Monodromy, Hecke orbits and Newton polygon strata

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We study the moduli spaces $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of principally polarized abelian varieties in characteristic p, and Newton polygon strata in this moduli space. We focus on a proof of Theorem (1.5).

Much of this research originated in the seminal paper:
[18] Yu. I. Manin –
–*The theory of commutative formal groups over fields of finite characteristic.* (1963).
The fascinating structures, studied by Manin forty years ago, reveal more and more their beautiful features to us..... after tenacious research.

In this talk we mainly discuss results on moduli of *principally* polarized abelian varieties, leaving aside questions on other components of $\mathcal{A}_g \otimes \mathbb{F}_p$. In this note we freely use notions, notations and results of [12], [13], [17], [18], [22], [24], [29], [30], [31]. Various tools used have been described in:
F. Oort – *Stratifications and foliations of moduli spaces*, Seminar Yuri Manin, Bonn, 30 - VII - 2002;   see http://www.math.uu.nl/people/oort/

1 Newton polygon strata

(1.1) As said, we consider $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$. We consider symmetric Newton polygons. As usual, σ denotes the "supersingular polygon", i.e. all slopes of σ are equal to 1/2; see (3.1).

For a symmetric Newton polygon ξ we write

$$W_\xi = \mathcal{W}_\xi(\mathcal{A}) = \{(A, \lambda) \in \mathcal{A} \mid \mathcal{N}(A) \prec \xi\}.$$

By Grothendieck-Katz we know that $W_\xi \subset \mathcal{A}$ is a *Zariski-closed subset*; see [7], see [13], Th. 2.3.1 on page 143. These are called the *Newton polygon strata.*

(1.2) **Notation.** Let us write $\pi_0(W_\xi)$ for the set of irreducible components of $W_\xi \otimes \overline{\mathbb{F}_p}$.

(1.3) By [17] we know that the supersingular locus W_σ has "many components" (component in this note = geometrically irreducible component). In fact, see [17], 4.9:

$$\#(\pi_0(W_\sigma)) = H_g(p,1) \quad \text{if} \quad g \quad \text{is odd},$$

$$\#(\pi_0(W_\sigma)) = H_g(1,p) \quad \text{if} \quad g \quad \text{is even}.$$

See (3.10) for the connection between $\pi_0(W_\sigma)$ and these class numbers $H_g(p,1)$ and $H_g(1,p)$; for g fixed and $p \to \infty$ these numbers "go to infinity".

(1.4) **Question.** *What can be said about the number $\#(\pi_0(W_\xi))$ of irreducible components of W_ξ for other Newton polygon strata?*
This seems an endless task. However:

(1.5) **Theorem.** *For every $\xi \supsetneq \sigma$ the locus $W_\xi \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is* geometrically irreducible. I.e.

$$\xi \neq \sigma \quad \Rightarrow \quad \#(\pi_0(W_\xi)) = 1.$$

This was expected as Conjecture 8B in [28].

In this talk I sketch a proof for this theorem.

2 Monodromy and Hecke orbits (following Ching-Li Chai)

(2.1) In this section we fix a prime number p, we fix an integer $N \geq 3$ (to be used to define level structures), such that p does not divide N, and we choose a prime number ℓ not dividing pN. We fix an algebraically closed field $k \supset \mathbb{F}_p$. We write $\mathcal{B} = \mathcal{A}_{g,1,N} \otimes k$, the moduli space of principally polarized abelian varieties over an extension of k with a symplectic level-N-structure. In this note $\mathcal{A}$ and $\mathcal{B}$ wil be used.

(2.2) **Hecke-ℓ-orbits.** Consider $[(A,\lambda)] = x \in \mathcal{A}$; suppose that (A,λ) is defined over a field $K \supset \mathbb{F}_p$; or, consider $[(A,\lambda,\beta)] \in \mathcal{B}$. Consider all diagrams

$$(A,\lambda) \otimes \Omega \xleftarrow{\varphi} (C,\zeta) \xrightarrow{\psi} (B,\mu),$$

where:
Ω is an algebraically closed field containing K,
(C,ζ) is a polarized abelian variety over Ω,
$\varphi : C \to A$ is an isogeny such that $\varphi^*(\lambda) = \zeta$,
(B,μ) is a principally polarized abelian variety over Ω, and
$\psi : C \to B$ is an isogeny such that $\psi^*(\mu) = \zeta$;
we assume that φ and ψ have a degree which is a power of ℓ;
moreover we assume that A, B and C have symplectic level-N-structures
compatible with φ and ψ.
For Hecke correspondences also see [5], VII.3.

(2.3) Notation.
$$\mathcal{H}_\ell \cdot x = \{[(B,\mu)] = y \in \mathcal{A}\},$$
where all moduli points $[(B,\mu)] = y$ are considered which can be constructed from $[(A,\lambda)] = x$ via a diagram as above, with the conditions mentioned. Analogously $\mathcal{H}_\ell \cdot [(A,\lambda,\beta)] \subset \mathcal{B}$.

This set $\mathcal{H}_\ell \cdot x \subset \mathcal{A}$ is called the Hecke-ℓ-orbit of the point x.

A subset $S \subset \mathcal{A}$ is called *Hecke-ℓ-stable* if for every $x \in S$ we have $\mathcal{H}_\ell \cdot x \subset S$.

(2.4) Let $Z \subset \mathcal{A}$ be a locally closed subset. Let Z^0 be an irreducible component of Z. Let $\eta \in Z^0$ be the generic point. Let $A \to Z$ be the universal abelian scheme restricted to Z. Let
$$\rho_{A,\ell} : \pi_1(Z^0, \overline{\eta}) \quad \longrightarrow \quad \mathrm{Sp}(T_\ell, <,>_\ell)$$
be the ℓ-adic monodromy representation in the Tate-ℓ-group of A_η. Identify the Tate-ℓ-group of A_η over $\overline{\eta}$ with $\mathbb{Z}_\ell^{2g}$ with the standard pairing.

(2.5) Theorem (C.-L. Chai). *Choose notation as above. Let $Z \subset \mathcal{B}$ be a locally closed subscheme, smooth over* $\mathrm{Spec}(k)$*, such that:*

Z is Hecke-ℓ-stable, and

the Hecke-ℓ-action on the set $\pi_0(Z)$ is transitive, and

$\eta \notin W_\sigma$ (equivalently: Z contains a non-supersingular point).

Then:
$$\rho_{A,\ell} : \pi_1(Z^0, \overline{\eta}) \longrightarrow \mathrm{Sp}(T_\ell, <,>_\ell) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$$
is surjective, *and*
$$Z \textit{ is irreducible, i.e. } Z = Z^0.$$
See [2], 4.4.

3 A proof of (1.5)

We will see that (1.5) follows from (2.5) if we use all kind of facts known about strata and Hecke orbits. We use the word "component" instead of the expression "geometrically irreducible component".

A remark on the basic idea of the proof below: abelian varieties are considered; in families we "go to the boundary"; by this we do not mean that the abelian varieties degenerate, but that we "degenerate the p-structure", where the superspecial points are considered as the most special points.

(3.1) Tool 1: Supersingular abelian varieties. Some references: [38], [22], [36], [25], [26], [17].

Proposition / Definition. *Let K be a field, let $k = \overline{k}$ be an algebraically closed field containing K, and let E be an elliptic curve over K. The following are equivalent:*

(a) the ring $\mathrm{End}(E \otimes k)$ *has rank 4 over* $\mathbb{Z}$*;*

(b) the characteristic of K equals $p > 0$, and $\#(E(k)[p]) = \{0\}$;

(c) **(Definition)** *the elliptic curve is called* **supersingular**.

Proposition. *A supersingular elliptic curve E over a field K has the property $j(E) \in \mathbb{F}_{p^2}$; hence there exists an elliptic curve E' over $\mathbb{F}_{p^2}$ and an isomorphism $E' \otimes k \cong E \otimes k$.*
Indeed, $\mathrm{Ker}(F^2 : E \to E^{(p^2)}) = E[p]$. $\square$

As corollary we have:

$$\#(\{j(E) \mid E \quad \text{is supersingular}\}) < \infty.$$

Remarks. The exact number h_p of supersingular j-values has been computed (Hasse, Deuring, Igusa); this we will not need here.

Any two supersingular elliptic curves over k are isogenous. In fact, for any prime number ℓ different from p there is an ℓ-power isogeny between them.

For every prime number p there exists a supersingular elliptic curve defined over $\mathbb{F}_p$.

Definition. *An abelian variety A is called* **supersingular** *if there exsist a supersingular elliptic curve E and an isogeny $A \otimes k \sim E^g \otimes k$.*

We write $\sigma = \sigma_g$ for the Newton polygon where all slopes are equal to $\frac{1}{2}$; this will be called the *supersingular Newton polygon.*

Property. *An abelian variety A is supersingular iff its Newton polygon $\mathcal{N}(A) = \sigma$.*

Let G be a group scheme in characteristic $p > 0$. We write $a = a(G)$ for the integer such that $G[F] \cap G[V]$ is of rank p^a. In case K is perfect, we have

$$a(G) = \dim_K \mathrm{Hom}(\alpha_p, G).$$

P. Deligne proved, see [36], 1.6: for $g > 1$ and supersingular elliptic curves $E_1, \cdots, E_{2g}$ over k there exists an isomorphism

$$E_1 \times \cdots \times E_g \cong E_{g+1} \times \cdots \times E_{2g}.$$

Proposition / Definition. *Let K be a field, let $k = \overline{k}$ be an algebraically closed field containing K, and let A be a supersingular abelian variety over K of dimension $g \geq 2$. The following are equivalent:*

(a) $a(A) = \dim(A)$;

(b) the characteristic of K is positive, and for every supersingular elliptic curve E there exists an isomorphism $A \otimes k \cong E^g \otimes k$;

(c) **(Definition)** *the elliptic variety A is called* **superspecial**.

From now on we work over an algebraically closed field k of characteristic p.

(3.2) Step 1.
Claim. *For every symmetric Newton polygon ξ and every irreducible component W of W_ξ we have $W \cap W_\sigma \neq \emptyset$, i.e. W contains a supersingular point.*

(3.3) Tool 2: EO strata. For details, see [29]. We make use of the fact that finite, commutative group schemes of given rank annihilated by p are finite in number (up to isomorphism over an algebraically closed field); this was proved by Kraft and by Oort; indeed we need more, namely also the polarization has to be taken into account, see [29], Section 9. On $\mathcal{A}$ we can consider for a given geometric isomorphism class φ of $(A,\lambda)[p]$ the set S_φ thus defined. In this way we obtain the Ekedahl-Oort stratification of $\mathcal{A}$. Some of the basic facts:

the isomorphism types are characterized by "elementary sequences"; for every value of g these form a partially ordered, finite set (in fact there are two orderings, see [29], 14.3);

for every φ the locus $S_\varphi \subset \mathcal{A}$ is locally closed and quasi-affine;

the superspecial $S_{E^g[p]}$ has dimension zero, and all other strata have positive dimension;

consider $\mathcal{A}^*$, the "minimal compactification" (or, the Satake compactification); for every φ there is a naturally defined $T_\varphi \subset \mathcal{A}$, see [29], 6.1; *for every φ such that the dimension of S_φ is positive (i.e. φ not superspecial), the Zariski closure of S_φ contains a point in $\mathcal{A}$ not in S_φ.*

(3.4) Tool 3: Finite Hecke orbits (Chai).
Proposition. *A Hecke-ℓ orbit $\mathcal{H}_\ell{\cdot}x$ is finite iff A_x is supersingular.*
See [1], Prop. 1 on page 448.

Proof of Step one (3.2). Consider

$$\Gamma := \cup_{W\subset W_\xi, W\cap W_\sigma=\emptyset} \quad W,$$

the union over all irreducible components of W_ξ which contain no supersingular point (and we show that Γ is empty). From the fact that any component H' of a Hecke-ℓ^i-correspondence $\mathcal{A}_g \leftarrow H' \rightarrow \mathcal{A}_g$ is finite-to-finite outside characteristic ℓ we see that Γ as defined above is Hecke-ℓ stable.

Consider all EO-strata meeting Γ; let φ be an elementary sequence appearing on Γ which is minimal for the "$\subset$" ordering. Let $x \in \Gamma\cap S_\varphi$, hence $\mathrm{ES}(A_x) = \varphi$. Note that $x \notin W_\sigma$. Hence $\mathcal{H}_\ell{\cdot}x$ is not finite by (3.4). Note that $\mathcal{H}_\ell{\cdot}x \subset \Gamma \cap S_\varphi$. Hence $S_\varphi \cap \Gamma$ has positive dimension. By (3.3) this implies that there is point y in the closure of $\Gamma \cap S_\varphi \subset \mathcal{A}$ which is not in S_φ. This is a contradiction with minimality of φ. $\square$(3.2)

(3.5) Step 2.
Claim. *For every symmetric Newton polygon ξ and every component, $W \subset W_\xi$ there is a component $T \subset W_\sigma$ such that $T \subset W$.*

For every $\zeta \prec \xi$, ordering by inclusion of components gives a well-defined map $\pi_0(W_\zeta) \to \pi_0(W_\xi)$ which moreover is surjective: every component of W_ζ is contained in a unique component of W_ξ and every component of W_ξ contains at least one of W_ζ.

(3.6) Tool 4: Purity. See [12], 4.1: *In a family, if the Newton polygon jumps, it already jumps in codimension one.*

(3.7) Tool 5: Deformations with constant Newton polygon. *For any* principally *polarized abelian variety (A,λ) there exists a deformation with generic fiber (A',λ') with $\mathcal{N}(A) = \mathcal{N}(A')$ and $a(A') \leq 1$.*
See [12], (5.12), and [31], (3.11) and (4.1).

(3.8) Tool 6: Cayley-Hamilton. See [30], 3.5 and use (3.7). *In particular this shows that around any point* $x \in \mathcal{A}$*, with* $\mathcal{N}(A_x) = \zeta$ *and* $a(A_x) = 1$ *the Newton strata* $\{W_\xi^{/x} \mid \xi \succ \zeta\}$ *are formally smooth and nested like the graph of all Newton polygons below* ζ*. The dimension of* $W_\xi \subset \mathcal{A}$ *equals* $\mathrm{sdim}(\xi)$*, and the generic point of any component of* W_ξ *has a-number at most one.*

Remark. Using (3.7) and (3.8) we can prove a conjecture by Grothendieck, see [30] and [31].

For the definition of $\mathrm{sdim}(\xi)$ see [30], 3.3, or [31], 1.9. Note that different components of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ may have different dimensions: our formula using $\mathrm{sdim}(\xi)$ works for *principally polarized* abelian varieties.

Proof of Step two (3.5). We have seen by (3.2) that W contains a supersingular point. By Purity, (3.6), we know that Newton polygons jump in codimension one; by (3.8) we know what the dimensions are of all strata defined by symmetric Newton polygons; in particular the difference of the codimensions of W_σ and of W_ξ in $\mathcal{A}$ is precisely the length of the longest chain of symmetric Newton polygons between ξ and σ. Combining these two we conclude: W *contains a component of* W_σ: by purity a component of $W_\xi \cap W_\sigma$ has at least codimension $\mathrm{sdim}(\xi) - \mathrm{sdim}(\sigma)$ in W_ξ, hence dimension at least $\mathrm{sdim}(\sigma)$ and, being contained in W_σ which is pure of dimension $\mathrm{sdim}(\sigma)$ the statement follows.

For an inclusion $T \subset W$ we choose a point $x \in T$ with $a(A_x) = 1$, which exists, by (3.8), or by [17], (4.9.iii). Around this point $x \in T \subset W$ we apply (3.8). This ends the roof of Step 2.

$\square$(3.5)

Notation. For $g \in \mathbb{Z}_{>1}$ and $j \in \mathbb{Z}_{\geq 0}$ we write $\Lambda_{g,j}$ for the set of isomorphism classes of polarizations μ on the superspecial abelian variety $A = E^g \otimes k$ such that $\mathrm{Ker}(\mu) = A[F^j]$; here E is a supersingular elliptic curve defined over $\mathbb{F}_p$. Note that $\Lambda_{g,j} \xrightarrow{\sim} \Lambda_{g,j+2}$ under $\mu \mapsto F^t{\cdot}\mu{\cdot}F$.

(3.9) Tool 7: Characterization of components of W_σ**.** *There is a canonical bijective map*

$$\pi_0(W_\sigma) \quad \xrightarrow{\sim} \quad \Lambda_{g,g-1}.$$

See [17], 3.6 and 4.2; this uses [22], 2.2 and 3.1.

(3.10) Tool 8: Transitivity. *The action of* $\mathcal{H}_\ell$ *on* $\pi_0(W_\sigma)$ *is transitive.*
By (3.9) the problem is translated into a question of transitivity of the set of isomorphism classes of certain polarizations on a superspecial abelian variety. Use [4], pp. 158/159 to describe the set of isomorphism classes of such polarizations. Use the strong approximation theorem, see [35], Theorem 7.12 on page 427.

(3.11) Tool 9 = (2.5) (Chai).

The end of the proof of (1.5). We show that $\mathcal{W}(\mathcal{A}_{g,1,N} \otimes \mathbb{F}_p)$ is geometrically irreducible for $\xi \neq \sigma$; from this the conclusion of (1.5) clearly follows. Write $\mathcal{B} = \mathcal{A}_{g,1,N} \otimes \mathbb{F}_p$. Indeed, by Step 2 we know that $\pi_0(W_\sigma) \to \pi_0(W_\xi)$ is surjective, and the same we conclude for

$\pi_0(\mathcal{W}_\sigma(\mathcal{B})) \to \pi_0(\mathcal{W}_\xi(\mathcal{B}))$. By (3.9) and (3.10) we conclude that the action of Hecke-ℓ on $\pi_0(\mathcal{W}_\xi(\mathcal{B}))$ is transitive. Hence by (2.5) we conclude that $\mathcal{W}_\xi(\mathcal{B})$ is geometrically irreducible for $\xi \neq \sigma$. This ends the proof of (1.5). $\square$

Not all references below are needed for this talk, but I include relevant literature for completeness sake.

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