Lifting an automorphism of a curve to characteristic zero

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Question. *Given a connected curve C_0, proper and smooth over a field K of characteristic p, given a subgroup $H \subset \mathrm{Aut}(C_0)$; can we lift the pair (C_0, H) to characteristic zero*?
((We shall see that the answer is "NO" in general; for cyclic groups we conjecture that the answer is "YES".))

Example (1) (Roquette). Consider the normalization of the completion of the curve given by the affine equation $Y^2 = X^p - X$. For $p \geq 5$ its genus equals $(p-1)/2$ and over an algebraically closed field $\#(\mathrm{Aut}(C_0)) = 2p{\cdot}(p^2-1)$. For $p \geq 5$ we see that $g \geq 2$ and $\#(\mathrm{Aut}(C_0)) > 84(g-1)$. By the Hurwitz bound we conclude that (C_0, H) cannot be lifted to characteristic zero for $H = \mathrm{Aut}(C_0)$.

Remark. By this example we knew that the Hurwitz bound $\#(\mathrm{Aut}(C)) \leq 84(g-1)$ for a curve of genus $g \geq 2$ does not hold in characteristic p.
Stichtenoth proved: $\#(\mathrm{Aut}(C)) \leq 16{\cdot}g^4$. Another bound was given by B. Singh.

Example (2). Take the curve defined in the previous example with $p = 5$; in this case $g = 2$. Consider automorphisms of this curve defined by

$$\beta(X) = X + 1, \quad \beta(Y) = Y,$$

and

$$\gamma(X) = -X, \quad \gamma(Y) = 2Y.$$

The group $N := <\beta> \cong \mathbb{Z}/5$ is normal in $H := <\beta, \gamma>$, because $\beta^{-1}\alpha\beta = \alpha^4$, and $G/N \cong <\gamma> \cong \mathbb{Z}/4$. The pair (C_0, H) cannot be lifted to characteristic zero.

Example (3). Let $\mathbb{F}_p \subset K$ and $a, b \in K$ such that $a \not\in \mathbb{F}_p(b)$ and $b \not\in \mathbb{F}_p(a)$. Define $\sigma_a \in \mathrm{Aut}(\mathbb{P}^1_K)$ by $\sigma_a(X) = X + a$, where X is an affine coordinate on $\mathbb{P}^1_K$, and analogously $\sigma_b \in \mathrm{Aut}(\mathbb{P}^1_K)$. Then $H := <\sigma_a, \sigma_b> \cong \mathbb{Z}/p \times \mathbb{Z}/p$. For $p > 2$ we see:

$$\mathbb{Z}/p \times \mathbb{Z}/p \quad \not\subset \quad \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}).$$

Hence $(\mathbb{P}^1_K, H)$ cannot be lifted to characteristic zero.

Perhaps we fail because we do not consider hyperbolic *ordinary* curves? For abelian varieties the ordinary ones in characteristic p "behave like" abelian varieties in characteristic zero.

Definition. An abelian variety A of dimension g over $K \supset \mathbb{F}_p$ is called *ordinary* if $A[p](k) \cong (\mathbb{Z}/p)^g$. A curve is called *ordinary* if its Jacobian is ordinary.

Example (4). Here is an example of an *ordinary* curve and a subgroup of its automorphism group which cannot be lifted to characteristic zero. Let C_0 be the normalization of the completion of the affine curve given by the equation $(X^p - X)(Y^p - Y) = 1$; this is an ordinary curve with $g = (p-1)^2$ and $\#(\mathrm{Aut}(C_0)) = 2p^2(p-1)$. Let $H = \mathrm{Aut}(C_0)$. Note:

$$p \geq 41 \quad \Rightarrow \quad \#(\mathrm{Aut}(C_0)) > 84(g-1).$$

Hence for p large, the pair (C_0, H) cannot be lifted to characteristic zero.

What can be said about liftability of $(C_0, \mathrm{Aut}(C_0))$ to characteristic zero with C_0 as in the previous example, and p some small prime number ? Probably none of these curves with their full group of automorphisms can be lifted to characteristic zero.

Nakajima proved: for an ordinary curve of genus $g \geq 2$ we have $\#(\mathrm{Aut}(C)) \leq 84(g-1)g$.

Remark (5). Drinfeld modular curves are ordinary, and for high level the Hurwitz bound is violated; these give examples of ordinary curves which cannot be lifted with their automorphism group to characteristic zero.

Question (Katz). *Given an ordinary curve C_0; consider its Jacobian* $\mathrm{Jac}(C_0) = (J_0, \lambda_0)$, *which is an ordinary abelian variety. Let (J, λ) be the canonical lift in the sense of Serre and Tate. Is (J, λ) a Jacobian?*

The answer is "NO" in general, as was proved by Dwork - Ogus, by Oort - Sekiguchi, by De Jong - Moonen - Oort. By (4) and (5) we also see that the answer is "NO" in general.

Remark. It was proved by Garuti that for a curve C_0 and a subgroup $H \subset \mathrm{Aut}(C_0)$ there exists a singular model C_0' (cusps in the point of wild ramification) such that the pair (C_0', H) can be lifted to characteristic zero (actually the result is stronger).

Conjecture (FO, 1985). Suppose given a connected curve C_0, proper and smooth over a field K of characteristic p of genus $g > 1$; let $\beta \in \mathrm{Aut}(C_0)$; we expect:

$$\boxed{\textit{the pair } (C_0, G = <\beta>) \textit{ can be lifted to characteristic zero } (?)}$$

Particular cases which are proved:

Tame ramification. Grothendieck showed that a covering of curves which is tamely ramified (i.e. p does not divide any of the ramification indices) can be lifted to charactersitic zero.

Inertia of order p. By Sekiguchi - Oort - Suwa this conjecture is true if p^2 does not divide the order of β.

Inertia of order p^2. By Green - Matignon this conjecture is true is p^3 does not divide the order of β.

The global - local principle. For abelian coverings see Sekiguchi - Oort - Suwa, for arbitrary covers see Bertin - Mézard in order to conclude: *if for a Galois cover in characteristic p we can lift all germs (plus inertia-group) of the covering to characteristic zero, it follows that the covering can be lifted to characteristic zero.* Hence in order to prove the conjecture it suffices to show liftability of $\Gamma_0 \to \Gamma_0/(\mathbb{Z}/p^n)$ for every n, where Γ_0 is a germ of a smooth curve in characteristic p. At this moment this seems to be an open problem.

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