CM Jacobians

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0. Introduction. In this talk we discuss properties of the Torelli locus inside the moduli space of polarized abelian varieties over $\mathbb{C}$. We would like to compare "canonical coordinates" on $\mathcal{A}_g \otimes \mathbb{C}$ on the one hand and see how they relate to $T_g \subset \mathcal{A}_g$ on the other hand. In order to make this more precise we study a special case:

consider algebraic curves C over $\mathbb{C}$
such that its Jacobian variety $J(C) = \underline{\mathrm{Pic}}^0_C$ is a CM *abelian variety.*

We will see this is a fascinating problem in itself, but it also pins down essential features of the problem. For further explanation see below.

For results, a discussion of these ideas, and many references see:
[MO] = [34] B. Moonen & F. Oort, *The Torelli locus and special subvarieties.*
[CO] = [7] C.-L. Chai & F. Oort, *Abelian varieties isogenous to a Jacobian.*

1. Complex multiplication. We say an abelian variety A of dimension g over a field K admits *sufficiently many Complex Multiplications* (smCM) if $D = \mathrm{End}^0(A)$ contains a commutative semi-simple algebra $\Lambda \subset D$ of rank $\dim_{\mathbb{Q}}(\Lambda) = [\Lambda : \mathbb{Q}] = 2g$. In this case we say A is a CM abelian variety. A point $z \in \mathcal{A}_g(k)$ is called a CM point if z is the moduli point of a polarized abelian variety $z = [(A, \mu)]$ such that A is a CM abelian variety.

If an algebraic curve C has a Jacobian $J(C)$ which is a CM abelian variety we say C is a CM curve.

A CM field L is a finite extension of $\mathbb{Q}$ (a number field) that contains a subfield $L_0 \subset L$ such that L_0 is totally real, $[L : L_0] = 2$ and L/L_0 is totally complex.

If A is a simple abelian variety, and A is an CM abelian variety, and $\mathrm{char}(K) = 0$ then $\mathrm{End}^0(A) = D = L$ is a CM field. (This is no longer true in positive characteristic in general.)

2. A conjecture by Coleman.
Conjecture (1987, see [9], Conjecture 6). *For a given $g \geq 4$ the number of isomorphism classes of algebraic curves C of genus g over $\mathbb{C}$ such that $J(C)$ is a CM abelian variety is finite.*

In this conjecture by "an algebraic curve" we mean: an algebraic curve, proper and smooth over a field, and absolutely irreducible. Certainly, if we would study the larger class of stable curves there are many CM (stable) curves of any genus > 0.

Why is this difficult? How can we give examples? Some curves have automorphisms that generate a large enough CM algebra, e.g. the Fermat curves. However in this way for any fixed g at most a finite number of isomorphisms classes of curves of this type give a CM Jacobian. In general it is hard to see from properties of a given curve whether its Jacobian is a CM abelian variety. On the other hand known criteria which decide whether a given principally polarized abelian variety is a Jacobian does not enable us, it seems, to single out form the large class of CM abelian variety many CM Jacobians:

starting with a curve C it is hard to decide whether $J(C)$ is a CM abelian variety;

starring with an abelian variety A (maybe CM, principally polarized, $g > 3$) *it is hard to decide whether it is the Jacobian of an algebraic curve.*

Why is this interesting? We will see that CM points are "torsion points in canonical coordinates" on $\mathcal{A}_{g,1}(\mathbb{C})$, and this will give access to questions posed below: granting (AO) we see that infinitely many CM Jacobians for a given value of g show some of the "canonical coordinates" on $\mathcal{A}_g$ parametrize a subvariety in T_g.

Notation. We write $\mathcal{M}_g$ for the moduli space of curves (smooth and proper over the base scheme, with absolutely irreducible geometric fibers). By $C \mapsto (J(C), \lambda)$ we obtain a morphism

$$j : \mathcal{M}_g \longrightarrow \mathcal{A}_g, \quad T_g^0 = j(\mathcal{M}_g) \subset \mathcal{A}_g$$

and the image of this map is called the open Torelli locus. Its closure

$$\left(T_g^0\right)^{\mathrm{Zar}} =: T_g \subset \mathcal{A}_g,$$

is called the (closed) Torelli locus. Note:

$$1 \leq 3 \quad \Longrightarrow \quad T_g = \mathcal{A}_{g,1}; \quad \text{the dimension (over any field) equals: } 1,\ 3,\ 6,$$

$$3 < g \quad \Longrightarrow \quad T_g \subsetneqq \mathcal{A}_{g,1}; \quad \text{the dimensions are: } 3g-3 < g(g+1)/2.$$

The Coleman conjecture can be formulated as:

$$\boxed{\#(\mathrm{CM}(T_g^0)) \overset{?}{<} \infty \text{ for } g > 3.}$$

3. Examples. We now know this conjecture does not hold for $g \in \{4, 5, 6, 7\}$, see [20], [53], [44]. For a survey see [MO]. We explain one way to construct such examples as this was done by Johan de Jong and Rutger Noot in 1991:

Consider curves given by

$$C_\lambda : Y^5 = X(X-1)(X-\lambda).$$

It can be proved that for infinitely may values $\lambda \in \mathbb{C}$ the curve C_λ is a CM curve.

All infinite families of CM Jacobians constructed along this line (i.e., variable covers of $\mathbb{P}^1$) have been classified, see [51], [31]. *It seems other methods are necessary to decide whether the Coleman conjecture holds for some/any given value $g \geq 8$.*

4. Special subvarieties. We will not give a treatment of *Shimura varieties* in general, but only briefly give an ad hoc defintion in the case $\mathcal{A}_{g,1}(\mathbb{C})$. Statements below can be generalized to arbitrary Shimura varieties.

We write

$$\mathfrak{h}_{\mathfrak{g}} = \{\tau \in (\mathbb{C}^g)^g = \mathrm{Mat}(g \times g, \mathbb{C}) \mid \tau = {}^t\tau, \quad \mathrm{Im}(\tau) > 0\},$$

the Siegel upper half space. We know

$$\mathcal{A}_{g,1}(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Q}) \backslash \mathfrak{h}_{\mathfrak{g}},$$

under

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}.$$

A subvariety $Z \subset \mathcal{A}_g := \mathcal{A}_{g,1} \otimes \mathbb{C}$ is said to be a *special subvariety* if there is a CM point $z \in \mathcal{A}_{g,1}(\mathbb{C})$ and an algebraic group $H \subset \mathrm{Sp}_{2g}$ over $\mathbb{Q}$ such that

$$\begin{tikzcd} \tau \arrow[d, mapsto] & \tau \in \mathfrak{h}_{\mathfrak{g}} \arrow[d] & H(\mathbb{R})^+ \cdot \tau \arrow[l, hook'] \arrow[d] \\ z & z \in \mathrm{Sp}_{2g}(\mathbb{Q}) \backslash \mathfrak{h}_{\mathfrak{g}} = \mathcal{A}_{g,1}(\mathbb{C}) & Z(\mathbb{C}). \arrow[l, hook'] \end{tikzcd}$$

Sometimes a special subvariety is also called "a subvariety of Hodge type", "a modular subvariety", "an irreducible component of a Hecke translate of a sub-Shimura variety". Note that *a special point in $\mathcal{A}_{g,\mathbb{C}}$ is a special sub variety of dimension zero, and hence, by definition, the same as a* CM *point.*

Important observation. The set $\mathrm{CM}(Z(\mathbb{C}))$ of CM points contained in a special subvariety Z is dense in $Z(\mathbb{C})$ (in the classical topology) and $\mathrm{CM}(Z(\mathbb{C}))$ is Zariski-dense in Z.

Here are some cases of special subvarieties.
(4.a) As mentioned before: a zero-dimensional case consisting of one CM point.
(4.b) PEL Shimura sub varieties. In particular: Hilbert Modular varieties; these will show up in (7a) and in (9)
(4.c) All counterexamples for $4 \leq g \leq 7$ to (2) will be given by families of curves with some automorphisms creating a special subvariety; see [34], see [51], [31].
(4.d) Not every special subvariety of $\mathcal{A}_g$ comes from a PEL type Shimura variety. In [35] Mumford constructed one-dimensional special subvarieties of $\mathcal{A}_4$ where generically the endomorphism ring of the fiber is $\mathbb{Z}$. These subvarieties are partly well understood, see [40]. However see (11.4).

(4.e) For $g > 3$ the Torelli locus is *not a special subvariety.* In fact more is true, see [34], 4.5:

$$g > 3, \quad T_g \subset Z \subset \mathcal{A}_{g,1}, \quad Z \ \text{ is special} \quad \Longrightarrow \quad Z = \mathcal{A}_{g,1}.$$

5. Linearity. See [34] for more explanation and references.
We see that we can feel a special subvariety as an orbit under the action of a (linear) subgroup. This is reflected in the fact that special subvarieties have strong linearity properties (e.g. under the notion of local Serre-Tate parameters) and properties of being geodesic subvarieties, see the PhD-Thesis by R. Noot (1992) [37], [38], and see the PhD-Thesis by B. Moonen (1995) [27], [28], [29].

(5.a) Serre-Tate canonical coordinates. We give one example. Let $\kappa \leftarrow R \subset \mathrm{Quot}(R) = K \subset \mathbb{C}$ be a local ring in mixed characteristic with finite residue class field κ and field of fractions K. Let (A, μ) be a polarized abelian scheme over $S = \mathrm{Spec}(R)$ giving $z : S \to \mathcal{A}_g \otimes R$; suppose $A_0 = A \otimes_R \kappa$ is an ordinary abelian variety, and suppose that $(A_\eta = A \otimes_R K, \mu)$ is the Serre-Tate canonical lift of $(A, \mu) \otimes_R \kappa = (A_0, \mu)$. In this case we have an isomorphism

$$\mathcal{A}_{g,R}^{/z} \cong ((\mathbb{G}_m)_R^{g(g+1)/2})^{/1}$$

between the formal scheme obtained by completing $\mathcal{A}_g$ along z and a power of the formal group attached to the multiplicative group. In this case CM abelian varieties reducing to A_0 correspond to torsion points under this isomorphism. Moreover an algebraic subvariety "containing" z is special only if its completion corresponds to a translate by a torsion point of a linear formal subscheme.

(5.b) Any abelian variety over a finite field is a CM abelian variety, Tate 1966: [54]. Hence any Jacobian over a finite field is a CM Jacobian. However (for $g > 4$) the theory of "canonical lifts" does not provide us in this way with many CM Jacobians in characteristic zero, see [47]. see [11], see [27], IV Theorem 2.6 and see (10):

(5.c) Theorem (Moonen-De Jong-Oort). *Fix g. Let $U_0 = T_g^0(\overline{\mathbb{F}_p})^{\mathrm{ord}}$ be the set of all ordinary Jacobians defined over a finite field. For every $x_0 \in U_0$ let x^{can} be the moduli point of the canonical lift of this principally polarized abelian variety.*

$$(\{x^{\mathrm{can}} \mid x_0 \in U_0)^{\mathrm{Zar}} = \mathcal{A}_{g,1}.$$

We see that for $g \geq 4$ "most" ordinary Jacobians do not give a Jacobian under the canonical lift.

The analogy with the Manin-Mumford conjecture is striking, and the following was conjectured.

6. The AO conjecture.

Conjecture (Yves André, FO; see [1], [42], [43]). *Suppose $Z \subset \mathcal{A}_g \otimes \mathbb{C}$ is an algebraic subvariety then:*

$$\boxed{\mathrm{CM}(Z(\mathbb{C})) \text{ is Zariski-dense in } Z \overset{?}{\Longleftrightarrow} Z \text{ is special.}}$$

Note that "$\Leftarrow$" is clear. For a survey, see [41].

Many special cases (sometimes under GRH) have been proved (Y. André, B. Edixhoven, B. Moonen, A. Yafaev, Clozel-Ullmo, Pila, Pila-Tsimerman,...; see [41]). In [23] and [56] a proof of the AO Conjecture for arbitrary Shimura varieties, under assumption of the Generalized Riemann Hypothesis for CM fields, is announced. Pila and Tsimerman announce an unconditional proof for $\mathcal{A}_g$ with $g \leq 6$, see [50].

7. An expectation (see [43], § 5). *For large g (in any case $g \geq 8$), there does not exist a special subvariety $Z \subset \mathcal{A}_g$ with $\dim(Z) \geq 1$ such that $Z \subseteq T_g$ and $Z \cap T_g^0$ is nonempty.*

Any point in T_g^0 corresponds with the canonically polarized Jacobian of an absolutely irreducible, regular complete curve, and any point in T_g corresponds with the canonically polarized Jacobian of a "compact type" algebraic curve.

Note that for $1 \leq g \leq 3$ we have $\mathrm{CM}(T_g^0(\mathbb{C})) = \infty$; this shows that for any $g > 0$ we have $\mathrm{CM}(T_g(\mathbb{C})) = \infty$. The Coleman conjecture expects: $\mathrm{CM}(T_g^0(\mathbb{C})) < \infty$ for $g > 3$.

We have seen in (3) that there are special subvarieties of positive dimension in $\mathcal{A}_g \otimes \mathbb{C}$ for every $g \leq 7$. For larger g we do not know either to construct such examples, or to prove the expectation in that case.

Observe: *Let $g \in \mathbb{Z}_{>7}$ such that the expectation mentioned above holds for that value of g. Assume* OA. *Then*

$$\#\left(\mathrm{CM}(T_g^0)\right) < \infty,$$

i.e. the conjecture (2) *by Coleman holds for that value of g.* In short, for a fixed g:

$$\boxed{(7)\ \&\ (\mathrm{AO}) \quad \Longrightarrow \quad \#\left(\mathrm{CM}(T_g^0)\right) < \infty.}$$

Explanation. We like to know which part of the linear structure of $\mathcal{A}_{g,\mathbb{C}}$ "lives" in $T_{g,\mathbb{C}}^0$. Special subvarieties contained in the Torelli locus tell us the answer to this question, and the expectation says that this only should happen for small values of g.

Several papers are devoted to a proof of the expectation in special cases. Especially the paper [19], using [16], gives an important approach. For further references, see [34]. We mention a special case:

(7.a) Theorem. *Let $Z \subset \mathcal{A}_{g,\mathbb{C}}$ be a Hecke translate of a Hilbert modular subvariety of $\mathcal{A}_{g,\mathbb{C}}$, i.e., Z is a special subvariety of* PEL *type obtained from a totally real field L_0 of degree g. Assume $g \geq 4$. Then Z is not contained in T_g.*
This was proved by A.J. de Jong and S.-W. Zhang, see [21], with a restriction in case $g = 4$; this last case was settled in [4].

(7.b) Remark. For any $g \geq 4$ the Torelli locus $T_g \subset \mathcal{A}_{g,\mathbb{C}}$ is not a special subvariety; see (4.e).

8. Weyl CM fields. Fix $g \in \mathbb{Z}_{>0}$. Consider indices $\{1, \cdots, 2g\}$. A group W_g is defined, which containes the (normal) subgroup N generated by the transpositions $(12), \cdots, ((2i+1)(2i+2)), \cdots ((2g-1)(2g))$, this group is isomorphic with $(\mathbb{Z}/2)^g$, and the quotient W_g/N is given by the natural action of the symmetric group S_g permuting the pairs $\{\{1,2\}, \cdots \{2g-1, 2g\}\}$. We have an exact sequence

$$1 \to (\mathbb{Z}/2)^g \to W_g \to S_g \to 1,$$

and W_g is called the Weyl group. Note that $\#(W_g) = 2^g \cdot (g!)$. (Note: some authors denote this group by W_{2g}).

For a field extension $\mathbb{Q} \subset M$ we write $M^{\sim}$ for the Galois closure of $M/\mathbb{Q}$.

Lemma (8.a). *Suppose $\mathbb{Q} \subset L_0 \subset L$ are field exensions with $[L_0 : \mathbb{Q}] = g$ and $[L : L_0] = 2$. There is an inclusion* $\mathrm{Gal}(L^{\sim}/\mathbb{Q}) \hookrightarrow W_g$, *unique up to conjugation.* $\square$

Definition (8.b). A field L is called a Weyl CM field, if it is a CM field such that $\mathrm{Gal}(L^{\sim}/\mathbb{Q}) = W_g$.

Note that "most CM fields are Weyl CM fields", as was made explicit in [18], [8], [24], [25], [26], [13].

9. Finiteness of the number of Weyl CM Jacobians for $g \geq 4$.
Under a mild restriction we can prove an analog of the Coleman conjecture:
Theorem (Ching-Li Chai & FO). *Assume* (AO). *For any $g \in \mathbb{Z}_{\geq 4}$ we have:*

$$\#\left(\mathrm{WCM}(T_g(\mathbb{C}))\right) < \infty.$$

See [7], 3.7. Sketch of proof. Suppose for a given value of g we have $\#\left(\mathrm{WCM}(T_g(\mathbb{C}))\right) = \infty$. The Zariski closure of this set contains a positive dimensional subvariety. By (AO) this is a special subvariety. One proves this is a Hilbert modular variety, see [7], Lemma 3.5. By (7a) we conclude $g \leq 3$, a contradiction. □

Note that for any g the boundary $T_g - T_g^0$ contains infinitely many CM Jacobians (for $g > 1$). However none of these is a Weyl CM Jacobian.

Note that a hyperelliptic curve is not a Weyl CM curve. Note that any curve C of genus $g > 1$ with a non-trivial automorphism is not a Weyl CM curve.

10. Linearity at the boundary. We compare [11] (1986) with [17] (1994).

Consider:
(DO) The moduli space $\mathcal{A}_g \otimes \mathbb{Z}_{(p)}$ in mixed characteristics, view $\mathcal{A}_g \otimes \mathbb{Q}$ as "the interior" and consider

$$\mathcal{A}_g \otimes \mathbb{F}_p = \partial = \partial\left(\mathcal{A}_g \otimes \mathbb{Z}_{(p)}\right)$$

as its boundary. For any point $x = [(A, \lambda)]$ in the *ordinary locus* $\partial^o := (\mathcal{A}_g \otimes \mathbb{F}_p)^{\mathrm{ord}}$ we have the notion of a kind of tubular structure where the canonical lifts give directions "transversal to the boundary ∂^o". In [11] the question was asked: suppose moreover x lies in the Torelli locus $\mathcal{T}_g \otimes \mathbb{F}_p$ does the canonical lift stay in the Torelli locus? (Also see 5.c.)

(FvdP) Consider the moduli space $\mathcal{A}_{g,\mathbb{C}}$, let $\mathcal{A}_{g,\mathbb{C}}^{(*)}$ some nice kind of compactification (e.g. a toroidal compactification), and consider

$$\mathcal{A}_{g,\mathbb{C}}^{(*)} - \mathcal{A}_{g,\mathbb{C}} = \partial = \partial(\mathcal{A}_{g,\mathbb{C}})$$

as its boundary. We have a good moduli interpretation of points in this boundary. Again here one can introduce the notion of a "canonical lifting" from a point in the boundary (this can be done analytically, or in formal algebraic geometry). In [17] the question was asked: if a boundary point x lies in the Torelli locus of the boundary (i.e. it is the polarized Jacobian of a stable curve of "compact type") does the canonical lift stay in the Torelli locus?

In both cases the answer is: if $g \geq 4$, there is a non-empty open set U in ∂^o, respectively in ∂, such that for every $x \in U$ the canonical lift x^{can} does not land into T_g. Needless to say that, for these two very different theorems, in different fields the proofs are different. But the analogy is striking.

Also see [27], IV Theorem 2.6, [29], Theorem 6.6. for an explanation and a proof in mixed characteristic.

(10.a) Question. For a given $g > 3$ study the closure/boundary structure of special subvarieties and their boundaries in both settings. Study "canonical coordinates" and their behavior with respect to the Torelli locus in this setting. See [3]. How far can we use this (for non-compact special subvarieties) to give an answer to Expectation (7) ?

(10.b) Remark. It has been fruitful to consider the "boundary" for subsets of $\mathcal{A}_g$ not only in the case the abelian varieties in question degenerate, but also in cases where the abelian varieties do not degenerate, but the p-structure changes. E.g. in positive characteristic p lower p-rank abelian varieties can be considered as giving boundary points of moduli of ordinary abelian varieties, and in this sense supersingular abelian varieties, which by now are fairly well understood, give moduli points "deepest in the boundary".

11. Some questions.

Already raised: Expectation (7) and Question (10.a).

(11.1) How do we construct explicitly CM **curves?** Except some very special cases (e.g. curves with many automorphisms) we do not know how to construct explicitly CM curves.

(11.2) CM **curves with** $\mathrm{Aut}(C) = \mathbb{Z}$? Clearly any elliptic curve with $j(E) \neq 0, \neq 1728$ has no non-trivial automorphisms; hence there exist many CM elliptic curves with trivial automorphism group. Do we know examples of such curves for $g \geq 2$? Or do we know explicit examples, or can we prove the existence of CM curves with a small automorphism group?

(11.3) Do we know Weyl CM **curves of genus at least** 4 ? We showed that for a given $g \geq 4$ the number of Weyl CM curves of genus g is finite. But, do we know any examples? I expect there do exist such examples, but I do not know how to construct, or prove existence of such curves.

(11.4) Are there any Mumford-Shimura curves in T_4? In [35] Mumford constructed special subvarieties of $\mathcal{A}_{4,\mathbb{C}}$ of dimension one where the generic fiber abelian variety has endomorphism ring equal to $\mathbb{Z}$ (in particular this is not a PEL type Shimura variety). Clearly any Hecke translate of such a special subvariety again is special. Let us call these "Mumford-Shimura curves". Is any of these curves contained in T_4 ? If such a curve would exist, it would meet T_4^0. If so, we would prove the existence of infinitely many CM curves of genus 4 along this line. As T_4 is an ample divisor in $\mathcal{A}_4$ and any of the Mumford-Shimura curve is compact, any such curve has a non-empty intersection with T_4.

(11.5) Describe all special subvarieties contained in T_g.
See [34], 6.4. Can we describe / classify all special subvarieties of positive dimension contained in T_g ? Even for $g = 4$ we do not know such a description.

(11.6) Does there exist a special subvariety $Z \subset T_g$ with $\mathrm{End}(A) = \mathbb{Z}$ **generically** ? We have seen constructions of special subvarieties in the Torelli locus using curves with some automorphisms (then moving the curves, and showing they sweep out a special family). For $g \geq 4$ we do not know a special subvariety $Z \subset T_g$ such that for the generic point $\eta \in Z$ we have $\mathrm{End}(A \otimes \overline{k(\eta)}) = \mathbb{Z}$.

(11.7) Abelian varieties isogenous to no Jacobian. Fix an algebraically closed field. Choose $g > 3$. Is there an abelian variety over k not isogenous to any Jacobian?

- *Easy*: in case $k = \mathbb{C}$ a countability argument shows that for any $g > 3$ there is an abelian variety over $\mathbb{C}$ not isogenous to any Jacobian.

- *More involved*: in case $k = \mathbb{Q}^a$, an algebraic closure of $\mathbb{Q}$ indeed for any $g > 3$ there is an abelian variety over $\mathbb{Q}^a$ not isognous to any Jacobian; see [7]. [55].
- *Unknown*: in case $k = \overline{\mathbb{F}_p}$ it is not known whether for a given $g > 3$ there exists an abelian variety over k not isogenous to any Jacobian; clearly we do not have a good approach to this question, or to any generalization in the style of [7], 1.2: given a closed subset $X \subsetneqq \mathcal{A}_{g,k}$ does there exist an abelian variety whose Hecke orbit has empty intersection with X?

(11.8) Families of curves with automorphisms. See [51], [31], [34] Section 5 and 6.8. We have seen families of curves with automorphisms sweeping out a special subvariety. In the cases classified the quotient of these curves is rational, and the families are obtained by moving the branch points. Are there any special families of Jacobians given by deforming a covering $C \to G\backslash C = D$ not already described by the know examples?

(11.9) AO in positive characteristic? Is there a clear and useful analog of the AO Conjecture for $\mathcal{A}_{p,\mathbb{F}_p}$? One could involve linearity properties or something like that, but for the moment we have no clear ideas.

Note that the notion of *hypersymmetric points* seems to be of no help: Proposition (7.3) of [6] says that any algebraic curve in $(\mathcal{A}_{1,\mathbb{F}_p})^2$ contains a dense set of hypersymmetric points (although hypersymmetric points in positive characteristic seem the right analog of CM points in characteristic zero). Maybe, the analog is not about density of a certain type of points.

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