CM liftings of abelian varieties

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In this talk we study CM liftings of abelian varieties from a field in characteristic p (usually a finite field) to an integral domain in characteristic zero.

About 20 years ago Professor Borovoi asked me whether a CM lifting is possible for every abelian variety defined over a finite field.

At first I had some results, published in 1992.
The answer is "NO": in general we need an isogeny.
After that progress was slow. But now joint work Ching-Li Chai – Brian Conrad – FO completely answers this question. Especially new ideas by Brian Conrad and by Ching-Li Chai were important for this progress.

1 Introduction, definitions.

(1.1) smCM. For an abelian variety A over a field K of dimension g we say that A *admits sufficiently many complex multiplications*, smCM, if $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a commutative semi-simple algebra of rank $2g$ over $\mathbb{Q}$. Sometimes abbreviated by saying "A is a CM abelian variety".

Remarks.

- This is the maximal dimension such an algebra can have.
- Albert described the possible structures the endomorphism algebra of an abelian variety (over some field) can have. Albert, Shimura and Gerritzen proved that any "Albert algebra" appears in every characteristic as the endomorphism algebra of a simple abelian variety over an algebraically closed field.
- For a simple abelian variety A over a field of *characteristic zero* which admits smCM, $\mathrm{End}^0(A)$ is a field, in fact a CM field.
- However there are many abelian varieties, simple over $\mathbb{C}$, for which the endomorphism algebra is not commutative.

- There are many examples of a simple abelian variety A over a field, of characteristic p, such that A admits smCM and such that $\mathrm{End}^0(A)$ is a not a field.

An abelian variety A of dimension g over a field of characteristic zero is said to be of CM type if it admits smCM and if moreover a CM algebra $P \subset \mathrm{End}^0(A)$ of degree $2g$ over $\mathbb{Q}$ is given; this action of P can be given by a representation of P on the tangent space of A. We do not use the terminology "of CM type" for an abelian variety in positive characteristic.

(1.2) Over a finite field (Tate). Tate described the structure the endomorphism algebra of an abelian variety *over a finite field* can have. In particular: *every abelian variety over a finite field admits* smCM. See [18].

(1.3) An abelian variety over a field of characteristic zero with smCM can be defined over a number field. More generally:
Grothendieck proved that any abelian variety with smCM up to isogeny can be defined over a finite extension of the prime field. See [11], [25]
Caution. An abelian variety in characteristic p which admits smCM need not be defined over a finite field.

(1.4) We know that an abelian variety A over a field K is isogenous with a product of abelian varieties simple over K. We say that A is *isotypic* if there exists an abelian variety B simple over K and $\mu \in \mathbb{Z}_{>0}$ such that $A \sim_K B^\mu$.

Remark. If A is an abelian variety *over a finite field* κ and A is isotypic, and $\kappa \subset \kappa'$ is a field extension, then $A \otimes_\kappa \kappa'$ is isotypic.

(1.5) Definition (CML). Given an isotypic abelian variety B_0 of dimension g over a field $\kappa \supset \mathbb{F}_p$ we say that B_0 satisfies (CML), and we say that B is a CM *lifting* of B_0, if there exists a local domain R with characteristic 0 and residue field κ, an abelian scheme $\mathcal{B}$ over R equipped with an action $L \subset \mathrm{End}^0(\mathcal{B})$ by a CM field L with $[L:\mathbb{Q}] = 2g$, and an isomorphism $\mathcal{B} \otimes_R \kappa \cong B_0$ as abelian varieties over κ.

Caution. There are many cases where $L = \mathrm{End}^0(\mathcal{B})$ but $L \subsetneq \mathrm{End}^0(B_0)$.

(1.6) Remark. If B_0 is an abelian variety defined over a field K such that it cannot be defined over any finite subfield of K, then B_0 does not admit a CM lifting to characteristic zero (because every abelian variety of CM type in characteristic zero is defined over a number field). This gives many examples of an abelian variety in positive characteristic, having smCM, but not CM liftable to characteristic zero. In asking questions of a CM lifting in the sequel we will only consider abelian varieties defined over a finite field.

(1.7) CM lifting up to isogeny, up to extending the base field.
Theorem (Honda, 1968). *Given an abelian variety A over a finite field $\kappa = \mathbb{F}_q$, there exists a finite extension $\kappa \subset \kappa'$ and an isogeny $A \otimes_\kappa \kappa' \sim B_0$ such that B_0 can be lifted to an abelian variety B in characteristic zero with* smCM.

Caution: in general $\mathrm{End}^0(A)$, and $\mathrm{End}^0(A \otimes_\kappa \kappa') = \mathrm{End}^0(B_0)$, and $\mathrm{End}^0(B)$ can be different.

We could say: Every abelian variety satisfies (RIN), where
"R" stands for "up to extending the residue class field",
"I" stands for "up to isogeny", and
"N" stands for "lifting to a normal domain".

(1.8) Questions.
Is an isogeny necessary?
Is a field extension necessary?

(1.9) Theorem / Problem. The theorem of Honda just quoted is part of the "Honda-Tate theory". In that theory it is proven that a Weil q-number appears as the Weil number of an abelian variety over $\mathbb{F}_q$ (an eigenvalue of the q-Frobenius morphism):
Theorem (Honda, Tate)

$$\{\text{simple} \quad \mathrm{AV}/\mathbb{F}_q\}/\sim_{\mathbb{F}_q} \quad \xrightarrow{\sim} \quad \{\text{Weil} \quad q-\#\}/\sim .$$

All known proofs of that fact use CM-theory in characteristic zero. Se [19], [] **Question.** *Does there exist a proof of Theorem (1.7), in fact of Honda-Tate theory, not using methods of characteristic zero?*

2 An isogeny is necessary

(2.1) Theorem (FO, 1992). $\forall g \geq 3$, $\forall f \leq g-2$ *there exsits an abelian variety A over $\mathbb{F} := \overline{\mathbb{F}_p}$ of dimension g of p-rank equal to f such that A does not admit a* CM *lifting to characteristic zero.* See [13].

"'An isogeny is necessary, in general". In particular, in general an abelian variety over a finite field does not admit (CML).

(2.2) Remark. An example of B_0 as in (1.6) can be given by taking an abelian variety C over a finite field sucht that $\alpha_p \times \alpha_p \hookrightarrow C$, and taking a "generic quotient" $C/\iota(\alpha_p)$. The proof in [13] follows this line of thought, not taking "generic quotients", but choosing C carefully, taking quotients defined over $\mathbb{F}$ and showing that many of these do not admit a CM lift.

3 CM lifting to a normal domain

(3.1) Definition (IN). We say an abelian variety A over a finite field κ satisfies (IN) if there exists an *isogeny* $A \sim B_0$ such that B_0 can be CM lifted to a *normal* domain in characteristic zero.

(3.2) Theorem (Ching-Li Chai – Brian Conrad – FO). *There exist examples of an abelian variety over a finite field which do not sartisfy* (IN).
"For CM lifting to a *normal domain* up to isogeny, a field extension is necessary in general".

By Honda-Tate theory we can construct abelian varieties over finite fields having required p-adic properties. The key to the proof of the previous theorem is to construct an abelian variety which violates the "*residual reflex condition*".

(3.3) Example. Choose a prime number p with $p \equiv 2 \pmod 5$ or $p \equiv 3 \pmod 5$; equivalently: p is totally inert in the extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_5)$. Let $\pi := p{\cdot}\zeta_5$. This is a Weil p^2-number. Hence by Honda - Tate theory there exsits an abelian variety A, simple and defined over $\kappa = \mathbb{F}_{p^2}$ (and unique up to κ-isogeny) such that the p^2-Frobenius

$$\pi_A = (\mathrm{Frob}_{A^{(p)}}){\cdot}(\mathrm{Frob}_A) \in \mathrm{End}(A)$$

of A/κ is an algebraic integer conjugated to π.
Claim. *A does not satisfy* (NI).
Proof. One shows that $\dim(A) = 2$, and $\mathrm{End}^0(A) \cong \mathbb{Q}(\zeta_5)$. Suppose some abelian variety B_0 isogenous to A over κ could be CM lifted to an abelian variety B over a normal domain R of characteristic zero, with field of fractions M. Then $\mathrm{End}^0(B_M) \cong \mathbb{Q}(\zeta_5)$. We know that the field M contains the reflex field L of the CM type of B_M. We know that any reflex field of L is a CM field, contained in the Galois extension $\mathbb{Q}(\zeta_5) \supset \mathbb{Q}$. Hence, whatever the CM type is, we see that $L = \mathbb{Q}(\zeta_5)$. Hence $L = \mathbb{Q}(\zeta_5) \subset M$. The residue class field of any prime in M above p contains the residue class field of $\mathbb{Q}(\zeta_5)$ at p. *As p is inert in $\mathbb{Q}(\zeta_5)$, this residue class field is isomorphic with $\mathbb{F}_{p^4}$ on the one hand; on the other hand we know that the residue class field of the normal domain R is $\kappa \cong \mathbb{F}_{p^2}$. This contradiction shows that A does not satisfy* (IN). □

(3.4) Remark. The previous example is a supersingular abelian variety. However we also do have examples of an abelian variety A over a finite field, such that A does not admit a CM lift to a normal domain of characteristic zero, and such that the Newton polygon has exactly two slopes (hence no slopes equal to 1/2). Conclusion: there exist abelian varieties whose NP has no slopes equal to 1/2 which do not satisfy (IN).

4 The residual reflex condition is sufficient

(4.1) Let L be a CM field, and let p be a prime number. Complex conjugation induces an involution ι on L. Let C be an algebraic closure of $\mathbb{Q}_p$. A subset $\Phi_p \subset \mathrm{Hom}(L, C)$ is called a p-adic CM type if $\Phi_p \coprod \Phi_p{\cdot}\iota = \mathrm{Hom}(L, C)$.

(4.2) Let A be an abelian variety of CM type $\Phi \subset \mathrm{Hom}(L, \mathbb{C})$ over a field M in characteristic zero. Suppose that A has good reduction at a p-adic place ρ of M. Let $\mathcal{A}$ be the Néron model of A over the ring of integers of M_ρ. Write $\Phi_p \subset \mathrm{Hom}(L, \overline{\mathbb{Q}_p})$ for p-adic CM type determined by $\mathcal{A}$ and Φ. Let A_0 be the reduction modulo ρ of $\mathcal{A}$. Suppose A_0 is isotypic. Let $\pi = \pi_{A_0}$ be the Weil number determined by A_0. The Newton polygon can be read off from the p-adic values of π; the Shimura-Taniyama formula gives: for every p-adic valuation w of L we have

$$\frac{\mathrm{ord}_w(\pi)}{\mathrm{ord}_w(q)} = \frac{\#\{\phi \in \Phi_p \mid \phi \text{ induces } w \text{ on } L\}}{[L_w : \mathbb{Q}_p]}.$$

(4.3) Definition. Suppose given an abelian variety B_0 of dimension g over a finite field K. Suppose given a CM field $L \subset \mathrm{End}^0(B_0)$ with $[L:\mathbb{Q}] = 2g$. Suppose given a p-adic CM-type Φ_p for L. Write $\mathcal{R} = \mathcal{R}(L, \overline{\mathbb{Q}_p})$ for the reflex field. We say that (B_0, L, Φ_p) satisfies the *residual reflex condition* if:

- **(1)** The slopes of B_0 are given by the Shimura-Taniyama formula applied to (L, Φ_p).
- **(2)** The reflex field $\mathcal{R} \subset \overline{\mathbb{Q}_p}$ has a valuation ρ with residue class field $\kappa_\rho \subset \kappa$.

(4.4) Theorem (Ching-Li Chai – Brian Conrad – FO). *Let $\kappa = \mathbb{F}_q$. Consider (B_0, L, Φ_p), where (B_0, L) is a* CM *abelian variety over κ and Φ_p is a p-adic* CM *type for L. The triple (B_0, L, Φ_p) satisfies* (IN) *if and only if it satisfies the residual reflex condition.*

5 CM lifting up to isogeny without extending the base field.

Even if an abelian variety in characteristic p does not satisfy the residual reflex condition, such as in (3.3), this still leaves open the possibility that A over κ satisfies the following condition.

(5.1) Definition (I). We say an abelian variety A over a finite field κ satisfies (I) if there exists an isogeny $A \sim B_0$ over κ such that B_0 can be CM lifted to an *integral* domain in characteristic zero.

(5.2) Theorem (Ching-Li Chai – Brian Conrad – FO). *Any abelian variety A defined over a finite field κ satisfies* (I).
"A field extension is not necessary".

The theorem says: *there is an isogeny $A \sim_\kappa B_0$, and a* CM *abelian scheme $\mathcal{B}$ over a domain R in characteristic zero with $R \twoheadrightarrow \kappa$ such that $B \otimes_R \kappa \cong B_0$.* Note that we ask the residue class field of R to be κ, but we do not require R to be a normal domain.

We first show how this can be proven in the example constructed above.
Then we sketch briefly a proof in the general case.

(5.3) The Serre tensor construction. Let A be an abelian variety over a field K. Let Γ be a commutative ring with $1 \in \Gamma$, and $\Gamma \to \mathrm{End}(A)$; let M be a module of finite type over Γ. The Serre tensor construction produces an abelian variety $A \otimes_\Gamma M$ over K. For example let $D' \subset D$ be a commutative subalgebra of $D := \mathrm{End}^0(A)$; write $\Gamma = (\mathrm{End}(A) \cap D')$ contained in the ring $\mathcal{O} := \mathcal{O}_{D'}$ of elements in D' which are integral over $\mathbb{Z}$. Then there exists an abelian variety B, which will be denoted by the symbolic notation $B = A \otimes_\Gamma \mathcal{O}$, and an isogeny $A \sim_K B$ such that $\mathcal{O} \subset \mathrm{End}(B)$.

(5.4) Remark. In case A is an abelian scheme and N is a module *projective and of finite type* over $R \subset \mathrm{End}(A)$, the Serre tensor construction produces $A \otimes_R N$. For the general situation of an abelian scheme the condition "projective over R" is necessary in general. However for an abelian variety over a field just "of finite type" suffices.

(5.5) We use the defintion and properties of the "a-number": we write $a(G) = \dim_\kappa(\mathrm{Hom}(\alpha_p, G))$ for a group scheme G over a perferct field κ.

(5.6) We study Example (3.3), where $\pi = p{\cdot}\zeta_5$. Here $L = \mathbb{Q}(\pi) = \mathbb{Q}(\zeta_5)$ and A is a simple supersingular abelian variety over $\kappa = \mathbb{F}_{p^2}$ with $\pi_A \sim \pi$. *We show that this abelian variety A over $\kappa = \mathbb{F}_{p^2}$ satisfies* (I).

Step 1. If necessary, using the Serre tensor construction, we change A up to κ-isogeny into an abelian variety B_0 over $\kappa = \mathbb{F}_{p^2}$ to an abelian variety with $\mathcal{O}_L \subset \mathrm{End}(B_0)$. We are going to show that B_0 satifies (CML).

Claim. We have $a(B_0) = 2$. □

Step 2. Write $B'_0 = B_0 \otimes_\kappa \mathbb{F}$.
Claim. *There is an abelian variety C'_0, an $\mathcal{O}_L$-isogeny $C'_0 \to C'_0/\alpha_p \cong B'_0$, such that the Lie type of $(C'_0, \mathcal{O}_L)$ is self-dual* (see [5] for definitions and details). *In this case $a(C'_0) = 1$.* □
On notation: Instead of $(C'_0, \mathcal{O}_L)$ we should write something like $(C'_0, \gamma_0 : \mathcal{O}_L \to \mathrm{End}(C'_0))$; however we wil use shorter notation here.

We study $X_0 := C'_0[p^\infty]$, a p-divisible group over $\mathbb{F}$, with

$$\mathcal{O}_L \hookrightarrow \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathcal{O}_E \hookrightarrow \mathrm{End}(X_0); \quad E := L \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

Step 3. Theorem. *Suppose X_0 is a p-divisible group over $\mathbb{F}$, with an action $\mathcal{O}_E \subset \mathrm{End}(X_0)$ where E is an algebra of degree over $\mathbb{Q}_p$ equal to* $\mathrm{ht}(X_0)$. *Suppose that $(X_0, \mathcal{O}_E)$ has self-dual Lie type. Then there exists a* CM *type Φ_p for E and a lifting $(X, \mathcal{O}_E)$ over some local algebra R' finite over $W_\infty(\mathbb{F})$ such that the generic fiber of $(X, \mathcal{O}_E)$ is of* CM *type Φ_p.*
There are several ways of proving this. One can use Breuil-Kisin theory. One can also use results on CM liftings by Yu.

Step 4. Applying the previous step to $X_0 := C'_0[p^\infty]$ and applying the Serre-Tate theorem we achieve a formal CM lifting to a formal abelian scheme $(\mathcal{C}', \mathcal{O}_L)$ over R' lifting $(C'_0, \mathcal{O}_L)$.

Step 5. The formal abelian scheme $(\mathcal{C}', \mathcal{O}_L)$ over the p-adic ring R' is generically of CM type. One shows that this implies the formal abelian scheme is algebraizable, obtaining $(C', \mathcal{O}_L)$, a CM lifting of $(C'_0, \mathcal{O}_L)$.

Step 6. Enlarging, if necessary, the ring R' we can choose a point P of exact order p on the generic fiber $C'_M(M)$. Take the flat extension $\mathcal{N} \subset C'$ of the group scheme generated by P and define $\mathcal{B}' = \mathcal{C}'/\mathcal{N}$. The special fiber $N_0 \subset C'_0$ is a group scheme of rank p. Because $a(C'_0) = 1$ we see that $C'/\mathcal{N} \otimes_{R'} \mathbb{F} \cong B'_0$. Moreover $\mathcal{O}' := \mathbb{Z} + p{\cdot}\mathcal{O}_L \subset \mathrm{End}(\mathcal{B}')$. We see that $(\mathcal{B}', \mathcal{O}')$ is a CM lifting of $(B'_0, \mathcal{O}')$.

Step 7. Studying the local deformation functor of $(B_0, \mathcal{O}')$ and knowing that $(B'_0 := B_0 \otimes \mathbb{F}, \mathcal{O}')$ admits a CM lifting, we conclude that $(B_0, \mathcal{O}')$ admits a CM lifting to an integral domain of mixed characteristic $R \to \kappa$.

This finishes a proof that Example (3.3) satisfies (I).

(5.7) A proof of Theorem (5.2) follows very much this pattern, although there are some steps which are much more complicated in the general case. In Step 1 one has to choose $(B_0, \mathcal{O}_L)$ "as close as possible to a self-dual Lie type"; this can be done above "good places" of L by changing to a self-dual type, but at a "difficult place" of L only a "striped" Lie type can be achieved. See [5] for details. A choice of an $\mathcal{O}_L$-isogeny $C_0' \to C_0'/N_0 \cong B_0'$ as in Step 2 is involved. Steps 3-4-5 are pretty much the same as above. A choice of $\mathcal{N} \subset C'$ follows after a difficult computation (we use Raynaud's paper [16]). Once we have arrived at this point Steps 6-7 are as above. Please see [5] for details; this manuscript will find its place in [2].

(5.8) **Remark.** Suppose that $\mathcal{N}(A)$, the Newton polygon of A, has no slopes equal to 1/2. Then we can choose a CM lift $\mathcal{B}$ of some $B_0 \sim_\kappa A$ with $\mathcal{O}_L \hookrightarrow \mathrm{End}(\mathcal{B})$.

(5.9) **Some comments.** Questions above can be refined *by fixing the CM field which we want to be the CM field operating on the lifted abelian variety.*

Or, even stronger one can refine the questions *by taking the maximal order in a CM field and request that this order operates on the lifted abelian variety.* There are examples where condition (I) is not satisfied in this restricted situation.

6 Survey

Survey of that various definitions a about CM lifts.

(CML) Does an abelian variety defined over a finite field admit a CM lift?
The answer is: in general not. See Section 2.

(RIN) Does an abelian variety defined over a finite field admit a CM lift to a normal domain after extending the field and after applying an isogeny ?
The answer is: yes. This is the theorem by Honda. See Th. (1.7).

(R) Does an abelian variety defined over a finite field admit a CM lift after extending the base field?
The answer is: in general not. An isogeny is necessary in general. See Section 2.

(IN) Does an abelian variety defined over a finite field admit a CM lift to a normal domain after applying an isogeny ?
The answer is: in general not. We have given examples above. See Section 3.

(I) Does an abelian variety defined over a finite field admit a CM lift after applying an isogeny?
The answer is: yes. This is Theorem (5.2) above.

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