The fundamental group of an algebraic curve
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Preliminary version.

In this seminar we study *geometric properties* of algebraic curves, or of Riemann surfaces, with the help of an algebraic object attached: *the fundamental group*, either the algebraic fundamental group, as introduced by Grothendieck, or the topological fundamental group.

Here is the central idea of the seminar. To an algebraic curve X over a field K we can attach the fundamental group (either in the algebraic context or in the topological vein). This algebraic object gives information about the geometry of X. Commuting between topology and algebraic geometry we can determine rather precisely the structure of this group (at least in characteristic zero, or the prime-to-p-part in characteristic p). This tool enables us to prove theorems, find structures in arithmetic geometry and in algebraic geometry.

Please do not consider these notes as containing complete information on the topics introduced, but rather as a guide-line through the literature mentioned. We hope and expect students to work themselves through this beautiful material; and, of course both of us are available whenever you have questions, in case you want to talk about this material, or whatever.

In the first sections we gather together some basic information, connecting topological concepts with definitions and theorems in algebraic geometry. These are meant as an introduction to, and a basis for studying the following topics, fundamental ideas and theorems:

I: Good reduction Given a family (of algebraic curves, of Riemann surfaces, of abelian varieties, or whatever) over a punctured disc, or over the field of fractions of a discrete valuation ring; try to find a criterion which ensures that this family can be extended in such a way that the central fiber has "good reduction"; it turns out that the Galois representation obtained enables us to formulate precisely this property. In case of abelian varieties this was given by Serre and Tate; in that case the Galois representation, obtained from (co)homology, or from the Tate-ℓ-group, provides the tool necessary. We discuss the case of algebraic curves, where the Galois representation (or the monodromy representation) on the fundamental group of the algebraic curve (sitting over the generic point) provides the necessary information, as Takayuki Oda explained to us, see Section 10.

These notes provide material for topic (I). *The rest of the theory in these notes will be worked out in a later version of these notes.*

II: Correspondences The fundamental group of a hyperbolic Riemann Surface, can be considered as an abstract group; in this consideration, as abstract groups, two complete, non-singular curves of the same genus have isomorphic topological fundamental groups. But in fact $\pi_1^{\mathrm{top}}(X(\mathbb{C}))$ should be considered as a subgroup of $\mathrm{SL}(2,\mathbb{R})$, up to conjugation; in this way of looking at fundamental groups different curves may very well produce totally different fundamental groups; in this way we can study correspondences between algebraic curves in characteristic zero, as shown by S. Mochizuki, see Section 11.

III: The Anabelian Grothendieck Conjecture Is the (algebraic) concept of the fundamental group strong enough to tell the isomorphism type of a geometric object? This seems at first sight a surprising question. Grothendieck formulated around 1983 a conjecture, see [16], [17], [18]. Several cases of the "Grothendieck anabelian conjectures" have been proved by now. - Analogy with number theory (the Neukirch - Uchida theorem) is striking. This topic will be of great importance in the future, we think.

Advice. Please try to understand the prerequisites as formulated in the first nine sections of these notes. You need not follow our references: in many cases there are many books and papers containing descriptions of these concepts. - Probably we will use some of the first sessions of the seminar to discuss aspects of these prerequisites.

Please contact us if you have questions, if you feel you need more explanation, don't hesitate!

Then, please study the paper [40], or [4], and "read backwards", i.e. try to digest material, and look up references whenever necessary, find definitions and theory, or ask us whenever necessary.

If time permits we can discuss some of the prerequisites of (II) and of (III).

1 Topology

(1.1) Classification of real surfaces. References: [12]; [52], Chapter 1; [53].
Exercise. [This holds for all sections of these notes!] *Find better references.*

Formulate the theory of real surfaces. Understand the theory which describes connected, compact real surfaces. This classification gives that a connected, compact, orientable real surface is determined up to isomorphism by its *genus* $g \in \mathbb{Z}_{\geq 0}$. For $g = 0$ we have the "Riemann sphere"; for $g = 1$ we have the "torus", e.g. defined as the topological space $\mathbb{C}/(\mathbb{Z}\cdot 0 + \mathbb{Z}i)$. Understand well the way a connected, compact, orientable real surface can be defined by "glue and sciccors" method, i.e. such a surface can be cut by $2g$ circles, and conversely a $4g$-gon with appropriate identification of sides gives a a connected, compact, orientable real surface of genus $g > 0$; see (2.8). *Please work out and understand this theory very well.* This kind of "easy topology" will be a fundamental tool in understanding and in proving facts in (difficult) algebraic geometry.

(1.2) The topological fundamental group. See [50], [10], [52], [53].

We use topological spaces with "reasonable properties". For the definition of the topological fundamental group this is not necessary; however for the connection with universal covering spaces such conditions are needed. Without further mention we suppose that a topological space S is *connected, locally pathwise connected and locally simply connected.*

Remark Note that if $V/\mathbb{C}$ is an (irreducible) variety over the complex numbers then the complex space $V(\mathbb{C})$ satisfies the properties just mentioned, see [11], II.2.4 (Local structure of analytic varieties). We will often encounter regular varieties (non-singular varieties), and these give (over $\mathbb{C}$) locally Euclidean topological spaces, even better:

(1.3) Note that if $V/\mathbb{C}$ is a regular (irreducible) variety over the complex numbers then the complex space $V(\mathbb{C})$ satisfies the properties just mentioned:

$$(V \text{ is irreducible}) \Rightarrow (V(\mathbb{C}) \text{ is connected}),$$

for example, see [49], Vol. 2, page 126, Th. 1, and:

$$(V \text{ is regular}) \Rightarrow (V(\mathbb{C}) \text{ is locally Euclidean}),$$

hence in this case $V(\mathbb{C})$ is locally pathwise connected and locally simply connected.

(1.4) The topological fundamental group. Let S be a topological space, and let $s_0 \in S$ be a chosen (base) point. We define $\pi_1^{\mathrm{top}}(S, s_0)$ to be the *topological fundamental group* of (S, s_0); this is the set of homotopy classes of loops in (S, s_0) with composition as group law. Note (we supposed S to be connected) that two choices $s_0, s_1 \in S$ yield isomorphic groups $\pi_1^{\mathrm{top}}(S, s_0) \cong \pi_1^{\mathrm{top}}(S, s_1)$.
Exercices. *Formulate well the notion of* "homotopy equivalence". **(a)** *Show that we obtain a group.*
(b) *What is the topological fundamental group of S^1, the circle? Formulate the Brouwer fix-point-theorem, and give a proof of this theorem using the structure of $\pi_1(S_1, 1)$, and using the functorial properties of the topological fundamental group.*
(c) *What is the fundamental group of the torus $\mathbb{C}/(\mathbb{Z}{\cdot}0 + \mathbb{Z}{\cdot}i)$?*

(1.5) Warning. In what follows, either on topological fundamental groups, or on algebraic fundamental groups (see below) we should mention and write the notion of the base points, but often we will assume and then ignore the choice and the notation of the base point.
Exercise. *Let S be a connected topological space, and $s_0, s_1 \in S$. Show that there exists an isomorphism $\pi_1(S, s_0) \cong \pi_1(S, s_1)$ (choose a path in S from $s - 0$ to $s - 1$, and "conjugate" with this).*
Remark / Exercise. *Let $f : (S, s) \to (T, t)$ be a continuous map of pointed topological spaces; this induces $f_* : \pi_1(S, s) \to \pi_1(T, t)$ (covariant behavior). Mind the choice of base points!*

(1.6) The Universal covering space. Let S be a topological space (with properties mentioned above !). There exits a "universal covering" $S^\sim \to S$; this is a covering, $S^\sim$ is connected and simply connected; these properties characterize this covering up to $S-$homeomorphism.

Moreover $\pi := \pi_1^{\mathrm{top}}(S, s_0)$ acts on $S^\sim \to S$ and the orbit space of $S^\sim$ under the fundamental group π is S:

$$S^\sim \to \pi\backslash S^\sim = S.$$

Exercise. *Define well the notion of the category of coverings of S.*

(1.7) Warning. Let $f : V \to W$ be a covering in the sense of algebraic geometry; this is a finite morphism between algebraic varieties; suppose these are defined over $\mathbb{C}$ and consider the induced map $V(\mathbb{C}) \to W(\mathbb{C})$. This need not be a topological covering: in case the morphism f is *ramified* the related topological map is not a topological covering. Sometimes a warning is given by saying something like "consider a (ramified) cover" in order to distinguish the habits of algebraic geometers on the one hand, and the usage in topology on the other hand.

2 The topology of complex varieties.

(2.1) Notation. Let X be an algebraic variety over $\mathbb{C}$. We write $X(\mathbb{C})$ for the topological space obtained from $\mathbb{C}$ (with the "classical topology", do you know how to define this ?). We write $\pi_1^{\mathrm{top}}(X(\mathbb{C}))$ for its topological fundamental group (you see already the bad habit: we did choose a base point, used it, and then suppress it in the notation).

(2.2) Riemann Surfaces. A Riemann surface is a connected complex manifold of dimension one (surface: it two-dimensional as real toplogical space; please note the possible confusion). Sometimes the complex manifold and the underlying toplogical space are denoted by the same letter; if you say "Riemann Surface", please make clear whether you mean the complex manifold, to the topological space.

For a variety V over $\mathbb{C}$ we can define the complex variety V^{an}; this is the underlying topological space $V(\mathbb{C})$ with the sheaf of holomorphic functions derived from the structure of the $\mathbb{C}$-variety V. We have functorial correspondences:

$$V \quad \mapsto \quad V^{\mathrm{an}} \quad \mapsto \quad V(\mathbb{C}).$$

(2.3) Theorem. *The category of connected, irreducible regular algebraic curves over $\mathbb{C}$ and the category of connected Riemann surfaces* (as complex manifolds) *are equivalent.*
The difficult point: to show that on a Riemann surface there exists a meromorphic function which is non-constant; once you have that theorem, the rest of the proof is a good exercise!

(2.4) GAGA *Let V and W be complete varieties over $\mathbb{C}$. The natural map*

$$\mathrm{Mor}(V, W) \quad \xrightarrow{\sim} \quad \mathrm{Mor}(V^{\mathrm{an}}, W^{\mathrm{an}})$$

is bijective. See [45]. This theorem, for projective varieties, was first proved by Chow. Later, Grothendieck generalized this to schemes with an ample line bundle. Literature: [45], [48], [31], [11].

(2.5) Example. For a complex manifold of dimension one (a Riemann surface) there does exist an algebraic structure and it is uniquely determined by the complex structure.

However, there do exist (many) complex manifolds which are not "algebraizable". For example, see [49], II.VIII.1.4, page 163: choose $a, b\mathbb{C}$ algebraically independent over $\mathbb{Q}$. Define the lattice $\Lambda \subset \mathbb{C}^2$ generated as a free abelian group by the vectors $(1,0), (i,0), (0,1), (a,b)$; let T be the "complex torus" $T := \mathbb{C}^2/\Lambda$. Clearly it is a compact complex manifold of dimension two. It can be showed that T is not algebraizable.

Moreover, if a complex manifold (which is non-compact, of dimension at least two) is algebraizable, this structure need not be unique. For an example, see [44], page 108: *an extension V of an elliptic curve by $\mathbb{G}_a$, i.e. $V/\mathbb{G}_a = E$ and $W = \mathbb{G}_m \times \mathbb{G}_m$, clearly $V \ncong W$ as algebraic varieties (exercise !) defining isomorphic analytic manifolds $V^{\mathrm{an}} \cong W^{\mathrm{an}}$.*

For another example, see: [49], II.VIII.3.2.

(2.6) Let $X/\mathbb{C}$ be an algebraic curve, and $S := X(\mathbb{C})$ the related Riemann surface.

- Suppose $X \cong \mathbb{P}^1_{\mathbb{C}}$, the case $g = 0 = r$; then $\pi = \{1\}$ is trivial, and $S^{\sim} = S = X(\mathbb{C})$, the Riemann sphere.
- Suppose $X \cong \mathbb{P}^1_{\mathbb{C}} - \{\infty\}$, the case $g = 0, \quad r = 1$; then $\pi = \{1\}$ is trivial, and $S^{\sim} = S = X(\mathbb{C}) \approx \mathbb{C}$.
- Consider the case $g = 0$ and $r = 2$ or the case $g = 1$ and $r = 0$; in these cases $\pi_1^{\mathrm{top}}(S)$ is commutative (free of rank one, respectively two), and $S^{\sim} = X(\mathbb{C}) \approx \mathbb{C}$.
- In case $X/\mathbb{C}$ is hyperbolic, the fundamental group $\pi_1^{\mathrm{top}}(X(\mathbb{C}))$ is non-abelian, and the corresponding universal covering space is homeomorphic with the upper half plane:

$$X(\mathbb{C})^{\sim} \approx \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

(2.7) Suppose X is an algebraic curve over $\mathbb{C}$. Let $S = X(\mathbb{C})$ be the related Riemann surface. As a real manifold, S is orientable, and classification of real surfaces shows that S is completely described by g and r: a Riemann surface of genus g with r punctures, see [12]; see [10], Chapter 17. In this case the fundamental group

$$\pi_1^{\mathrm{top}}(S) \cong \Gamma_{g,r},$$

where $\Gamma_{g,r}$ is the group defined by: it is generated by

$$\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_r$$

satisfying

$$(\alpha_1 \cdot \beta_1 \cdot \alpha_1^{-1} \cdot \beta_1^{-1}) \cdot \quad \cdots \quad \cdot (\alpha_g \cdot \beta_g \cdot \alpha_g^{-1} \cdot \beta_g^{-1}) \cdot \gamma_1 \cdot \quad \cdots \quad \cdot \gamma_r.$$

Note: If $r > 0$, i.e. if $X(\mathbb{C})$ is not compact, then $\Gamma_{g,r} \cong F_{2g+r-1}$, the free group on $2g + r - 1$ generators.

(2.8) Write out the "word"

$$W = [\alpha_1,\beta_1]\cdots[\alpha_g,\beta_g] = (\alpha_1{\cdot}\beta_1{\cdot}\alpha_1^{-1}{\cdot}\beta_1^{-1}){\cdot}\quad\cdots\quad{\cdot}(\alpha_g{\cdot}\beta_g{\cdot}\alpha_g^{-1}{\cdot}\beta_g^{-1}).$$

Choose a $4g$-gon D in the real plane (the lengths of the sides are not important, we are doing topology). Choose your way to walk along D (counter-clockwise, or clockwise). Choose your way to read W (from right to left, or from left to right). Give in this way the sides of D appropriate names and orientations. Identify sides of D with the same name, respecting orientations. Shows this produces a topological space (the identifications give a well-defined process), it is compact, and this is a topological compact real surface of genus g.

(2.9) Exercise. *Let X be a complete algebraic curve over $\mathbb{C}$ (non-singular, irreducible) of genus g. Let $\pi = \pi_1^{\mathrm{top}}(X(\mathbb{C}))$. Describe $\pi_{\mathrm{ab}} = \pi^{(1)} := \pi/[\pi,\pi]$.*

Definition. For a group G we define the "lower central series" by:

$$\Gamma_1(G) := G, \quad \text{and by induction} \quad \Gamma_{n+1}(G) := [\Gamma_n(G), G].$$

(2.10) Exercise. *Let Let X be a complete algebraic curve over $\mathbb{C}$ (non-singular, irreducible) of genus $g = 2$. Let $\pi := \pi_1^{\mathrm{top}}(X(\mathbb{C}))$. Describe the abelian groups $\Gamma_i(G)/\Gamma_{i+1}(G)$ for $i = 1,2,3$ by generators (and relations ?).*

Apology/warning. Please do not confuse the (standard) notations $\Gamma_{g,r}$ and $\Gamma_n(G)$ defined above (they have totally different meanings !).

3 Algebraic curves and Riemann surfaces

(3.1) Riemann-Zeuthen-Hurwitz-Hasse. Literature:
[19], IV.2, see page 301;
[31], Chapter 7;
[13], Chapter 2, in particular pp. 216-219.

This theorem considers a finite morphism $f : Y \to X$ of complete, nonsingular, irreducible algebraic curves, and it expresses invariants defined by X, by Y and by f in one formula. The three references give three different proofs! Please understand each of these proofs. A difficulty which you will encounter in this study is that you might end up in circular reasonings if you do not have a clear idea what is defined and what is a theorem;

e.g. you could define the genus of an algebraic curve $X/\mathbb{C}$ as the topological genus of the Riemann surface $X(\mathbb{C})$ and then prove the Riemann-Zeuthen-Hurwitz theorem using topological considerations;

or, you can define the genus of X, the topological genus of $X(\mathbb{C})$, prove theorems on coverings, and then prove that the genus of X and the topological genus of $X(\mathbb{C})$ are equal. Just choose your way of doing this (we prefer the second way).

Then having the Riemann-Zeuthen-Hurwitz you can moreover consider the case of positive characteristic, study wild ramification, and using [20], [21], describe (ramified) coverings in positive characteristic.

Notations:

$g(X) = g := \dim_K \mathrm{H}^1(X, \mathcal{O}_X)$, the *genus* of a complete, absolutely irreducible regular curve X over a field K,

$e_P = e(P \mapsto Q)$, the *ramification index* of a morphism $f : Y \to X$ of algebraic curves over an algebraically closed field k at $P \in Y$, where $P \mapsto f(P) = Q \in X$. This is defined as $v_P(f^*(t))$, where $t \in \mathcal{O}_{X,Q}$ is a uniformizer at $Q \in X$; if f is separable, and e_P is divisible by the characteristic of k, we say that f is wildly ramified at $P \mapsto Q$.

$\delta_P = \delta(P \mapsto Q)$, the degree of the different at P is defined as $v_P((\partial/\partial s)(f^*(t)))$, where $s \in \mathcal{O}_{Y,P}$ is a uniformizer at $P \in Y$; the different $\delta(f)$ is defined as

$$\delta(f) = \sum_{P \in Y} \delta_P \cdot P.$$

Note this is a finite sum if f is separable.

(3.2) Theorem (Riemann-Zeuthen-Hurwitz). *Let $f : Y \to X$ be a separable, finite morphism of complete, absolutely irreducible regular curves over an algebraically closed field k. Then*

$$2g_Y - 2 = \deg(f)(2g_X - 2) + \deg(\delta(f)).$$

(3.3) Hyperbolic curves. We consider algebraic curves X over a field K. We assume X to be regular and absolutely irreducible (unless otherwise specified). We write X^c for the unique complete curve over K containing X as dense subscheme (the "compactification of X").

Definition / Notation. Consider an algebraic curve X over K, and an algebraically closed field $k = \overline{K}$. We write:
the *geometric genus*:

$$g = \dim_K H^1(X^c, \mathcal{O}_{X^c});$$

the *number of holes*:

$$r = \#((X^c - X)(k));$$

the "*Euler number*":

$$\chi = -2g + 2 - r.$$

Definition. *An algebraic curve X/K is called* hyperbolic *if $-\chi(X) > 0$.*
I.e. $(g = 0 \ \&\ r \geq 3)$,
or $(g = 1 \ \&\ r \geq 1)$,
or $g \geq 2$.

(3.4) Exercise. Let $X/\mathbb{C}$ be a complete curve (regular and irreducible) over $\mathbb{C}$. Show that $g(X)$ equals the topological genus of $X(\mathbb{C})$. [Hint: show it for $\mathbb{P}^1$; then produce a covering $X \to \mathbb{P}^1$, and compare Th. (3.2) with the topological equivalent for the covering $X(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.]

(3.5) Exercise. *Let X/K be an algebraic curve, and $k = \overline{K}$.* Then:

$$X \text{ is a hyperbolic curve} \iff \mathrm{Aut}(X_k) \text{ is a finite group.}$$

(3.6) Exercise. (1) *Let X/K and Y/K be regular algebraic curves over a field K. Let $f : Y \to X$ be a quasi-finite morphism. Then*

$$X \text{ is a hyperbolic curve} \quad \Longrightarrow \quad Y \text{ is a hyperbolic curve.}$$

(2) *If moreover f is finite and étale and K is of characteristic zero, or X is complete, then*

$$X \text{ is a hyperbolic curve} \quad \Longleftrightarrow \quad Y \text{ is a hyperbolic curve.}$$

(3) *Give an example of a finite, étale morphism of regular algebraic curves $f : Y \to X$ such that Y is hyperbolic and X is not hyperbolic.*

4 Some easy facts in group theory.

Let G be a group, $N \subset G$ a normal subgroup, write $\Gamma := G/N$ for the factor group, and consider the "exact sequence"

$$1 \to N \longrightarrow G \xrightarrow{p} \Gamma \to 1. \qquad (\xi)$$

For any group N we define

$$\mathrm{Out}(N) := \mathrm{Aut}(N)/\mathrm{Inn}(N);$$

note that the group $\mathrm{Inn}(N)$ of inner automorphisms of N indeed is a normal subgroup of $\mathrm{Aut}(N)$.

The exact sequence defines a homomorphism

$$R : \Gamma \longrightarrow \mathrm{Out}(N) \quad \text{by:} \quad R(\sigma')(x) := \sigma x \sigma^{-1},$$

for $\sigma' \in \Gamma$, $\sigma \in G$ with $p(\sigma) = \sigma'$ and $x \in N$. Note that indeed $\sigma x \sigma^{-1} \in N$, because N is normal in G, and the definition of $R(\sigma') \in \mathrm{Out}(N)$ does not depend on the choice of $\sigma \in G$.

If moreover N is an abelian group, the exact sequence defines a homomorphism

$$R : \Gamma \longrightarrow \mathrm{Aut}(N) \quad \text{by:} \quad R(\sigma')(x) := \sigma x \sigma^{-1},$$

for $\sigma' \in \Gamma$, $\sigma \in G$ with $p(\sigma) = \sigma'$ and $x \in N$.

(4.1) Here is one of the fundamental examples. Let X be a scheme (or a variety, if you wish) over a field K. The we obtain an exact sequence

$$1 \to N = \pi_1(X \otimes_K K^s) \longrightarrow G = \pi_1(X) \xrightarrow{p} \Gamma_K = \mathrm{Gal}(K^s)/K) \to 1$$

as will see. Hence we obtain

$$R : \quad \Gamma_K \quad \longrightarrow \quad \mathrm{Out}(N).$$

This representation will play a fundamental role.

(4.2) Suppose given an exact sequence $G/N = \Gamma$ (ξ) as above, in particular N is normal in G. We have seen that this exact sequence defines the representation $R : \Gamma \to \mathrm{Out}(N)$. Conversely:

Lemma. *Suppose Γ and N are given, and suppose that the center of N is trivial: $\mathcal{Z}(N) = \{1\}$. From Γ, N and R we can reconstruct G and $G/N = \Gamma$. If G and N are topological groups, and R is continuous, the $G/N = \Gamma$ thus constructed can be equipped with the structure of an exact sequence of topological groups.* [See [9], page 134.]
The essential step: As N has trivial center, $N \hookrightarrow \mathrm{Inn}(N)$ is injective, hence the square

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(N) \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{R} & \mathrm{Aut}(N)/\mathrm{Inn}(N) \end{array} \qquad g \mapsto (x \mapsto g{\cdot}x{\cdot}g^{-1})$$

is cartesian; this shows we can reconstruct the left hand upper corner from the rest of the data. $\square$

(4.3) Exercise. In the vein of the lemma: use the square in the diagram in order to define R (for this you need not assume the center of N to be trivial).

5 The algebraic fundamental group

The notion of a *pro-finite group.* Let $\{G_i\}$ be a projective system of finite groups. Let $\pi = \lim_{\leftarrow} G_i$ be the projective limit. Equip all of the finite groups with the discrete topology (every subset is closed and is open). Equip π with the limit topology, i.e. the coarsest which make all projections $\pi \to G_i$ continuous. Or: a subbasis for the topology on π is given by the inverse images of subsets in G_i under all $\pi \to G_i$.

(5.1) Examples. 1) Let G be a group, and consider the projective system of all $\{G/N\}$ where $N \subset G$ is a finite index normal subgroup. The pro-finite group $\pi = \lim_{\leftarrow} G/N$ is called the *pro-finite completion of* G. Notation: $\pi = G^\wedge$.
Warning. There is a natural map $G \to \pi = G^\wedge$; this map need not be injective (e.g. take $G = \mathbb{Q}^+$).
Show that the pro-finite completion $\mathbb{Z}^\wedge$ is isomorphic with the product $\Pi_p \mathbb{Z}_p$, the product taken over all natural prime numbers p.

2) Let K be a field, and let K^s be a separable closure. Then $\mathrm{Gal}(K^s/K)$ is a pro-finite group: take the projective limit over the Galois groups of all finite separable, normal extension of K. The group $\mathrm{Gal}(K^s/K)$, sometimes denoted by Γ_K is called the absolute Galois group of K.
Reminder: Every closed subgroup $H \subset \mathrm{Gal}(K^s/K)$ defines a field extension $K \subset (K^s)^H$, invariants under H, and the main theorem of (infinite) Galois theory tells you that this is an inclusion inverting, bijection map between the set of closed subgroups of $\mathrm{Gal}(K^s/K)$ and the set of intermediate fields of $K \subset K^s$.

Show that for any finite field $K = \mathbb{F}_q$ its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ is isomorphic with the pro-finite completion $\mathbb{Z}^\wedge$, with topological generator $x \mapsto x^q$.

3) **Exercise.** Show that in a pro-finite group "open" is the same as "closed + finite index". Show that in a pro-finite group every closed subset is the intersection of open subgroups.

(5.2) Study the notions of a morphism being **flat**, **unramified**, or **étale = flat + unramified**. See [SGA1] = [14]; see [HAG] = [19]; see [2].

(5.3) Exercise. Let X be a projective scheme. Let G be a finite group. Suppose given an injective homomorphism $G \to \mathrm{Aut}(X)$. Show the quotient $G\backslash X$ exists (give appropriate definitions of "the quotient"); see [44], III.12.

We say that a finite morphism $Y \to X$ is *Galois with group* G, if there exists $G \subset \mathrm{Aut}_X(Y)$ such that $Y \to G\backslash Y = X$.

(5.4) Theorem / Definition (Grothendieck 1960: the fundamental group). *Let X be a connected scheme; let $s : \mathrm{Spec}(\Omega) \to S$ be a geometric point. There exists a pro-finite group* $\pi = \pi_1(S, s)$ called the fundamental group of (S, s) *and an equivalence between:*

(i) *the category of finite étale covers $Y \to X$ with Y connected, and*

(ii) *the category of finite sets (with discrete topology) with continuous π-action.*

The fundamental group as defined by Grothendieck is sometimes called the "algebraic fundamental group" or the "étale fundamental group" in case we want to emphasize a distinction with the topological fundamental group.

In practice this group $\pi_1(S, s)$ can be constructed by taking a projective system of all finite, étale Galois covers $Y_i \to G - i\backslash Y_i = X$, with Y_i connected, and constructing π as the projective limits of these finite groups G_i. - Make all definitions and notions used precise! - We will see that the fundamental group is a disguise of both the Galois group and of the topological fundamental group.

(5.5) Example. Let K be a field. There is an isomorphism, uniquely defined up to inner conjugations:

$$\pi_1(\mathrm{Spec}(K)) \quad \cong \quad \Gamma_K = \mathrm{Gal}(K^s/K).$$

(5.6) Theorem. Here is one of the fundamental facts. *Let K be a field, let $k \supset K$ be an algebraic closure, let $X \to \mathrm{Spec}(K)$ be a scheme over K, which is algebraic and geometrically connected over K (e.g. this is the case if we consider a variety V/K). Let $\overline{a}$ be a base point for $\overline{X} = X \otimes_K k$, i.e. a morphism $\overline{a} : \mathrm{Spec}(\Omega) \to \overline{X}$ where Ω is an algebraically closed field; let $a : \mathrm{Spec}(\Omega) \to X$ and $b : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)$ the induced base points (obtained by composition of morphisms). Hence we obtain morphisms of pointed schemes:*

$$(\overline{X}, \overline{a}) \longrightarrow (X, a) \longrightarrow (\mathrm{Spec}(K), b);$$

these induce an **exact sequence**

$$1 \to \pi_1(\overline{X}, \overline{a}) \longrightarrow \pi_1(X, a) \longrightarrow \pi_1(\mathrm{Spec}(K), b) \cong \mathrm{Gal}(K^s/K) \to 1.$$

See SGA1, Th. IX.6.1.

(5.7) Notation. Let $\pi = \lim_{\leftarrow}\{G_i \mid i \in I\}$ be a pro-finite group. Let ℓ be a prime number. Let $I(\ell)$ be the set of indices such that $\#(G_i)$ is a power of ℓ. Define

$$\pi(\ell) = \lim_{\leftarrow}\{G_i \mid i \in I(\ell)\}.$$

Let p be a prime number. Define $I^{(p)}$ as the set of indices such that $\#(G_i)$ is not divisible by p. Define

$$\pi^{(p)} = \lim_{\leftarrow}\{G_i \mid i \in I^{(p)}\}.$$

[Sometimes the notations $\pi(\ell)$ and $\pi^{(p)}$ are confused / interchanged, or whatever; please be systematic in this notation; please be careful when reading literature with other notations.]

(5.8) Exercise. *Show these pro-finite groups are well-defined. Show the existence of natural maps $\pi \to \pi(\ell)$ and $\pi \to \pi^{(p)}$. Show the kernels of these maps to be characteristic in π (a characteristic subgroup $H \subset \pi$ is by definition invariant under all automorphisms of π; here we mean invariant under all continuous automorphisms of π).*

Let $G/\pi = \Gamma$ be an exact sequence of pro-finite groups. This defines $R : \Gamma \to \mathrm{Out}(\pi)$. Show that this naturally defines exact sequence of pro-finte groups $G'/\pi(\ell) = \Gamma$ and $G''/\pi^{(p)} = \Gamma$ and naturally defines

$$R(\ell) : \Gamma \to \mathrm{Out}(\pi(\ell)) \quad \text{and} \quad R^{(p)} : \Gamma \to \mathrm{Out}(\pi^{(p)})$$

(write out the relevant definitions and diagrams).

(5.9) Remark. The following facts enable us to obtain information on algebraic fundamental group via topological considerations, to have more precise information about fundamental groups.

Let X be a scheme proper over an algebraically closed field k and let $k \subset \Omega$ be an inclusion, with Ω algebraically closed. Then

$$\pi_1(X) \xrightarrow{\sim} \pi_1(X \otimes \Omega).$$

Let X be an algebraic curve over an algebraically closed field $k \supset \mathbb{F}_p$. Then there exists an isomorphism

$$\Gamma^{\wedge}_{g,r}(\ell) \xrightarrow{\sim} \pi_1(X)(\ell);$$

here g is the genus of X, and r is "the number of holes" and ℓ is a prime number different from p.

Let X be a complete algebraic curve of genus g over an algebraically closed field $k \supset \mathbb{F}_p$. There exists a surjection

$$\Gamma^{\wedge}_{g,0}(p) \xrightarrow{\sim} \pi_1(X)(p).$$

Let X be an algebraic curve over an algebraically closed field $k \supset \mathbb{F}_p$. Write $\pi_1^{\mathrm{tame}}(X)$ for the pro-finte group classifying all finite, étale covers of X which extend to covers of X^c which are at most tamely ramified (ramification indices not divisible by p). There exists a surjection

$$\Gamma^{\wedge}_{g,r}(p) \xrightarrow{\sim} \pi_1^{\mathrm{tame}}(X)(p).$$

For all these facts, see [14].

6 The algebraic and topological fundamental group

(6.1) Comparison Theorems for étale morphisms, see SGA1 XII.5. *Let X be a $\mathbb{C}$-scheme, locally of finite type, and let $X^{\mathrm{an}} = X(\mathbb{C})$ be the corresponding analytic space* [in our case, we will often consider an algebraic curve over $\mathbb{C}$, and $X^{\mathrm{an}} = X(\mathbb{C}) = S$ will be a Riemann surface.] EGA1, Th. XII.5.1 *says that the category of finite étale morphisms above X and the same above X^{an} are equivalent.*

Note that this theorem is not so very difficult for algebraic curves, or for complete varieties.

(6.2) From this we conclude, see EGA1 Cor. XII.5.2 that for a connected, locally of finite type $\mathbb{C}$-scheme $X/\mathbb{C}$ we have the following comparison:

$$\pi_1^{\mathrm{top}}(X(\mathbb{C})) \quad \longrightarrow \quad \left(\pi_1^{\mathrm{top}}(X)\right)^\wedge \quad \xrightarrow{\sim} \quad \pi_1(X),$$

i.e. *the pro-finite completion of the topological fundamental group of $X(\mathbb{C})$ is equal to the fundamental group of X.*

(6.3) Here is a corollary of this comparison theorem, using the extension theorem of morphisms. Let k be an algebraically closed field of characteristic zero. Let X/k be an algebraic curve of genus g with $r(X) = r$ (the number of "punctures, the number of geometric points needed to "compactify" X). Then

$$\pi_1(X) \quad \cong \quad (\Gamma_{g,r})^\wedge\,;$$

for the notation $\Gamma_{g,r}$, see (2.7).

(6.4) Remark/Example. Note that we use topological considerations in order to describe the structure of an algebraic-geometric object like the algebraic fundamental group of a curve. As far as I know there is proof of the previous fact directly. For example, we now know that

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}) \cong (\mathbb{Z} * \mathbb{Z})^\wedge$$

but we do not know a proof of this without topological considerations (and, as Gabber told me, one can prove that this isomorphism suffices to determine the structure of the fundamental group of any algebraic curve along "algebraic" lines).

(6.5) Open Problem. *Can we determine the fundamental group of any algebraic curve along algebraic lines?*

(6.6) Remark. Let $\mathrm{char}(K) = p > 0$. Suppose $X = \mathbb{A}^1_K = \mathbb{P}^1_K - \{\infty\}$. The fundamental group $\pi_i(X)$ is "very large"; in fact the precise structure of this group is still "unknown".

7 Monodromy

(7.1) Exercise. *Let $K = \mathbb{Q}(t)$; let $n \in \mathbb{Z}_{>0}$. Let $f := X^n - t \in K[X]$. Let $L = L_n := \Omega^f_K$ be the splitting field of f over K. Show:*

- *There is a primitive n-th root of unity $\zeta = \zeta_n$ contained in L.*

- *Define $C = C_n = K(\zeta) \subset L$. Show that $C = \mathbb{Q}(\zeta)(t)$. Show C/K is Galois. Show that a choice of ζ gives an isomorphism* $\mathrm{Gal}(C/K) =: \Gamma \cong (\mathbb{Z}/n)^*$. *[Here R^* denotes the group of units in R.]*

- *Show that a choice of a zero of f in L,and a choice of ζ gives an isomorphism* $\mathrm{Gal}(L/C) =: N \cong \mathbb{Z}/n$.

- *Consider the exact sequence*

$$1 \to N \cong \mathbb{Z}/n \longrightarrow G := \mathrm{Gal}(L/K) \xrightarrow{p} \Gamma \cong (\mathbb{Z}/n)^* \to 1.$$

Use the same ζ for the two isomorphisms, and determine the representation $R : \Gamma \cong (\mathbb{Z}/n)^* \to \mathrm{Aut}(N \cong \mathbb{Z}/n)$ determined by this exact sequence (what could it be ...?).

(7.2) Fact (see [61], IV.3.2): *the algebraic closure of $\mathbb{Q}((t))$ equals:*

$$\overline{(}\mathbb{Q}((t))) \quad = \quad \cup_n \quad \mathbb{Q}((t))(\sqrt[n]{t}).$$

(7.3) Exercise (Grothendieck). *Let $K = \mathbb{Q}((t))$, and let*

$$k := \overline{K} \quad \supset \quad C := \overline{\mathbb{Q}}((t) \quad \supset \quad K.$$

We have

$$N := \mathrm{Gal}(k/C) \cong \widehat{\mathbb{Z}}, \qquad \Gamma_K := \mathrm{Gal}(C/K) \cong \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}),$$

and we have an exact sequence

$$1 \to N \longrightarrow G := \mathrm{Gal}(k/K) \xrightarrow{p} \Gamma_K \to 1.$$

Let $N \in \mathbb{Z}_{>0}$, suppose ℓ is a prime number, and suppose

$$\rho : \mathrm{Gal}(k/K) = G \longrightarrow \mathrm{GL}(N, \mathbb{Z}_\ell)$$

is a homomorphism of groups into the multiplicative group of invertible $N \times N$-matrices over $\mathbb{Z}_\ell$. Let $1 \in \mathbb{Z} \subset \widehat{\mathbb{Z}} = N$. Prove that every eigenvalue of $\rho(1)$ is a root of unity.

For a more general form of this fact, see [47], the Proposition on page 515.

8 Geometric class field theory

(8.1) Motivation / topological formulation: Exercise. Suppose X is a complete curve over $\mathbb{C}$ of genus g; let $X \to J = \mathrm{Jac}(X)$ be its Jacobian.
(1) *Show that we have isomorphisms:*

$$\left(\pi_1^{\mathrm{top}}(X(\mathbb{C}))\right)_{\mathrm{ab}} \quad \xrightarrow{\sim} \quad \pi_1^{\mathrm{top}}(J(\mathbb{C})) \quad \cong \quad \mathrm{H}_1(X(\mathbb{C}), \mathbb{Z}) \quad \cong \quad \mathbb{Z}^{2g}.$$

(2) *Let $\varphi : Y \to X$ be a finite abelian Galois cover of complete algebraic curves over $\mathbb{C}$. Show there exists an isogeny $\beta : B \to J$ of abelian varieties and a pull-back diagram:*

$$\begin{array}{ccc} Y & \longrightarrow & B \\ \varphi \downarrow & & \downarrow \beta \\ X & \longrightarrow & J. \end{array}$$

(8.2) Discussion: Albanese versus Picard. For a regular, complete variety V over a field K, and the choice of $P \in V(K)$ we can define the Albanese variety of V:

$$\tau : V \quad \longrightarrow \quad \mathrm{Alb}(V) = A_V;$$

this is the universal solution of mapping V into an abelian variety; see [24], II.3

For a complete variety V over a field K one can define its Picard variety (the connected component of a "universal solution to the problem of parameterizing divisor classes on V"), see [24], II.3, VI.1 and VI.4 for the case V is complete and regular (much more general theorems about the existence of Picard schemes exist; we will not go into those details here). In case V is regular, we obtain an abelian variety $\mathrm{Pic}_V = P_V$, and a divisor class (the Poincaré bundle) on the product $V \times P_V$ describing the parameterization of divisor classes.

For a regular, complete variety V over a field K, with $V(K) \neq \emptyset$ both notions are defined, and in fact the abelian variety A_V is isomorphic with the dual abelian variety of P_V:

$$A_V \quad \cong \quad (P_V)^t,$$

see [24], Th. 1 on page 148. As $(P_V)^{tt} \cong P_V$ we also obtain $(A_V)^t \cong P_V$ (double duality on abelian varieties is the identity: Cartier and Nishi; for references, see [41], Section 20).

Suppose A_V admits a principal polarization; in that case $A_V \cong (A_V)^t \cong P_V$; in this case, sometimes we "identify" the Albanese and the Picard variety; however we should realize that A_V comes with $\tau : V \to A_V$, and P_V comes with the Poincaré class. For a complete, nonsingular algebraic curve X these abelian varieties have a principal polarization, "the canonical polarization", we write $A_X = \mathrm{Jac}(X) = P_X$, and we call this "the Jacobian variety"; however, in applications, please realize well which aspect of the Jacobian is used. In the exercise above, and in the application below we use the Albanese property $X \to J = \mathrm{Alb}(X)$.

(8.3) Class field theory for algebraic curves. In [GA]:
J-P. Serre – *Groupes algébriques et corps de classes,*
we find a formulation of class field theory of curves over an arbitrary algebraically closed ground field $k = \overline{k}$.

Let C be a complete (regular, irreducible) algebraic curve over k, and let $X \subset C$ be an open part. Let $D \to C$ be a finite separable morphism, which is a finite Galois cover with **abelian** Galois group Π, and which is unramified over X
(note: conversely an étale cover of X can be extended to a finite cover $C = X^c$).

Then, see [GA] Chapter V, Chapter VI, especially see VI.11, Proposition 9 on page 126:
– there exists an effective divisor m on C with support on $C - X$, a "module" in the terminology of [GA],
– we can define a curve C_{m} (which is singular if the degree of m is at least two), and

$$J_{\mathsf{m}} := \mathrm{Jac}(C_{\mathsf{m}}) = \mathrm{Pic}^0_{C_{\mathsf{m}}},$$

the "generalized Jacobian of C_{m}",

– a choice of $P_0 \in X$ gives $i : X \to J$,

– and there exists an isogeny of group schemes $G \to J = G/\Pi$ and a commutative diagram

$$\begin{array}{ccccc} D \supset & Y & \longrightarrow & G \\ & \downarrow & & \downarrow \\ C \supset & X & \longrightarrow & J. \end{array}$$

We see that Π is a quotient of $\pi_1(J_{\mathsf{m}})$; the pull-back by $X \to J$ of $G \to X$ gives a Galois cover $Y \to \Pi\backslash Y = X$, which after going to the regular compactifications, gives the (possibly ramified) cover $D \to \Pi\backslash D = C$.

(8.4) For higher dimensional class field theory see [23], Lemma 5 on page 308.

9 Properties of good reduction for abelian varieties.

Situation: K is a field, v is a discrete valuation on K, we consider K^s, a perfect closure of K, and an extension $\overline{v}$ of v to K^s. Let $I(\overline{v}) \subset \Gamma_K := \mathrm{Gal}(K^s/K)$ be the inertia subgroup defined by $\overline{v}$.

We fix a prime number ℓ different from the residue characteristic of v (and hence different from the characteristic of K).

Let A be an abelian variety over K. We consider

$$T_\ell(A) \quad := \quad lim_{\leftarrow} A[\ell^i](K^s),$$

the Tate-ℓ-group of A.

Exercise. *Show that $T_\ell(A) = \pi_1(A \otimes K^s)(\ell)$.*

This is a pro-finite group, isomorphic to $T_\ell(A) \cong (\mathbb{Z}_\ell)^{2g}$, where $g = \dim A$, with a continuous action $\rho = \rho_{A,\ell} : \Gamma_K := \mathrm{Gal}(K^s/K) \quad\longrightarrow\quad \mathrm{Aut}(T_\ell(A))$. [Give two proofs for the existence of this representation.]

(9.1) Theorem (Criterion of Ogg-Néron-Shafarevich; see [47], Theorem 1 on page 493): *The abelian variety A has good reduction at v iff $\rho_{A,\ell}(I(\overline{v})) = \{1\}$.*

See [47], see [15] for further properties.

10 I: Properties of good reduction (following Takayuki Oda)

We will discuss a criterion of good reduction for curves as formulated by Takayuki Oda, see [4]; also see [39], [40].

These papers contain a lot of results, and some notations are complicated. We propose to do the "easiest case". [In the seminar we will first consider the case of equal-characteristic zero. And the case of complete curves.]

(10.1) Exercise. Construct an example of K, v as above, and a complete, regular algebraic curve X over K with Jacobian $\mathrm{Jac}(X) = J$, such that J has good reduction at v and X has bad reduction at v.

(10.2) Here is the central result:
Theorem (Takayuki Oda). *Suppose K is a field with a discrete valuation v, and residue class field k. Let X be a complete curve over K (proper over* $\mathrm{Spec}(K)$*, regular and absolutely irreducible) of genus $g \geq 2$. Choose a prime number ℓ different from the characteristic of k. Then we obtain a representation*

$$\rho_{X,v,\ell} = \rho : I(\overline{v}) = I \longrightarrow \mathrm{Out}(\pi_1(X \otimes \overline{K})(\ell)).$$

We have:

$$X \ \ \textit{has good reductions at} \ \ v \quad \Longleftrightarrow \quad \rho(I) = \{1\}.$$

In fact:

$$X \ \ \textit{has good reductions at} \ \ v \quad \Longleftrightarrow \quad \rho_{X,v,\ell,4}(I) = \{1\},$$

where

$$\rho_{X,v,\ell,n} : I \longrightarrow \mathrm{Out}(\Pi/\Gamma_n(\Pi)).$$

Here $\Pi = \pi_1(X \otimes \overline{K})(\ell))$ and $\Gamma_n(\Pi)$ is the n-th step in the lower central series, defined by: $\Gamma_1(\Pi) = \Pi$ and the next steps are defined inductively as commutator subgroups: $\Gamma_{n+1}(\Pi) := [\Gamma_n(\Pi), \Pi]$.

(10.3) **Exercise.** *Show that the kernel $\pi_1(X \otimes \overline{K}) \to \pi_1(X \otimes \overline{K})(\ell)$ is a characteristic subgroup (i.e. invariant under automorphisms). Show the ρ in the theorem is well-defined.*

(10.4) **Suggestion.** Follow first the paper [40]. In that paper it is assumed that we work over a number field; show that is an unnecessary assumption. The crucial case is the case $g = 2$. Make a careful study of the geometric fundamental group of a complete curve over genus $g = 2$. Study its fundamental group either topological or étale. Study the "lower central series" in this group. Make a careful analysis as in [40], pp. 1.13. If you understand this completely, this is the "computation part", then as later will be proved, the rest of the theorem is more of a general-nonsense-proof.

11 II: Correspondences on algebraic curves (following S. Mochizuki)

We will discuss the contents of the paper [28]. In this section all base fields are supposed to be *of characteristic zero.*

(11.1) **Exercise.** *Let $f : X \to Y$ be an étale, finite covering of algebraic curves* (mind, over a field of characteristic zero). *Then:*

$$X \textit{ is hyperbolic} \Longleftrightarrow Y \textit{ is hyperbolic.}$$

Here is an interesting question. Consider algebraic curves over $\mathbb{C}$. Say that Y_1 and Y_2 are "isogenous" if there exist étale, finite coverings $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$. Show this is an equivalence relation. What can be said about the curves in one equivalence class having a fixed Euler characteristic (having a fixed genus if we consider complete curves)? Mochizuki proves that for *hyperbolic curves* this set is finite in every isogeny-equivalence class for every choice of an Euler characteristic.

The paper [28] is not so difficult te read, except for the fact that a result by Margulis, see [26], Theorem 27 on page 337, Lemma 3.1.1(v) on page 60, and a result by Takeuchi, see [57], Theorem 2.1, are used.

We could accept these results as "black boxes" and then try to onderstand the arguments and results of [28]. Beautiful ideas!

12 Anabelian groups

Material of this section will be expanded. Probably we will write a proof of (12.3).

(12.1) Definition. A topological group G is called "anabelian" if for every closed, finite index subgroup $H \subset G$ has trivial center: $\mathrm{Cent}(H) = \{1\}$. [A group G is called "anabelian" if for every finite index subgroup $H \subset G$ has trivial center.]

(12.2) An example: *For every number field $[K : \mathbb{Q}] < \infty$ the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ is anabelian,* [38], 12.1.6.

Here is another example:

(12.3) Exercise. *Let k be an algebraically closed field, and let X be a* **hyperbolic** *curve over k. The fundamental group $\pi_1(X)$ is anabelian.*
For the case that X is complete, see [9], Lemma 1 on page 133.
As an appetizer, you might first try the case $k = \mathbb{C}$ and X is a complete curve of genus $g \geq 2$.

(12.4) Exercise. Let k be an algebraically closed field of characteristic *zero*, and $k \subset \mathbb{C}$. Let X be an algebraic curve over k, of genus g and r holes, i.e. $r = \#(X^c(k) - X(k))$. We write $\pi := \pi_1(X)$. Show:
(a) $\pi = \{1\}$ *iff* $(g, r) = (0, 0), (0, 1)$.
(b) π *is non-trivial abelian iff* $(g, r) = (0, 2), (1, 0)$.
(c) *the following are equivalent*:
 (c1) π *is non-abelian,*
 (c2) π *is anabelian,*
 (c3) X *is hyperbolic.*

Suggestion: formulate the previous exercise for $\pi := \pi_1^{\mathrm{top}}(X(\mathbb{C}))$, and prove the new version to be correct.

(12.5) Exercise. Let k be an algebraically closed field of characteristic $p > 0$. Let X be an algebraic curve over k, of genus g and r holes, i.e. $r = \#(X^c(k) - X(k))$. We write $\pi := \pi_1(X)$. Show:
(a) $\pi = \{1\}$ *iff* $(g, r) = (0, 0)$.
(b) π *is non-trivial abelian iff* $(g, r) = (1, 0)$.

(c) *Show that* π *is non-abelian in the following cases:* $(0,1)$ *and* $(0,2)$.
(d) *the following are equivalent* (! mind, $\mathrm{char}(k) = p > 0$):
(c1) π *non-abelian,*
(c2) π *is anabelian,*
(c3) $(g,r) = (0,1)$ *or* $(g,r) = (0,2)$ *or* X *is hyperbolic.*

(12.6) Remark. we have shown that for an algebraic curve X over an algebraicallly closed field k and $\pi := \pi_1(X)$ we have:
π *is non-abelian* $\Longleftrightarrow$ π *is anabelian.*

(12.7) Exercise. Let k be an algebraically closed field. Let X be an algebraic curve over k. We write $\pi := \pi_1(X)$. Let ℓ be a prime number different from the characteristic of k. Show:

$$X \textit{ is hyperbolic} \Longrightarrow \pi_1(X)(\ell) \textit{ is anabelian.}$$

13 III: The anabelian Grothendieck conjectures (following Nakamura, Tamagawa and Mochizuki)

Here we have a choice which form of the anabelian Grothendieck conjectures we are going to study. For the case of algebraic curves over finite fields, over number field, or over sub-p-adic fields there are four options:
III.a Rational hyperbolic curves, following Nakamura; see [34].
III.b Affine curves over finite fields and over number fields, following Tamaga; see [58].
III.c Complete curves over number fields following Mochizuki; see [27] (this paper uses the paper by Tamagawa in III.b).
III.d Hyperbolic curves over sub-p-adic fields following Mochizuki; see [29] (this paper reproves te main result of III.c).
None of these papers is elementary...!

(13.1) Possible confusion. Grothendieck defines:
"**anabelian schemes**": “Schemata (von endlichem Typ) über K, die Geometrie von X vollstaňdig durch die (profinite) Fundamentalgruppe $\pi_1(X,\xi)$ bestimmt ist... ” (Letter to Faltings, June 1983, see [17]).

We defined a group to be anabelian, see (12.1).

One might expect (the anabelian Grothendieck conjectures) that a scheme over a field of finite type over its prime field with an anabelian geometric fundamental group is determined by its fundamental group. This is true for algebraic curves; here “hyperbolic” $\Leftrightarrow$ “geometric π_1 is anabelian”. However for higher dimensions it is not clear that the condition the geometric fundamental group being anabelian is the right condition.

Therefore it seems wise to distinguish:
The Grothendieck anabelian philosophy: *Try to find a class of schemes (varieties, fields, ...) such that these are determined by their fundamental groups.* Such objects can be called “anabelian objects in the Grothendieck philosophy”.
And:

Anabelian geometric fundamental groups.
And ask:
Question. *Which varieties should be baptized " anabelian in the Grothendieck philosophy"?*

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