \quad \mapsto \quad \mathcal{N}(X) \quad =: \quad \mathcal{N}(A).$$

For an abelian variety A over a finite field Manin proves that the Newton polygon of an abelian variety is symmetric in the sense that the isogeny factors $G_{m,n}$ and $G_{n,m}$ appear with the same multiplicity; or in other words: a slope β appears with the same multiplicity in $\mathcal{N}(A)$ as the slope $1-\beta$. For an abelian variety over any field of positive characteristic the symmetry of $\mathcal{N}(A)$ follows from [43], Theorem 19.1.

(1.4) Manin then formulates his celebrated
Manin Conjecture [31], Conjecture 2 on page 76:

a Newton polygon ξ comes from an abelian variety $\iff$ *ξ is symmetric.*

(1.5) This conjecture was proved using the Honda-Tate theory, see [69], page 98. For a pure characteristic p proof, see [51], 5.3.

(1.6) A symmetric Newton polygon is called
supersingular if all slopes are equal to 1/2. A Newton polygon is called
ordinary if only the slopes 0 and 1 appear. A symmetric Newton polygon ξ is called
almost supersingular if it is the unique one "one step away" from supersingular:
if $g = 2r$, we have $\xi = (r, r-1) + (1,1) + (r-1, r)$;
if $g = 2r+1$, we have $\xi = (r+1, r) + (r, r+1)$.

(1.7) The p-rank stratification. For an abelian variety A we define the p-rank to be equal to $f(A) = f$ if $\#(A[p](k)) = p^f$. This gives a stratification of $\mathcal{A}_g \otimes \mathbb{F}_p$, by declaring that V_f contains all $[(B, \mu)]$ such that $f(B) \leq f$. It is not difficult to see that $V_f \subset \mathcal{A}_g \otimes \mathbb{F}_p$ is *closed*. It is a surprising and difficult theorem (suggested by D. Mumford, proved by P. Norman and F. Oort):

$$\dim(V_f) = \frac{1}{2}g(g+1) - (g-f),$$

see [40], Theorem 4.1. Note that all components have this dimension. This is surprising and unexpected, if you realize that Hecke correspondences blow up and down in general, but that the dimensions of these strata remain the same under such correspondences.

2 Barsotti-Tate groups: A conjecture by Grothendieck

(2.1) Basic references: [15], [19], [22]. For a scheme S and and integer $h \in \mathbb{Z}_{\geq 0}$ we define a p-divisible group (or, a Barsotti-Tate group) $X \to S$ as a union, an inductive limit, of commutative finite flat group schemes $X_n \to S$ such that the rank of $X_n \to S$ equals p^{hn}, such that X_n is annihilated by p^n, and such that for i and j there is an exact sequence $X_i = \mathrm{Ker}(\times p^i : X_{i+j} \to X_{i+j})$; in this case $X_{i+j}/X_i \cong X_j$.

For an abelian scheme $A \to S$ we study $A[p^i] := \mathrm{Ker}(\times p^i : A \to A)$; the system defined by $X_i := A[p^i]$ is a p-divisible group.

This concept is a strong tool in studying abelian varieties in characteristic p. One can "feel" a p-divisible group as a terrific substitute of p-adic homology, as a characteristic p substitute for Tate-ℓ-groups. There are many equivalent properties. Such as: a Tate-ℓ-group is "defined over the prime field" and "basically a p-divisible group, up to isogeny is defined over the prime field". Or: deforming an abelian variety its Tate-ℓ-group "stays geometrically constant" (i.e. constant up to a Galois twist); we will see that an analogous statement holds for p-divisible groups holds over central leaves.

A basic example/surprise. Let E_0 be an ordinary elliptic curve over a field $k \supset \mathbb{F}_p$. Let $E \to S$ be an equicharacteristic-p-deformation over an integral base, $0 \in S$, $E_0 = E \times_S 0$, and let $\eta \in S$ be the generic point. Then we have

$$T_\ell(E_0) \cong T_\ell(E_\eta) \otimes \overline{k(\eta)}$$

and we have

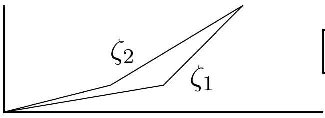
$$E_\eta[p^\infty] \otimes_{k(\eta)} \overline{k(\eta)} \cong E_0[p^\infty] \otimes_k \overline{k(\eta)}.$$

Families of p-divisible groups can be studied, see [19], [75], [64], [54]. Proving the conjecture by Grothendieck recorded below gives access to studying the boundary of Newton polygon strata (the main missing piece of information in Manin's thesis), and constructing central leaves, studying properties using a slope filtration (idea basically due to Grothendieck and Zink), access to the Hecke Orbit Problem becomes available. We see that the topic "families of p-divisible groups" lies in the heart of these developments.

(2.2) In 1970 Grothendieck observed that" Newton polygons go up" under specialization. In order to study this and related questions we introduce a notation:

(2.3) For Newton polygons we introduce a partial ordering.

We write $\zeta_1 \succ \zeta_2$ if ζ_1 is "below" ζ_2,
i.e. if no point of ζ_1 is strictly above ζ_2.



$\boxed{\zeta_1 \succ \zeta_2}$

Note that we use this notation only if Newton polygons with the same endpoint are considered.

If S is a base scheme, and $\mathcal{X} \to S$ is a p-divisible group over S we write

$$\mathcal{W}_\zeta(S) := \{s \in S \mid \mathcal{N}(\mathcal{X}_s) \prec \zeta\} \subset S$$

and

$$\mathcal{W}^0_\zeta(S) := \{s \in S \mid \mathcal{N}(\mathcal{X}_s) = \zeta\} \subset S.$$

(2.4) **Theorem** (Grothendieck and Katz; see [25], 2,3,2). *$\mathcal{W}_\zeta(S) \subset S$ is a closed set.*
As the set of Newton polygons of a given height is finite we conclude: $\mathcal{W}^0_\zeta(S) \subset S$ is a *locally closed set.*

(2.5) **Notation.** Let ξ be a symmetric Newton polygon. We write $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$.

(2.6) We have seen that "Newton polygons go up under specialization". Does a kind of converse hold? In 1970 Grothendieck conjectured the converse. In [15], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: "$\cdots$ *The wishful conjecture I have in mind now is the following: the necessary conditions $\cdots$ that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular deformation $\cdots$ we want to know if every sequence of rational numbers satisfying $\cdots$ these numbers occur as the sequence of slopes of a fiber of G as some point of S.*"

(2.7) **Theorem** (conjectured by Grothendieck, Montreal 1970). *Let K be a field of characteristic p, and let G_0 be a p-divisible group over K. We write $\mathcal{N}(\mathcal{G}_0) =: \beta$ for its Newton Polygon. Suppose given a Newton Polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation G_η of G_0 such that $\mathcal{N}(\mathcal{G}_\eta) = \gamma$.*
For a proof see [21], [51], [53].

This the "unpolarized case". An analogous formulation of the conjecture for quasi-polarized p-divisible groups is not true in general (for an arbitrary degree of the polarization); in [24], 6.10 we find an example of a supersingular abelian variety of dimension three with a polarization of degree p^2, which cannot be deformed to a polarized abelian variety having Newton polygon equal to $(2,1) + (1,2)$. Many more examples can be given.
However the analogous conjecture in the *principally* polarized case is true:

(2.8) **Theorem** (the principally polarized variant of the conjecture by Grothendieck, Montreal 1970). *Let K be a field of characteristic p, and let (G_0, λ_0) be a* principally *quasi-polarized p-divisible group over K. We write $\mathcal{N}(G_0) = \beta$ for its Newton Polygon. Suppose given a symmetric Newton Polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation (G_η, λ) of (G_0, λ_0) such that $\mathcal{N}(\mathcal{G}_\eta) = \gamma$.*
For a proof see [21], [51], [53].

Using a theorem by Serre and Tate, we see that from (2.8) the analogous theorem for principally polarized abelian varieties holds true.

3 EO strata

EO $\boxed{\text{Ekedahl-Oort strata are given by } A[p]/\cong .}$
Main reference: [52].

(3.1) We call a finite group scheme G a BT_1 group scheme if $\mathrm{Im}F_G = \mathrm{Ker}V_G$ and $\mathrm{Im}V_G = \mathrm{Ker}F_G$. In particular this implies that G is annihilated by p. Group schemes annihilated by p were classified by Kraft, see [27], and later, independently by Oort. A crucial fact is:

Fix $h \in \mathbb{Z}_{\geq 0}$. The set of isomorphism classes of finite group schemes annihilated by p of rank equal to p^h over an algebraically closed field k is finite.

Moreover one can give a definition of a "quasi-polarization" on a BT_1 group scheme; this is not so difficult if $p > 2$, but more care has to be taken in case $p = 2$, for details see [52]. A principally polarized abelian variety (A, λ) gives rise to a quasi-polarized BT_1 group scheme $(A, \lambda)[p]$. In all cases one proves that *the set of isomorphism classes of quasi-polarized* BT_1 *group schemes of fixed rank over an algebraically closed field k is* finite; especially see [52], Section 9.

(3.2) A quasi-polarized BT_1 group scheme (G, λ) is uniquely determined by its "elementary sequence". This is a set of positive integers $\{\varphi(1), \cdots, \varphi(g)\}$ with the properties $\varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1$ for $0 \leq i < g$, where $\varphi(0) := 0$. This sequence is obtained by producing on G a "final filtration", see [52], Section 2; the elementary sequence describes for every step in the filtration its image under V. We write $\mathrm{ES}(G, \lambda)$ for the elementary sequence of (G, λ).

(3.3) Choose an elementary sequence φ; equivalently: choose an isomorphism class of a quasi-polarized BT_1 group scheme (G, λ) over an algebraically closed field. Let $(A, \lambda) \to T$ be a principally polarized abelian scheme over a base field in characteristic p. We write:

$$\mathcal{S}_\varphi(T) = \{t \mid \mathrm{ES}((A, \lambda)_t[p]) = \varphi\}.$$

This turns out to be a locally closed set in T, see [52], Proposition 3.2. We write

$$S_\varphi = \mathcal{S}_\varphi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p),$$

which is called the Ekedahl-Oort stratum determined by φ. Note that $[(A, \lambda)] = x \in \mathcal{A}_{g,1}(K)$ iff for any algebraically closed field $k \supset K$ we have $(A, \lambda)[p] \otimes k \cong (G, \lambda) \otimes k$. In [52] several properties of these strata are proven (quasi-affine, dimension, etc.).

4 Foliations

Fol $\boxed{\text{Central leaves are given by } (A, \lambda)[p^\infty]/\cong .}$

(4.1) Let X be a p-divisible group and let $\mathcal{Y} \to S$ be a p-divisible group over a base S. We write

$$\mathcal{C}_X(S) = \{s \in S \mid \exists \Omega, \exists \mathcal{Y}_s \otimes \Omega \cong X \otimes \Omega\}.$$

Choose a quasi-polarized p-divisible group (X, λ). Let $(\mathcal{Y}, \mu) \to S$ be a quasi-polarized p-divisible group over a base S. We write

$$\mathcal{C}_{(X,\lambda)}(S) = \{s \in S \mid \exists \Omega, \exists (\mathcal{Y}, \mu)_s \otimes \Omega \cong (X, \lambda) \otimes \Omega\}.$$

(4.2) Theorem, see [54], Th. 2.3.

$$\mathcal{C}_X(S) \subset \mathcal{W}^0_{\mathcal{N}(X)}(S)$$

is a closed set.

A proof can be given using (4.3), (4.4) and (4.5).

(4.3) Definition. *Let S be a scheme, and let $X \to S$ be a p-divisible group. We say that X/S is* geometrically fiberwise constant, *abbreviated* gfc *if there exist a field K, a p-divisible group X_0 over K, a morphism $S \to \mathrm{Spec}(K)$, and for every $s \in S$ an algebraically closed field $k \supset \kappa(s) \supset K$ containing the residue class field of s and an isomorphism $X_0 \otimes k \cong_k X_s \otimes k$.*
The analogous terminology will be used for quasi-polarized p-divisible groups and for (polarized) abelian schemes.

See [54], 1.1.

(4.4) Theorem (T. Zink & FO). *Let S be an integral,* normal *Noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group with* constant Newton polygon. *Then there exists a p-divisible $\mathcal{Y} \to S$ and an S-isogeny $\varphi : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}/S$ is* gfc.

See [75], [64], 2.1, and [54], 1.8.

(4.5) Theorem. *Let S be a scheme which is integral, and such that the normalization $S' \to S$ gives a noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group; let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $\mathcal{X} \to S$ is* gfc. *Then there exists a finite surjective morphism $T_n = T \to S$, such that $\mathcal{X}[p^n] \times_S T$ is constant over T.*

See [54], 1.3.

(4.6) Note that we gave a "point-wise" definition of $\mathcal{C}_X(S)$; we can consider $\mathcal{C}_X(S) \subset S$ as a closed set, or as a subscheme with induced reduced structure; however is this last definition "invariant under base change"? It would be much better to have a "functorial definition" and a nature-given scheme structure on $\mathcal{C}_X(S)$.

Note that the proof of this theorem is quite involved. One of the ingredients is the notion of "completely slope divisible p-divisible groups" introduced by T. Zink, and theorems on p-divisible groups over a normal base, see [75] and [64].

Considering the situation in the moduli space with enough level structure in order to obtain a fine moduli scheme, we see that $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p)$ is regular.

We write $\mathcal{C}_x$ for the irreducible component of $C(x) := \mathcal{C}_{(A,\lambda)[p^\infty]}\mathcal{A}$ passing through $[(A, \lambda)] = x$; we say that $\mathcal{C}_x$ is the *central leaf* passing through x.
Remark (Chai & FO). In fact, for $\mathcal{N}(A) \neq \sigma$, i.e. A is not supersingular, it is known that $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A})$ is *geometrically irreducible;* see (9.8).

(4.7) Isogeny correspondences, unpolarized case. Let $\psi : X \to Y$ be an isogeny between p-divisible groups. Then there exist finite surjective morphisms

$$\mathcal{C}_X(\mathrm{Def}(X)) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_Y(\mathrm{Def}(Y)).$$

(4.8) Corollary. *The dimension of $\mathcal{C}_X(\mathrm{Def}(X))$ only depends on the isogeny class of X.*

(4.9) Isogeny correspondences, polarized case. Let $\psi : A \to B$ be an isogeny, and let λ respectively μ be a polarization on A, respective on B, and suppose there exists an integer $n \in \mathbb{Z}_{>0}$ such that $\psi^*(\mu) = n{\cdot}\lambda$. Then there exist finite surjective morphisms

$$\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [54], 3.16.

(4.10) Corollary. The dimension of $\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$ only depends on the isogeny class of (X, λ).
Remark / Notation. In fact, this dimension only depends on the isogeny class of X. We write

$$c(\xi) := \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)\right), \quad X = A[p^\infty], \quad \xi := \mathcal{N}(X);$$

this is well-defined: all irreducible components have the same dimension.

A proof of all previous results on isogeny correspondences, and the independence of the dimension of the leaf in an isogeny class can be given using (4.3), (4.4) and (4.5); see [54], 2.7 and 3.13.

(4.11) Hecke-α-orbits. Consider $[(A, \lambda)] = x \in \mathcal{A}(K)$ Consider all diagrams

$$(A, \lambda) \otimes \Omega \xleftarrow{\varphi} (C, \zeta) \xrightarrow{\psi} (B, \mu),$$

where:
Ω is an algebraically closed field containing K,
(C, ζ) is a polarized abelian variety over Ω,
$\varphi : C \to A$ is an isogeny such that $\varphi^*(\lambda) = \zeta$,
(B, μ) is a principally polarized abelian variety over Ω, and
$\psi : C \to B$ is an isogeny such that $\psi^*(\mu) = \zeta$;
Ker(φ) and Ker(ψ) are geometrically successive extensions of the finite group scheme α_p; in this case we write $[(B, \mu)] \in \mathcal{H}_\alpha(x)$

(4.12) For a given point $[(A, \mu)] = x$ one can study $\mathcal{H}_\alpha(x)$, see (4.11). In general *this is not a (locally) closed set in* $\mathcal{A}_g \otimes \mathbb{F}_p$: the "number of components" can be infinite. The union of "components" of $\mathcal{H}_\alpha(x)$ containing x however is a closed set, which we denote by $I(x)$. An irreducible component of $I(x)$ is called an *isogeny leaf*; for details see [54]; related notion: Rapoport-Zink spaces. It turns out that

central leaves and isogeny leaves give, up to a finite morphism,
a product structure on every component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$,

see [54], Theorem 5.3.

(4.13) We see that all central leaves belonging to a symmetric Newton polygon ξ have the same dimension (independent of the degree of the polarization). This dimension will be denoted by $c(\xi)$ or by $\mathrm{cdp}(\xi)$. For any irreducible component W of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ we see that for $x \in W$ we have $c(\xi) + \dim(I(x)) = \dim(W)$. We will write $i(\xi) := \mathrm{sdim}(\xi) - c(\xi)$; this is the dimension of any isogeny leaf in the *principally* polarized case. Here are some examples:

(4.14) As illustration we record for $g = 4$ the various data considered:

NP	ξ	f	sdim(ξ)	$c(\xi)$	$i(\xi)$	ES($H(\xi)$)
ρ	$(4,0)+(0,4)$	4	10	10	0	$(1,2,3,4)$
$f=3$	$(3,0)+(1,1)+(0,3)$	3	9	9	0	$(1,2,3,3)$
$f=2$	$(2,0)+(2,2)+(0,2)$	2	8	7	1	$(1,2,2,2)$
β	$(1,0)+(2,1)+(1,2)+(0,1)$	1	7	6	1	$(1,1,2,2)$
γ	$(1,0)+(3,3)+(0,1)$	1	6	4	2	$(1,1,1,1)$
δ	$(3,1)+(1,3)$	0	6	5	1	$(0,1,2,2)$
ν	$(2,1)+(1,1)+(1,2)$	0	5	3	2	$(0,1,1,1)$
σ	$(4,4)$	0	4	0	4	$(0,0,0,0)$

Here $\rho \succ (f=3) \succ (f=2) \succ \beta \succ \gamma \succ \nu \succ \sigma$ and $\beta \succ \delta \succ \nu$. The notation ES, encoding the isomorphism type of a BT_1 group scheme, is as in [52]; the number f indicates the p-rank.

(4.15) For $g = 5$ and $f = 0$ we obtain 5 possible Newton polygons (a totally ordered set in this case):

ξ	sdim(ξ)	$c(\xi)$	$i(\xi)$	ES($H(\xi)$)
$(4,1)+(1,4)$	10	9	1	$(0,1,2,3,3)$
$(3,1)+(1,1)+(1,3)$	9	7	2	$(0,1,2,2,2)$
$(2,1)+(2,2)+(1,2)$	8	4	4	$(0,1,1,1,1)$
$(3,2)+(2,3)$	7	3	4	$(0,0,1,1,1)$
$(5,5)$	6	0	6	$(0,0,0,0,0)$

(4.16) **Theorem.** *A geometrically fiberwise constant p-divisible group X/S over an (excellent) integral normal base has a natural slope filtration.*

(4.17) **Corollary.** *Let $X \to C$ be a p-divisible group over a central leaf. Then X/S admits a slope filtration.*

For the proof of the theorem one uses first [64], in order to have an isogeny $\psi : Z \to X/S$ from a completely slope divisible p-divisible group , called Z, to the given one X/S. Let $\mathrm{Ker}(\psi) = N \to S$. By induction on the isoclinic parts of the filtration $Z_1 \subset Z_2 \subset \cdots Z$, it suffices to show that $N \cap Z_1$ is flat over S.

Going to a finite cover $T \to S$, using [54], Theorem 1.3, further using that completely slope divisible implies geometrically fiberwise constant (or using directly [54] Lemma 1.4), and using [54] Lemma 1.10, and [54] Lemma 1.9 we see that the fibers of $N_T \cap Z_{1,T}$ have constant rank, hence the same for the fibers of $N \cap Z_1$, hence this is flat. This proves the layer $Z_1 \subset Z$ descends to a sub-p-divisible $X_1 \subset X$, which is the lowest isoclinic part; finish by induction.

(4.18) **Example.** Here is an example where there does not exist a slope filtration. Choose $g = 3$, and $\xi = (2,1) + (1,2)$. Choose a principally polarized abelian variety (A, λ) with $\mathcal{N}(A) = \xi$ and $a(A) = 2$. For $[(A,\lambda)] = x$ we see that $I(x)$ is a curve, which locally at x consists of two branches; over one branch there does exist a slope filtration of the deformed p-divisible group; over the other branch the natural slope filtration does not exist.

In most cases: over an open Newton polygon stratum the natural slope filtration does not exist.

5 Monodromy and Hecke orbits (following Ching-Li Chai)

(5.1) In this section we fix a prime number p, we fix an integer $N \geq 3$ (to be used to define level structures), such that p does not divide N, and we choose a prime number ℓ not dividing pN. We fix an algebraically closed field $k \supset \mathbb{F}_p$. We write $\mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$, the moduli space of principally polarized abelian varieties over an extension of k with a symplectic level-N-structure.

(5.2) Hecke-ℓ-orbits. Fix a prime number ℓ different from p. Consider $[(A,\lambda)] = x \in \mathcal{A}$; suppose that (A,λ) is defined over a field $K \supset \mathbb{F}_p$; we say that (B,μ) is in the Hecke-ℓ-orbit of (A,λ), notation $[(B,\mu)] \in \mathcal{H}_\ell(x)$, if there exists an isogeny $\varphi : A \to B$ and an integer n which is a power of ℓ such that $\varphi^*(\mu) = n{\cdot}\lambda$.

This set $\mathcal{H}_\ell(x) \subset \mathcal{A}$ is called the Hecke-ℓ-orbit of the point x. A subset $S \subset \mathcal{A}$ is called *Hecke-ℓ-stable* if for every $x \in S$ we have $\mathcal{H}_\ell(x) \subset S$.

For Hecke correspondences also see [12], VII.3.

(5.3) Let $Z \subset \mathcal{A}$ be a locally closed subset. Let Z^0 be an irreducible component of Z. Let $\eta \in Z^0$ be the generic point. Let $A \to Z$ be the universal abelian scheme restricted to Z. Let

$$\rho_{A,\ell} : \pi_1(Z^0, \overline{\eta}) \quad \longrightarrow \quad \mathrm{Sp}(T_\ell, <,>_\ell)$$

be the ℓ-adic monodromy representation in the Tate-ℓ-group of A_η. Identify the Tate-ℓ-group of A_η over $\overline{\eta}$ with $\mathbb{Z}_\ell^{2g}$ with the standard pairing.

(5.4) Theorem (C.-L. Chai). *Choose notation as above. Let $Z \subset \mathcal{B}$ be a locally closed subscheme, smooth over* $\mathrm{Spec}(k)$*, such that:*

Z is Hecke-ℓ-stable, and

the Hecke-ℓ-action on the set $\pi_0(Z)$ is transitive, and

$\eta \notin W_\sigma$ (equivalently: Z contains a non-supersingular point).

Then:

$$\rho_{A,\ell} : \pi_1(Z^0, \overline{\eta}) \longrightarrow \mathrm{Sp}(T_\ell, <,>_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$$

is surjective, *and*

$$Z \textit{ is irreducible, i.e. } Z = Z^0.$$

See [4], 4.4.

6 NP: dimensions

In 1977 Tadao Oda and I made a conjecture computing the expected dimension of the supersingular locus $W_\sigma \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, see [41]. In 1998 Ke-Zheng Li and I proved this, see [30]. In [47] I indicated what the dimensions of the Newton polygon strata in $\mathcal{A}$ should be. In [53] I prove a more general result which reproves the supersingular case. Here are the results giving these formulas.

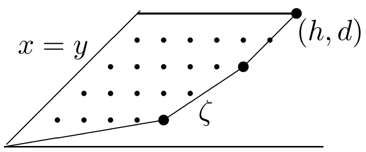
(6.1) The dimension of Newton polygon strata, unpolarized case. We fix integers $h \geq d \geq 0$, and we write $c := h - d$. We consider Newton Polygons ending at (h,d). For such a NP ζ we write:

$$\Diamond(\xi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < d, \quad y < x, \quad (x,y) \ \text{ on or above } \ \zeta\},$$

and we write

$$\boxed{\dim(\zeta) := \#(\Diamond(\zeta)).}$$

Example:



$\zeta = 2 \times (1,0) + (2,1) + (1,5) =$

$= 6 \times \frac{1}{6} + 3 \times \frac{2}{3} + 2 \times \frac{1}{1}; \quad h = 11.$

Here $\dim(\zeta) = \#(\Diamond(\zeta)) = 22$.

(6.2) Theorem see [51], Th. 3.2 and [53], Th. 2.10. *Let X_0 be a p-divisible group over a field K; let $\zeta \succ \mathcal{N}(X_0)$. Then:*

$$\dim(\mathcal{W}_\zeta(\mathrm{Def}(X_0))) \quad = \quad \dim(\zeta).$$

For polarized abelian varieties and for quasi-polarized p-divisible groups one can try to determine the dimension of the Newton polygon strata. For an arbitrary polarization see (12.2). For *principal polarizations*:

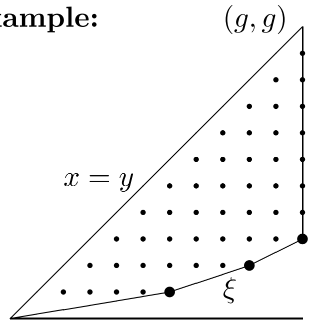
(6.3) The dimension of Newton polygon strata, principally polarized case. We fix an integer g. For every *symmetric* NP ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g, \quad (x,y) \ \text{ on or above } \ \xi\},$$

and we write

$$\boxed{\mathrm{sdim}(\xi) := \#(\triangle(\xi)).}$$

Example:



$\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)) = \#(\triangle(\xi))$

$\xi = (5,1) + (2,1) + 2{\cdot}(1,1) + (1,2) + (1,5),$

g=11; slopes: $\{6 \times \frac{5}{6}, 3 \times \frac{2}{3}, 4 \times \frac{1}{2}, 3 \times \frac{1}{3}, 6 \times \frac{1}{6}\}$.

This case: $\dim(\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)) = \mathrm{sdim}(\xi) = 48$.

(6.4) Theorem, see [51], Th. 3.4 and [53], Th. 4.1. *Let ξ be a symmetric Newton polygon. Then:*

$$\dim\left(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)\right) = \mathrm{sdim}(\xi).$$

(6.5) We sketch a proof of the Grothendieck conjecture, and a proof of the computation of dimensions of Newton polygon strata.

For a group scheme G over a field L of characteristic p we define $a(G)$ to be the dimension of the K-vector space $\mathrm{Hom}(\alpha_p, G_K)$, where $K \supset L$ is a perfect field. If G is local-local, the a-number is the minimal number of generators of the Dieudonné module of G_K.

The central and most difficult part of the proof is the fact:

(6.6) Theorem. *For a given p-divisible group X_0 over a field K, there is a deformation $X \to T$ over an integral base, $0 \in T(K)$, with $X_0 = X \otimes_T 0$, with generic point $\eta \in T$ such that $a(X_\eta) \leq 1$ and $\mathcal{N}(X_\eta) = \mathcal{N}(X_0)$.*
For a given principally quasi-polarized p-divisible group (X_0, λ) there exist a deformation $(X, \lambda) \to T$ with constant Newton polygon and $a(X_\eta) \leq 1$.

On first shows this to be true for a simple p-divisible group, i.e. a group isogenous with some $G_{m,n}$. This is done by constructing a "catalogue" containing all such groups, for details see [21]. Then one has the task to show this base T is irreducible. A tricky computation plus the general theorem "Purity" then finishes the proof in the case of a simple p-divisible group; for details see [21]. To find this proof took me 7 years.

Then one studies deformations of p-divisible groups with a filtration; for details see [53]. In this way Theorem (6.6) is proved.

Then one sees that Newton polygon strata are regular at points where $a(X) = 1$; this is done by the technique of "Cayley-Hamilton": in that case the deformation theory with constant Newton polygon is a smooth problem, and the strata are exactly nested as given by the Newton polygon partial ordering; for details see [51]. In this way we can can read off the dimension of any Newton polygon stratum. We can write down explicitly a deformation to a given Newton polygon, and a proof of the Grothendieck conjecture follows.

7 NP: irreducibility

(7.1) The number of supersingular elliptic curves is large for p large. For the case of elliptic curves in characteristic p this is classical. By Hasse, Deuring and Igusa we know:

$$\sum_{j(E) \text{ is ss}} \frac{1}{\#(\mathrm{Aut}(E))} = \frac{p-1}{24};$$

this implies that the number of supersingular j-invariants is asymptotically $p/12$ for $p \to \infty$.

For other values of g, the dimension of the abelian varieties in considerations we have an analogous situation:

(7.2) Example. For a scheme X over a field we denote by $\Pi_0(X)$ the number of geometrically irreducible components of X. Note that the supersingular locus has "many components" for fixed g and large p; see [30], 4.9:

$$\#(\Pi_0(W_\sigma)) = H_g(p,1) \quad \text{if} \quad g \quad \text{is odd},$$

$$\#(\Pi_0(W_\sigma)) = H_g(1,p) \quad \text{if} \quad g \quad \text{is even}.$$

Note that for g fixed, and $p \to \infty$, indeed $\#(\Pi_0(W_\sigma)) \to \infty$. We see that W_σ is geometrically reducible for $p \gg 0$, e.g. for $p > 11$ if $g = 2$ (see [23], 5.8), for $p > 2$ if $g = 3$ (see [24], 6.8).

One might wonder what the number of irreducible components is for an arbitrary Newton polygon stratum; it seems quite a job to compute the number of components of all Newton polygon strata. However, as it will turn out: *any non-supersingular* NP-*stratum* W_ξ *is irreducible,* see (7.4) below. In fact:

(7.3) Theorem. *For every g, for every prime number $\ell \neq p$, for every N as above, and for every symmetric Newton polygon ξ, the action of $\mathcal{H}_\ell$ on the set of irreducible components $\Pi_0(W_{N,\xi})$ of the Newton polygon stratum $W_{N,\xi} \subset \mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$ is* transitive.
For details see [55].

Using the monodromy result by Chai in Section 5 we obtain:

(7.4) Theorem. *For every $\xi \supsetneq \sigma$ the locus $W_{N,\xi} \subset \mathcal{B}$ is* irreducible; i.e.

$$\xi \neq \sigma \quad \Rightarrow \quad \#(\Pi_0(W_{N,\xi})) = 1.$$

This was conjectured in [50], Conjecture 8B. This answers [66], 3.8.

8 EO: dimensions and irreducibility

(8.1) In [52] Theorem 1.2 we find:

$$\dim(S_\varphi) \quad = \quad |\varphi| \quad := \quad \varphi(1) + \cdots + \varphi(g).$$

In [34], [36], [70], [71] we find alternative approaches to this computation.

(8.2) Lemma (FO, unpublished). *Let $g = 2r+1$, respectively $g = 2r$; let φ be an elementary sequence. Then:*

$$S_\varphi \quad \subset \quad W_\sigma \quad \Longleftrightarrow \quad \varphi(r) = 0.$$

(8.3) In [52], 14.1 I conjectured that the EO strata not contained in the supersingular locus should be irreducible. This has been proved now:

(8.4) Theorem (T. Ekedahl and G van der Geer, see [11], Theorem 11.5). *Any* EO-stratum *S_φ inside $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ not contained in the supersingular locus is geometrically irreducible.*

Also, I conjectured that EO strata contained in the supersingular locus should be reducible for p large. This has been proved:

(8.5) Theorem (S. Harashita). *For every $S_\varphi \subset W_\sigma$ the number of geometric components of S_φ equals a certain class number; for p large this number is bigger than 1, i.e. for $p \gg 0$ the locus S_φ is geometrically* reducible.
See [16].

9 Foliations: dimensions and irreducibility

We have seen that the dimension of a central leaf, in the unpolarized case written as $\mathrm{cdu}(\zeta)$, respectively $\mathrm{cdp}(\xi)$ in the polarized case, only depends on the Newton polygon (and not on the degree of the polarization) see (4.8), (4.10). Hence we are looking for a formula allowing us to compute such a dimension.

(9.1) Historical remark. In summer 2000 I gave a talk in Oberwolfach on foliations in moduli spaces of abelian varieties. After my talk, in the evening of Friday 4-VIII-2000 Bjorn Poonen asked me several questions, especially related to the problem I raised to determine the dimensions of central leaves. Our discussion resulted in Problem 21 in [10]. His expectations coincided with computations I had made of these dimensions for small values of g. Then I jumped to the conclusion what those dimensions for general g could be; that is what was proved later, and reported on here, see (9.3), (9.6). I thank Bjorn Poonen for his interesting questions; our discussion was valuable for me.

(9.2) Notation. Let ζ be a Newton polygon, and $(x,y) \in \mathbb{Q} \times \mathbb{Q}$. We write:
$(x,y) \prec \zeta$ if (x,y) is on or above ζ,
$(x,y) \precneqq \zeta$ if (x,y) is strictly above ζ,
$(x,y) \succ \zeta$ if (x,y) is on or below ζ,
$(x,y) \succneqq \zeta$ if (x,y) is strictly below ζ.

Let ζ be a Newton polygon. Suppose that the slopes of ζ are $1 \geq \beta_1 \geq \cdots \geq \beta_h \geq 0$; this polygon has slopes $\beta_h, \cdots, \beta_1$ (non-decreasing order), and it is lower convex. We write ζ^* for the polygon starting at $(0,0)$ constructed using the slopes $\beta_1, \cdots, \beta_h$ (non-increasing order); note that ζ^* is upper convex, and that the beginning and end point of ζ and of ζ^* coincide.

We write:

$$\Diamond(\zeta;\zeta^*) := \{(x,y) \in \mathbb{Z} \mid (x,y) \prec \zeta, \quad (x,y) \succneqq \zeta^*\}; \qquad \boxed{\mathrm{cdu}(\zeta) := \#\left(\Diamond(\zeta;\zeta^*)\right)};$$

"cdu" = <u>d</u>imension of <u>c</u>entral leaf, <u>u</u>npolarized case.

Remark. One can show:

$$\#(\Diamond(\zeta;\zeta^*)) = \sum_{1 \leq i < h} \left(\zeta^*(i) - \zeta(i)\right).$$

See [62] for details.

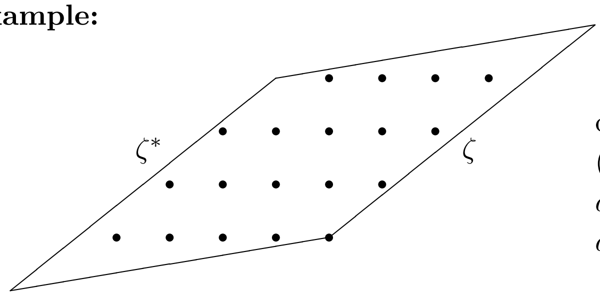
We suppose $\zeta = \sum_{1 \leq i \leq u} \mu_i \cdot (m_i, n_i)$, written in such a way that $\gcd(m_i, n_i) = 1$ for all i, and $\mu_i \in \mathbb{Z}_{>0}$, and $i < j \Rightarrow (m_i/(m_i+n_i)) > (m_j/(m_j+n_j))$. Write $d_i = \mu_i \cdot m_i$ and $c_i = \mu_i \cdot n_i$ and $h_i = \mu_i \cdot (m_i+n_i)$; write $\nu_i = m_i/(m_i+n_i) = d_i/(d_i+c_i)$ for $1 \leq i \leq u$. Note that the slope $\nu_i = \mathrm{slope}(G_{m_i,n_i}) = m_i/(m_i+n_i) = d_i/h_i$: this Newton polygon is the "Frobenius-slopes" Newton polygon of $\sum (G_{m_i,n_i})^{\mu_i}$. Note that the slope ν_i appears h_i times; these slopes with these multiplicities give the set $\{\beta_j \mid 1 \leq j \leq h := h_1 + \cdots + h_u\}$ of all slopes of ζ.

(9.3) Theorem (Dimension formula, the unpolarized case.) *Let X_0 be a p-divisible group, $D = \mathrm{Def}(X_0)$; let Y be a p-divisible group with $\mathcal{N}(Y) \succ \mathcal{N}(X_0)$;*

$$\dim(\mathcal{C}_Y(D)) \quad = \quad \mathrm{cdu}(\beta).$$

See [5], 7.10. For details see [62].

Example:



$\dim(\mathcal{C}_X(D)) = \#((\diamond(\zeta;\zeta^*)).$
$(\frac{4}{5} - \frac{1}{6}){\cdot}5{\cdot}6 = 19,$
$d_1h_2 - d_2h_1 = 4{\cdot}6 - 1{\cdot}5 = 19;$
$d_1c_2 - d_2c_1 = 4{\cdot}5 - 1{\cdot}1 = 19.$

(9.4) Remark/Example. Suppose ζ is isoclinic (i.e. all slopes are the same). We see that for an isoclinic Newton polygon $\mathrm{cdu}(\zeta) = 0$. Hence the theorem says that there are no non-trivial deformations with constant Newton polygon in the isoclinic case; this was already proved in [21], Corollary 2.17.

(9.5) Notation, the polarized case. Let ξ be a symmetric Newton polygon. For convenience we adapt notation to the symmetric situation:

$$\xi = \mu_1{\cdot}(m_1, n_1) + \cdots + \mu_s{\cdot}(m_s, n_s) + r{\cdot}(1,1) + \mu_s{\cdot}(n_s, m_s) + \cdots + \mu_1{\cdot}(n_1, m_1)$$

with:
$m_i > n_i$ and $\gcd(m_i, n_i) = 1$ for all i,
$1 \leq i < j \leq s \Rightarrow (m_i/(m_i + n_i)) > (m_j/(m_j + n_j))$,
$r \geq 0$ and $s \geq 0$.
We write $d_i = \mu.m_i$, and $c_i = \mu.n_i$, and $h_i = d_i + c_i$. Write $g := \left(\sum_{1 \leq i \leq s}(d_i + c_i)\right) + r$ and write $u = 2s + 1$.

We write:

$$\triangle(\xi;\xi^*) := \{(x,y) \in \mathbb{Z} \mid (x,y) \prec \xi, \quad (x,y) \succneqq \xi^*, \quad x \leq g\}; \qquad \boxed{\mathrm{cdp}(\zeta) := \#\left(\triangle(\zeta;\zeta^*)\right)};$$

"cdp" = <u>d</u>imension of <u>c</u>entral leaf, <u>p</u>olarized case.

Remark. One can show:

$$\#(\triangle(\xi;\xi^*)) = \sum_{1 \leq i \leq g} \left(\xi^*(i) - \xi(i)\right).$$

See [62] for details.

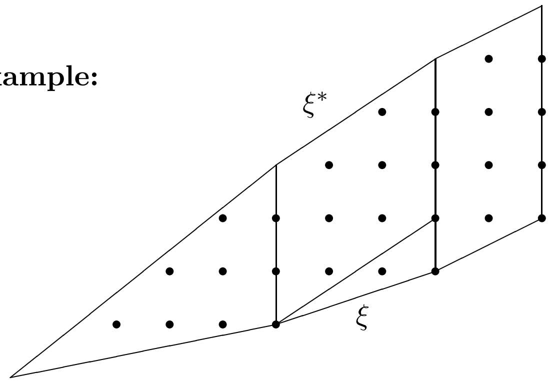
Write $\xi = \sum_{1 \leq i \leq u} \mu_i{\cdot}(m_i, n_i)$, i.e. $(m_j, n_j) = (n_{u+1-j}, m_{u+1-j})$ for $s < j \leq u$ and $r(1,1) = \mu_{s+1}(m_{s+1}, n_{s+1})$. Write $\nu_i = m_i/(m_i + n_i)$ for $1 \leq i \leq u$; hence $\nu_i = 1 - \nu_{u+1-i}$ for all i.

(9.6) Theorem (Dimension formula, the polarized case.) *Let (A, λ) be a polarized abelian variety. Let $(X, \lambda) = (A, \lambda)[p^\infty]$; write $\xi = \mathcal{N}(A)$; then*

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A} \otimes \mathbb{F}_p)\right) \quad = \quad \mathrm{cdp}(\xi).$$

See [5], Proposition 7.12. For details, see [62].

Example:



$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \sum_{0<i\leq g} (\xi^*(i) - \xi(i))$.

slopes $1/5, 4/5, h = 5$: $\quad \frac{1}{2}4{\cdot}5 - \frac{1}{2}1{\cdot}2 = 9$,

slopes $1/3, 2/3, h = 3$: $\quad \frac{1}{2}2{\cdot}3 - \frac{1}{2}1{\cdot}2 = 2$,

$(d_1 - c_1)h_2 = 3{\cdot}3 = 9$,

$(d_1 + d_2 - c_1 - c_2)r = 4{\cdot}2 = 8$,

$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)) = \#\,(\triangle(\zeta;\zeta^*)) = 28$.

(9.7) Remark/Question. Suppose $\xi_1 \gneqq \xi_2$. By the theorem we see that in this case $\mathrm{cdp}(\xi_1) > \mathrm{cdp}(\xi_2)$. Is there a direct proof for this? It might be that this can be answered in another way if we have more information on the question as formulated in (12.7).

(9.8) Theorem (Ching-Li Chai & FO, details will appear in [8]). *Let $x \in \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ be* non-supersingular. *The central leaf $C(x)$ is geometrically irreducible.*
Various methods are applied. We use irreducibility of non-supersingular Newton polygon strata, see (7.4), and we use the notion of hypersymmetric abelian varieties, see [7].

10 Hecke orbits

Basic references: [3], [5].

(10.1) Let $x \in \mathcal{A}_g \otimes \mathbb{C}$. The closure of the Hecke orbit $\mathcal{H}(x)$ equals $\mathcal{A}_g \otimes \mathbb{C}$; here "closure" can either be taken in the complex topology or in the Zariski topology.

In positive characteristic the analogous statement is not true in general. As an easy example, choose a moduli point $e \in \mathcal{A}_1 \otimes \mathbb{F}_p$. If the related elliptic curve is *ordinary*, then $\mathcal{H}(e)$ is dense in $\mathcal{A}_1 \otimes \mathbb{F}_p$ (and this is not difficult to show); however if the related elliptic curve is *supersingular*, then we see that $\mathcal{H}(e)$ in every component of $\mathcal{A}_1 \otimes \mathbb{F}_p$ consists of the finite set of elliptic curves with that degree of polarization; hence $\mathcal{H}(e)$ is not dense in any of the components of $\mathcal{H}(e)$. However a deep result by Chai is stimulating:

(10.2) Theorem (Chai, 1995). *For every* ordinary $x \in \mathcal{A}_g \otimes \mathbb{F}_p$ *the Hecke-ℓ-orbit is dense in $\mathcal{A}_g \otimes \mathbb{F}_p$.* See [3].

In 1993 I formulated the following conjecture:

(10.3) Conjecture. *Suppose $[(A,\mu)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$, and write $\xi = \mathcal{N}(A)$. Then*

$$\mathcal{H}(x) \quad \text{is Zariski-dense (?) in} \quad \mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [50], 15A.

In 2004 Ching-Li Chai and I proved this conjecture, where one final detail was filled in by Chia-Fu Yu:

(10.4) Theorem (the Hecke Orbit Conjecture).

$$(\mathcal{H}(x))^{\mathrm{Zar}} = \mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p).$$

Note that the Hecke Orbit Conjecture is true if x is supersingular: every irreducible component of $\mathcal{W}_\sigma(\mathcal{A}_g \otimes \mathbb{F}_p)$ is an isogeny leaf, and $\mathcal{H}_\ell$ operates transitively on $\Pi_0(W_\sigma)$.

(10.5) The proof of the general case consists of three completely different steps:

(I) *Any irreducible component of an open Newton polygon stratum is the product of a central leaf and an isogeny leaf, up to a finite map*, see (4.12); hence:

$$\mathcal{H}_\ell(x) \textit{ is dense in } C(x) \quad \Rightarrow \quad \mathcal{H}(x) \textit{ is dense in } \mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p).$$

We see it suffices to prove the Hecke-ℓ-Orbit Conjecture.

(II) (The discrete form of the Hecke Orbit Conjecture). *For every non-supersingular $x \in \mathcal{A}_g \otimes \mathbb{F}_p$ the central leaf passing through x is geometrically irreducible; i.e. $C(x) = C_x$.* This is precisely (9.8).

(III) (The continuous form of the Hecke Orbit Conjecture). *For every (non-supersingular) $x \in \mathcal{A}_g \otimes \mathbb{F}_p$ its Hecke-ℓ-orbit $\mathcal{H}_\ell(x)$ is dense in the irreducible component C_x of the central leaf $C(x)$ passing through x.* This last step is quite involved, and we refer to [5] for a survey.

We see that these three steps prove the Hecke Orbit Conjecture. We see that the methods in the various steps are are rather different. The most technical and most difficult step is the continuous form of the Hecke Orbit Conjecture; it requires a lot of different techniques. For a survey we refer to [5], until the proof is completely written up and published in [8].

11 Minimal p-divisible groups and simple finite group schemes

Basic references: [60], [63].

(11.1) A p-divisible group X can be seen as a tower of building blocks, each of which is isomorphic to the same finite group scheme $X[p]$. Clearly, if X_1 and X_2 are isomorphic then $X_1[p] \cong X_2[p]$; however, conversely $X_1[p] \cong X_2[p]$ does in general not imply that X_1 and X_2 are isomorphic. Can we give, over an algebraically closed field in characteristic p, a condition on the p-kernels which ensures this converse? Here are two known examples of such a condition: consider the case that X is *ordinary*, or the case that X is *superspecial* (X is the p-divisible group of a product of supersingular elliptic curves); in these cases the p-kernel uniquely determines X.

(11.2)

EO In [52] we have defined a natural *stratification* of the moduli space of polarized abelian varieties in positive characteristic: moduli points are in the same stratum if and only if the corresponding p-kernels are geometrically isomorphic. Such strata are called EO-strata.

Fol In [54] we define in the same moduli spaces a *foliation*: moduli points are in the same leaf if and only if the corresponding p-divisible groups are geometrically isomorphic; in this way we obtain a foliation of every open Newton polygon stratum.

Fol $\subset$ EO The observation $X \cong Y \Rightarrow X[p] \cong Y[p]$ shows that any leaf in the second sense is contained in precisely one stratum in the first sense; the main result of [60] "*X is minimal if and only if $X[p]$ is minimal*", shows that *a stratum* (in the first sense) *and a leaf* (in the second sense) *are equal* if we are in the minimal, principally polarized situation:

(11.3) For coprime integers $m, n \in \mathbb{Z}_{\geq 0}$ we define $H_{m,n}$ (see [21], 5.2. Basically this is a p-divisible group , isogenous to $G_{m,n}$, defined over $\mathbb{F}_p$ such that $\mathrm{End}(H_{m,n} \otimes \overline{\mathbb{F}_p})$ is the maximal order in $\mathrm{End}^0(G_{m,n} \otimes \overline{\mathbb{F}_p}$. This determines $H_{m,n}$ up to isomorphism over $\overline{\mathbb{F}_p}$. For a Newton polygon $\zeta = \sum(m_i, n_i)$ we write $H(\zeta) = \prod H_{m_i,n_i}$. Such a p-divisible group is called *minimal*. For every Newton polygon there is up to isomorphism over any algebraically closed field exactly one minimal p-divisible group in the isogeny class defined by that Newton polygon.

(11.4) Theorem see [60], Theorem 1.2. *Suppose X is a p-divisible group over k, and suppose there exist a Newton polygon ζ and an isomorphism $X[p] \cong H(\zeta)[p] \otimes k$. Then $X \cong H(\zeta) \otimes k$.*
Remark. Note that we did not put any restriction a priori on the Newton polygon of X. One of the strong aspects of the theorem is the fact that $X[p] \cong H(\zeta)[p] \otimes k$ forces $\mathcal{N}(X) = \zeta$.

(11.5) Corollary. *Let $[(A, \mu)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$. Suppose $\varphi = (H(\zeta)[p]/\cong)$, and $x \in S_\varphi$. Then $C_x = S_\varphi$.*
Remark. Of course $X \cong Y \Rightarrow X[p] \cong Y[p]$. Hence, for any $[(A, \mu)] = x$ we have $C_x \subset S_{A[p]}$: any central leaf is contained in a unique EO-stratum. The corollary says that in the minimal situation these two concepts coincide. Also see (12.8).

(11.6) Here is another way of phrasing the results. Starting from a p-divisible group X we obtain a BT_1 group scheme:

$$[p] : \{X \mid \text{a } p-\text{divisible group}\}/\cong_k \quad \longrightarrow \{G \mid \text{a } \mathrm{BT}_1\}/\cong_k; \quad X \mapsto G := X[p].$$

This map is known to be surjective; see [19], 1.7, see [52], 9.10; it is not difficult to construct a section for this map, e.g. see [63], 2.5. It is the main theorem of [60] that the fiber of this map over (G up to $\cong_k$) is precisely one p-divisible group X if G is minimal.

(11.7) Work over an algebraically closed field $k \supset \mathbb{F}_p$. Consider BT_1 group schemes. Can we classify the simple ones (in the category of BT_1 group schemes)? In [63] we find:

$$G \textit{ is } \mathrm{BT}_1\textit{-simple} \quad \Longleftrightarrow \quad G \textit{ is indecomposable and minimal.}$$

12 Remarks, questions and conjectures

(12.1) Components of a Newton polygon stratum. We have seen how to compute the dimension of a Newton polygon stratum in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$. However in the non-principally polarized case the number derived need not be the dimension of a NP-stratum. For example, the dimension of W_σ, the supersingular locus, for $g = 3$ equals $[g^2/4] = 2$. However $\mathcal{W}_\sigma(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$ does have components of dimension 2 and of dimension 3; see [24]; for more examples see [30], Chapter 12.

The way that Hecke-p-correspondences blow up and down is still not fully understood in general.

(12.2) Conjecture. *Let ξ be a symmetric Newton polygon; suppose its p-rank is $f(\xi) = f$; this means: $\xi = (f, 0) + \sum(m_i, n_i) + (0, f)$ with $m_i > 0$ and $n_i > 0$. Then (?) $\mathcal{W}_\sigma(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$ does have an irreducible component of dimension equal to $g(g+1)/2 - (g-f) = g(g-1)/2 + f$.* Note that his is the maximal possible by [40], Th. 4.1.

Note the curious fact that moduli spaces of polarized abelian varieties with fixed Newton polygon (or the same of polarized p-divisible groups) under Hecke correspondences can change in dimension, as we have seen. But components of moduli spaces (deformation spaces) of (unpolarized) p-divisible groups with fixed Newton polygon all have the same dimension, as we have seen in (6.2), although also here Hecke correspondences blow up and down in a rather wild way.

(12.3) Complete subvarieties. Moduli spaces of abelian varieties are non-complete (non-compact). One can ask what the maximal dimension is of a complete subvariety. In [50],14.A we find the conjecture that

(12.4)

$$\textit{a complete subvariety } W \textit{ of } \mathcal{A}_g \otimes \mathbb{C} \textit{ for } g \geq 3 \textit{ has } \dim(W) < g(g-1)/2,$$

i.e. the codimension of W in $\mathcal{A}_g \otimes \mathbb{C}$ at least $g+1$ for $g \geq 3$. Also see [49], 2.3G. This conjecture was proved to be true in 2003 by Keel and Sadun, see [28].

In positive characteristic however indeed there is a subvariety of codimension g: the locus V_0 of abelian varieties of p-rank equal to zero is complete, see [44], Theorem 1.1, and has this codimension: Koblitz in the principally polarized case, see [40], Theorem 4.1 for the general case.

(12.5) Conjecture (see [50], 14.B). *Let $g \geq 3$ and let $W \subset \mathcal{A}_g \otimes \mathbb{F}_p$ be a complete subvariety of dimension $g(g-1)/2$; then* (?) the generic point of W corresponds with an abelian variety of p-rank zero.

Suppose this conjecture holds, i.e. indeed V_0 would be the only complete subvariety of this dimension; the irreducibility of $V_0 \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ for $g \geq 3$, see (7.4), and computations of Chow classes by G. van der Geer, see [13], would give a new proof of (12.4)

(12.6) Abelian varieties isogenous to a Jacobian. For more information, and for references, see [59]. Fix some g, and choose some algebraically closed field k. *Is every abelian variety A over k isogenous with the Jacobian of a compact type algebraic curve?* If $g \leq 3$ the answer is "yes": in those cases the closed Torelli locus $T_g \otimes k$ equals $\mathcal{A}_{g,1} \otimes k$. If $g \geq 4$ and $k = \mathbb{C}$: there does exist an abelian variety of this dimension over $\mathbb{C}$ not isogenous with a Jacobian.

(12.6).1 **Expectation.** *If $g \geq 4$ and either $k = \overline{\mathbb{Q}}$ or $k = \overline{\mathbb{F}_p}$ there does exist an abelian variety of this dimension over k not isogenous with a Jacobian.*

(12.6).2 **Question.** *Fix g; which Newton polygon occur as the Newton polygon of a Jacobian? I.e., determine the set of all Newton polygons ξ such that $W_\xi \cap (T_g \otimes \mathbb{F}_p) \neq \emptyset$.*

(12.6).3 **Expectation.** *There does exist g and ξ for which $W_\xi \cap (T_g \otimes \mathbb{F}_p) = \emptyset$.* For details see [59].

(12.6).4 **Expectation.** *We expect that $(3,2) + (2,3)$ does not appear on T_5.*
More generally. *Let $g \geq 5$. We expect that the almost supersingular Newton polygon does not appear on the closed Torelli locus.*

(12.6).5 **Conjecture.** *Suppose C_1 and C_2 are non-singular complete curves whose Jacobian has Newton polygon ξ_1, respectively ξ_2. We conjecture there do exist $P_i \in C_i$ such that the stable curve $C_1 \cup_{P_1=P_2} C_2$ admits a deformation to a smooth curve with Newton polygon $\xi_1 \cup \xi_2$.*
Notation: $\xi_1 \cup \xi_2$ is the Newton polygon obtained by taking all slopes in ξ_1 and in ξ_2, and ordering these slopes in non-decreasing order.
See [59].

(12.6).6 **Question.** *Is it true that for any g the supersingular locus $W_\sigma \cap T_g$ has a component of dimension at least $g-1$? Or that all of these components have dimension equal to $g-1$?*
I discussed this with Carel Faber in August 2005, and we both thought that at least one of these questions could have a positive answer.

If the previous conjecture holds, for every g there does exist a component of the supersingular locus of T_g of dimension at least $g-1$.

Boundary points of a central leaf. Suppose we have Newton polygons $\xi_1 \supsetneqq \xi_2$. Let $x \in C_x \subset W^0_{\xi_1}$ be a central leaf. Consider the intersection $C(x)^{\mathrm{Zar}} \cap W^0_{\xi_2}$, i.e. boundary points of C_x with Newton polygon ξ_2. We would like to have information on this intersection. Experiments show that in general a point $y \in W^0_{\xi_2}$ need not be in the closure of any leaf in $W^0_{\xi_1}$. It seems that certain points in $W^0_{\xi_2}$ act as kind of "attractor" for being in the boundary of leaves in $W^0_{\xi_1}$. We do not have a general conjecture to offer, but we would like to know an answer to:

(12.7) Question. *For every $\xi_1 \succ \xi_2$ and every $x \in C_x \subset W^0_{\xi_1}$ describe $C(x)^{\mathrm{Zar}} \cap W^0_{\xi_2}$.*

When is a central leaf = EO-stratum ? We have seen that for a minimal p-divisible group X, and $\varphi = (X[p]/\cong)$ the central leaf, in a universal deformation space, or in $\mathcal{A}_g \otimes \mathbb{F}_p$, coincide. We think that the converse should be true:

(12.8) Conjecture. *Let* $[(A,\mu)] = x \in \mathcal{A}_g \otimes \mathbb{F}_p$*, and let* $\varphi = ((A,\mu)/\cong)$*. We know that* $C_x \subset S_\varphi$*. We conjecture that:*

$$C_x = S_\varphi \quad \Longleftrightarrow \quad A[p^\infty] \quad \text{is minimal.}$$

The same conjecture can be stated for the unpolarized case.

Elementary sequences on a Newton polygon stratum. Choose a symmetric Newton polygon ξ. Which elementary sequences appear on W^0_ξ? We do not see a general rule for this. Write $\varphi_\xi = \mathrm{ES}(H(\xi)[p])$ for the elementary sequence determined by $H(\xi)[p]$.

(12.9) Conjecture.

$$W^0_\xi \cap S_\varphi \neq \emptyset \quad \Longrightarrow \quad \varphi_\xi \subset \varphi.$$

Here we use the notation $\varphi_1 \subset \varphi_2$ as in [52], 14.3.

(12.10) Conjecture. *Suppose* $x = [(A,\mu)]$ *is not minimal; write* $\varphi = ((A,\mu)[p]/\cong)$*. We conjecture that* $\dim(S_\varphi \cap I(x)) > 0$.
It seems that this conjecture, if true, would imply that (12.9) is true.

A question which is slightly different:

(12.11) Question. *Suppose given a symmetric Newton polygon* ξ*. Determine the set of elementary sequences* φ *such that the generic point of* S_φ *is contained in* W^0_{xi}*.*

In [17] S. Harashita starts an interesting research partially answering this question: in that paper we find, given φ the first slope of the Newton polygon whose open stratum contains the generic point of S_φ. Hopefully this method can be extended in such a way that it can answer the question just posed.

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