Open problems in Algebraic Geometry

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All problems in this list have been reviewed by at least one referee. In this process many useful suggestions and new references have come up. We thank all referees for their valuable comments.

Problem 1. Proposed by Dick Gross.

Question. Are there any examples of complex abelian varieties of dimension 4 with Mumford-Tate group isogenous to a $\mathbb{Q}$-form of $\mathbb{G}_m \times (\mathrm{SL}_2)^3$ which are Jacobians of curves?

Explanation. The simplest example of a "Shimura subvariety" $S \subset A_{g,1}$ which is not of PEL-type occurs for $g = 4$. The generic Mumford-Tate group in this case is isogenous to a $\mathbb{Q}$-form of $\mathbb{G}_m \times (\mathrm{SL}_2)^3$. The resulting varieties S are 1-dimensional and complete; they parametrize abelian 4-folds X with (generically) $\mathrm{End}(X) = \mathbb{Z}$ but with a certain "exceptional" Hodge class in $H^4(X^2)$. See [Mu], [MZ] and [No].

One would like to understand the intersection of the Shimura curves S obtained in this manner with the Torelli locus. See also [Oo], section 7, where it is asked whether there exists an S as above which is fully contained in the (closed) Torelli locus.

References

[MZ] B.J.J. Moonen and Yu.G. Zarhin, *Hodge classes and Tate classes on simple abelian fourfolds*, Duke Math. J. **77** (1995), 553–581.

[Mu] D. Mumford, *A note of Shimura's paper "Discontinuous groups and abelian varieties"*, Math. Ann. **181** (1969), 345–351.

[No] R. Noot, *On Mumford's families of abelian varieties*, to appear in J. Pure Appl. Algebra.

[Oo] F. Oort, *Some questions in algebraic geometry*, June 1995.

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Problem 2. Suggested by Richard Taylor to Dick Gross.

What is the connection between the weight $(k_1, k_2, \ldots, k_d)$ of a Hilbert modular form for $\mathrm{GL}_2(F)$ and the restriction of the mod $\mathfrak{p}$ Galois representation to the decomposition groups at the primes dividing p in the totally real field F? (Here $\mathfrak{p}$ is a prime dividing p in the ring of coefficients of the form.) For $F = \mathbb{Q}$ the answer to the question is known; a reference is Bas Edixhoven, *The weight in Serre's conjectures on modular forms*, Invent. math. **109** (1992), 563–594.

Problem 3. Proposed by Dick Gross.

Here is a problem, which I think would shed a lot of light on arithmetic geometry (if we could see how to approach it). What is the zeta function of the moduli space of stable curves of genus g over $\mathbb{Z}$?

Problem 4. Proposed by Frans Oort.

Conjecture. Let (X, λ) be a generic supersingular abelian variety of dimension $g \geq 2$. Let k be an algebraically closed field containing a field of definition of X. Then $\mathrm{Aut}\big((X, \lambda) \otimes k\big) = \{\pm 1\}$.

Open problem. Suppose $g \in \mathbb{Z}_{\geq 3}$, and p is a prime number. Suppose there exists a supersingular curve of genus g in characteristic p. Is it true that there exists a supersingular curve C of genus g over $\overline{\mathbb{F}}_p$ such that $\mathrm{Aut}(C) = \{e\}$? (See also problem number 19 in this list.)

Problem 5. Proposed by Richard Pink.

A finite simple group is called *sporadic* if it is neither abelian, nor alternating, nor of Lie type (including Suzuki and Ree). By the classification of finite simple groups, there exist precisely 26 isomorphism classes of such sporadic groups. For many qualitative statements about finite groups, however, it is enough to know that their number is finite. Unfortunately, the proof of the full classification theorem is extremely involved, inaccessible to non-experts, and does not provide a conceptual explanation for the mere finiteness.

For certain applications an even weaker result on sporadic groups suffices. For example, for any set H of positive integers consider the series $\zeta_H(s) := \sum_{n \in H} n^{-s}$. If Spor is the set of orders of sporadic finite simple groups, the classification implies that $\zeta_{\mathrm{Spor}}(s)$ is really a finite sum. But perhaps one can prove by easy elementary means that Spor is sparse. Specifically: *Prove that $\zeta_{\mathrm{Spor}}(s)$ converges for $\mathrm{Re}(s) > 0$, without using the classification of finite simple groups.*

Michael Larsen considered the above series in the following situation. Let Hur be the set of orders of finite groups that are generated by three elements x, y, z of orders 2, 3, and 7, respectively, and satisfying $xyz = 1$. From classical results of Hurwitz it follows that these are precisely the numbers $84(g-1)$ for which there exists a compact connected Riemann surface of genus $g \geq 2$ possessing this theoretically maximal number of automorphisms.

Larsen proves that $\zeta_{\mathrm{Hur}}(s)$ converges for $\mathrm{Re}(s) > 1/3$ but has a singularity at $s = 1/3$. Thus, loosely speaking, these values of g are about as common as perfect cubes. In the course of his proof, Larsen needs the fact that $\zeta_{\mathrm{Spor}}(s)$ has abscissa of convergence less than $1/3$. It would be nice to eliminate the dependence on the classification of finite simple groups at this point. It would also be nice to determine the set Hur in more detail, or at least the part that is responsible for the singularity at $s = 1/3$.

Reference

[1] M.J. Larsen, How often is $84(g-1)$ achieved?, Preprint April 2000, e-mail: `larsen@math.indiana.edu`.

Problem 6. Proposed by Richard Pink.

Let $\mathbb{C}$ be an algebraically closed complete normed field of characteristic $p > 0$. Consider the punctured open unit disc $D^\bullet := \{\zeta \in \mathbb{C} : 0 < |\zeta| < 1\}$. Let q be a power of p, and let $\sigma\colon D^\bullet \to D^\bullet$ denote the morphism $\zeta \mapsto \zeta^q$. Let $\mathcal{M}$ be a vector bundle of rank n on $D^\bullet$ in the sense of rigid analytic geometry. Consider an $\mathcal{O}_{D^\bullet}$-linear isomorphism $\tau\colon \sigma^*\mathcal{M} \to \mathcal{M}$. *Conjecture: $\mathcal{M}$ can be extended to a vector bundle $\overline{\mathcal{M}}$ on the full open disc $D := D^\bullet \cup \{0\}$ such that τ extends to a homomorphism $\sigma^*\overline{\mathcal{M}} \to \overline{\mathcal{M}}(r \cdot [0])$ for some integer r.*

If $\mathcal{M}$ is free, this conjecture can be phrased in elementary terms, as follows. Let R denote the ring of Laurent series $\sum_{i\in\mathbb{Z}} a_i z^i$ with $a_i \in \mathbb{C}$ which converge on $D^\bullet$, and let $R' \subset R$ be the subring consisting of Laurent series with finite principal part, i.e., which have no essential singularity at $z = 0$. Consider the endomorphism ${}^\sigma\big(\sum_{i\in\mathbb{Z}} a_i z^i\big) := \sum_{i\in\mathbb{Z}} a_i^q z^i$ of R. It is extended coefficientwise to $n \times n$-matrices over R. *Conjecture: For every $A \in \mathrm{GL}_n(R)$ there exists $B \in \mathrm{GL}_n(R)$ such that $B^{-1} \cdot A \cdot {}^\sigma B \in \mathrm{GL}_n(R')$.*

Explanation: The automorphism σ of $D^\bullet$ is properly discontinuous, so the quotient $D^\bullet/\sigma^{\mathbb{Z}}$ exists as a locally ringed space. The Grothendieck topology defining the rigid analytic structure of $D^\bullet$ also descends to the quotient, although not in the usual rigid analytic sense, because σ acts non-trivially on the base field $\mathbb{C}$. Nevertheless, the quotient may be viewed as a strange kind of curve, sharing some properties with $\mathbb{P}^1$ and others with elliptic curves. The pair $(\mathcal{M}, \tau)$ may be viewed simply as a vector bundle on $D^\bullet/\sigma^{\mathbb{Z}}$.

For any integer r let $\mathcal{O}_{D^\bullet}(r)$ denote the structure sheaf $\mathcal{O}_{D^\bullet}$ together with the isomorphism ${}^\tau f(z) := z^{-r} \cdot {}^\sigma f(z)$. Under some extra assumptions on $\mathbb{C}$ one can easily show that every $(\mathcal{M}, \tau)$ of rank 1 is isomorphic to $\mathcal{O}_{D^\bullet}(r)$ for unique r. In any case $\mathcal{O}_{D^\bullet}(1)$ should play the role of the standard twisting sheaf from algebraic geometry. Specifically, let us abbreviate $\mathcal{M}(r) := \mathcal{M} \otimes \mathcal{O}_{D^\bullet}(r)$. Then the above conjecture for given $\mathcal{M}$ is equivalent to the statement that $\mathcal{M}(r)$ is "generated by global sections" for all $r \gg 0$, or again to the vanishing of $H^1(D^\bullet/\sigma^{\mathbb{Z}}, \mathcal{M}(r))$ for all $r \gg 0$. Thus many of the standard techniques of algebraic geometry can be brought to bear on such $\mathcal{M}$.

The problem arose in studying uniformizability of t-motives, where the desired property holds by construction of the respective $\mathcal{M}$. In phrasing the above problem as a conjecture, the proposer simply follows his intuition from algebraic geometry; he has no other evidence for it. For further information please contact the proposer.

Problem 7. Proposed by Richard Pink.

Let X be an irreducible smooth projective curve over an algebraically closed field k of characteristic p. Let $\mathcal{F}$ be a constructible étale sheaf of $\mathbb{F}_\ell$-vector spaces on X. If $\ell \neq p$, the formula of Grothendieck-Ogg-Shafarevich expresses the Euler characteristic $\chi(X,\mathcal{F})$ in terms of the generic rank of $\mathcal{F}$, the genus g_X of X, and purely local information at the points where $\mathcal{F}$ is not lisse. Assume now that $\ell = p$. Then the p-rank of a curve does not depend on such data alone; hence there cannot be such a general formula for the Euler characteristic. However, the p-rank of a curve is bounded by its genus, so one can nevertheless try to bound the Euler characteristic of arbitrary $\mathcal{F}$ in a non-trivial way. To be precise:

Problem: Determine the sharpest possible lower bound for the Euler characteristic $\chi(X,\mathcal{F})$ that involves only the generic rank of $\mathcal{F}$, the genus of X, and local information on $\mathcal{F}$.

In [2, Prop. 7.1] such a bound is given under certain rather strong restrictions on the local ramification in $\mathcal{F}$. For simplicity we illustrate this result only in an easy special case. Let $\pi\colon Y \to X$ be an irreducible finite Galois covering with Galois group G, and set $\mathcal{F} := \pi_*\mathbb{F}_p$. Then $\chi(X,\mathcal{F}) = \chi(Y,\mathbb{F}_p) = 1 - h_Y$, where $h_Y := h^1(Y,\mathbb{F}_p)$ is the p-rank of Y. The stabilizer G_y of any closed point $y \in Y$ acts on the local ring $\mathcal{O}_{Y,y}$ and its maximal ideal $\mathfrak{m}_{Y,y}$. Let $G_{y,i}$ denote the kernel of its action on $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}^{i+1}$. Since k is algebraically closed, we have $G_{y,0} = G_y$. The wild inertia subgroup is $G_{y,1}$. Let g_X and g_Y denote the genus of X and Y, respectively. The Hurwitz genus formula asserts that

$$1 - g_Y = (1 - g_X)\cdot|G| + \sum_{y\in Y}\sum_{i\geq 0}\frac{1-|G_{y,i}|}{2}.$$

Now consider the condition

$$(*) \qquad G_{y,2} = 1 \quad \text{for all } y \in Y.$$

If this holds, the inequality $h_Y \leq g_Y$ and the genus formula imply

$$(\dagger) \qquad 1 - h_Y \geq (1 - g_X)\cdot|G| + \sum_{y\in Y}\left(1 - \frac{|G_y| + |G_{y,1}|}{2}\right).$$

It was proved by Nakajima [1] that $(*)$ holds whenever Y is ordinary, and the inequality is then an equality. If $(*)$ holds but Y is not ordinary, the inequality is strict. In the general case of non-ordinary Y there are—ceteris paribus—two effects: On the one hand the genus of Y becomes larger, giving room for h_Y to increase. On the other hand the "defect" $g_Y - h_Y$ will also tend to increase. How do these opposing tendencies compare to each other? Does one of them dominate the other? For example:

Question: Does the inequality $(\dagger)$ hold without the assumption $(*)$? More generally, does the inequality in [2, Prop. 7.1] hold without the assumption $(*)$?

The proposer tends to expect a positive answer, because the right hand side of $(\dagger)$, in its sheaf theoretic form of [2, Prop. 7.1], is the only reasonable formula that obeys the requirements stated in the problem above and that is also additive in short exact sequences. But he has no other evidence for it. During the conference the proposer made a bet with Frans Oort about this for a bottle of wine. Everybody is invited to help settle our bet.

References

[1] S. Nakajima, p-ranks and automorphism groups of algebraic curves, *Transact. Amer. Math. Soc.* **303** (1987), 595–607.

[2] R. Pink, Euler-Poincaré Formula in equal characteristic under ordinariness assumptions, *Manuscripta Mathematica* **102** (2000) 1–24; or to obtain through the URL `http://www.math.ethz.ch/~pink/preprints.html`

Problem 8. Proposed by Chia-Fu Yu.

This question may interest the people who are interested in rigid analytic geometry or Hilbert modular forms. Let F be a totally real field of degree d and p a rational prime inert in F. Let B be a quaternion algebra over F such that B is unramified outside p and the infinite places and every archimedean place is ramified. Let $*$ be a positive involution of B and O_B be a maximal order which is stable under the involution. We define an algebraic group G over $\mathrm{Spec}(\mathbb{Z})$ as follows:

$$G(R) = \left\{ g \in (O_B \otimes R)^\times \;\middle|\; g^* g \in R^\times \right\},$$

where R is a commutative ring.

Question: Is there any relation between the double coset

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}) \tag{$*$}$$

and the space of Hilbert modular forms of certain weights and level? If it is so, is there any explanation from geometry for this?

The motivation of this question is the following (when $d = 1$). Let D be the quaternion algebra over $\mathbb{Q}$ ramified exactly at $\{p, \infty\}$. Taking $G = D^\times$, the double coset $(*)$ is in bijective correspondence with the set of supersingular elliptic curves, up to isomorphism over $\overline{\mathbb{F}}_p$. Deligne and Rapoport showed that $X_0(p)$ has semi-stable reduction at p and $X_0(p) \otimes \overline{\mathbb{F}}_p$ has 2 irreducible components with transversally intersection at supersingular points and each component is isomorphic to the j-line under projection. Each supersingular point gives an "annulus" in the generic fibre $X_0(p) \otimes \mathbb{Q}_p$ as a rigid analytic curve. Therefore we have

$$\mathrm{genus}\big(X_0(p) \otimes \mathbb{Q}_p\big) + 1 = \text{the number of supersingular elliptic curves}.$$

Hence, we have:

$$\dim S_2\big(\Gamma_0(p)\big) + 1 = \# G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}).$$

There might be a natural surjective map

$$\Gamma\big(G(\mathbb{Q})\backslash G(\mathbb{A}_f)/G(\hat{\mathbb{Z}}), \mathbb{C}\big) \longrightarrow S_2\big(\Gamma_0(p)\big),$$

which can possibly be deduced from the global Jacquet-Langlands correspondence on GL_2.

In a recent work of Stamm, he showed that Hilbert-Blumenthal moduli scheme with $\Gamma_0(p)$-level, when p is inert in F, has semi-stable reduction. (See H. Stamm, *On the reduction of the Hilbert-Blumenthal-moduli scheme with $\Gamma_0(p)$-level structure*, Forum Math. **9** (1997), 405–455.) Is it possible to conclude any information to my question from the rigid analytic point of view?

Remarks.
(1) It was pointed out to me that the case $d = 1$ was already done by J-P. Serre. The reference is: Jean-Pierre Serre, *Two letters on quaternions and modular forms (mod p)*, Israel J. Math. **95** (1996), 281–299; reprinted as Nr. 169 in Serre's Oeuvres (Collected papers), Volume IV; Springer-Verlag, 2000. An approach to forms (mod p) on groups other than GL_2 and their connections with Galois representations was developed by B. Gross. See B. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93.

(2) A connection between the space of functions on the double coset $(*)$ and the space of Hilbert modular forms of weight 2 is a consequence of the Jacquet-Langlands correspondence on $\mathrm{GL}_{2,F}$. A reference is: S. Gelbart, *Automorphic forms on adèle groups*, Ann. math. Studies **83** (1975). I would like to thank R. Taylor for the discussion on the connection.

Problem 9. Proposed by Shinichi Mochizuki.

A *hyperbolic curve of type* (g, r) (i.e., genus g, with r marked points, where $2g - 2 + r > 0$) in characteristic $p > 2$ is called *"hyperbolically ordinary"* (cf. [1], Chapter II, §3, Definition 3.3) if it admits at least one *indigenous bundle* (i.e., a bundle of projective lines over the curve, equipped with a connection satisfying certain properties — cf. [1], Chapter I, §2, Definition 2.2) which is *nilpotent* (i.e., whose p-curvature, as a 2 by 2 matrix, has square equal to zero — cf. [1], Chapter II, §2, Definition 2.4) and *ordinary* (i.e., lies in the locus of the moduli stack of hyperbolic curves equipped with a nilpotent indigenous bundle which is *étale* over the moduli stack of hyperbolic curves — cf. [1], Chapter II, §3, Definition 3.1). Unlike the case for *elliptic curves* (or abelian varieties), where, in every positive characteristic, it is well known that there exist *nonordinary curves*, for *hyperbolic curves*, there exist cases — for instance, $(g, r) = (1, 1)$, $p = 5$ — in which *every hyperbolic curve of that type in that characteristic is hyperbolically ordinary* (cf. [2], Chapter IV, §1, the second Remark following Theorem 1.4).

Then the question is: Is it the case that all hyperbolic curves of all types (g, r) in all characteristics $p > 2$ are hyperbolically ordinary (cf. the list of "Open Problems" in [2], Introduction, §2.1, Problem 1)?

References
[1] S. Mochizuki, A Theory of Ordinary p-adic Curves, *Publ. RIMS, Kyoto Univ.* **32**, No. 6, pp. 957-1151 (1996).

[2] S. Mochizuki, *Foundations of p-adic Teichmüller Theory*, AMS/IP Studies in Advanced Mathematics **11**, American Mathematical Society/International Press (1999).

Problem 10. Proposed by Shinichi Mochizuki.

Let X_0 be a *hyperbolic curve* over a *finite field* k. It is known that any hyperbolic curve of type (g,r) in characteristic $p>2$ admits $\leq p^{3g-3+r}$ distinct *nilpotent ordinary indigenous bundles* (cf. the Problem above; [1], Chapter II, §2, Theorem 2.3, and the discussion following this theorem). Moreover, for each choice of such a nilpotent ordinary indigenous bundle $\mathcal{P}_0$ on X_0, there is associated to $(X_0,\mathcal{P}_0)$ a *canonical lifting* of this pair to a pair $(X,\mathcal{P})$ over the ring of Witt vectors $W(k)$ (cf. [1], Chapter III, §3, Theorem 3.2). This canonical lifting may be thought of as the *hyperbolic curve analogue of the Serre-Tate canonical lifting of an ordinary abelian variety* (cf. [2], Introduction, §0.9, the discussion surrounding Theorem 0.3).

Then the question is: When, if ever, is this canonical lifting $(X,\mathcal{P})$ defined over a number field (cf. the list of "Open Problems" in [2], Introduction, §2.1, Problem 7)?

(References as in the previous problem.)

Problem 11. Proposed by Ben Moonen and Frans Oort.

Let k be an algebraically closed field of characteristic $p>0$. By a BT_1 over k we mean a finite commutative k-group scheme $\mathcal{G}$, killed by p, with the property that the sequence

$$\mathcal{G} \xrightarrow{F_{\mathcal{G}}} \mathcal{G}^{(p)} \xrightarrow{V_{\mathcal{G}}} \mathcal{G}$$

is exact. (This is equivalent to the condition that $\mathcal{G}$ occurs as the p-kernel of a Barsotti-Tate group.) Let us say that $\mathcal{G}$ is of type (d,f) if $\mathcal{G}$ has rank p^d and $\mathrm{Ker}(F_{\mathcal{G}})$ has rank p^f. Let $W \cong \mathfrak{S}_d$ be the Weyl group of $G := \mathrm{GL}_d$. Let X be the conjugacy class of maximal parabolic subgroups of G which arise as stabilizers of an f-plane in k^d. This conjugacy class gives rise to a subgroup $W_X \subset W_G$. In [Mo] it is shown that there is a natural bijection

$$\left\{\begin{matrix}\text{isomorphism classes of } \mathrm{BT}_1\text{'s} \\ \text{over } k \text{ of type } (d,f)\end{matrix}\right\} \xrightarrow{\sim} W_X\backslash W_G\,. \qquad (*)$$

Now suppose given (d_1,f_1) and (d_2,f_2) with $(d_1+d_2,f_1+f_2)=(d,f)$. Writing $W_{X_i}\subset W_{G_i}$ for the Weyl groups associated to the pair (d_i,f_i), we obtain a map

$$\left(W_{X_1}\backslash W_{G_1}\right)\times\left(W_{X_2}\backslash W_{G_2}\right) \longrightarrow W_X\backslash W_G \qquad (**)$$

given by taking the product of group schemes. (I.e., via the bijections as in $(*)$ we send $(\mathcal{G}_1,\mathcal{G}_2)$ to $\mathcal{G}_1\times\mathcal{G}_2$.)

Problem. Describe the map $(**)$ purely in terms of Weyl groups.

Let w and w' be two W_X-cosets in W_G. Let us write $w' \preccurlyeq w$ if we can specialize a BT_1 of type w to a BT_1 of type w', i.e., if there is a local domain R of characteristic p and a BT_1 $\mathcal{G}$ over $\mathrm{Spec}(R)$ such that the geometric generic fibre is of type w and the geometric special fibre is of type w'.

Problem. Describe the partial ordering $\preccurlyeq$ purely in terms of Weyl groups.

Every coset $w \in W_X\backslash W_G$ has a distinguished representative $\dot{w}$ of minimal length. The most obvious guess would be that $\preccurlyeq$ is the Bruhat ordering on these distinguished representatives. An example of Oort (see [Oo], Section 14) shows that this is *not* the case. In this example we have three BT_1's $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{H}$ such that (a) $\mathcal{G}_2$ specializes to $\mathcal{G}_1$, and the corresponding Weyl group elements are comparable in the Bruhat ordering, (b) the Weyl group elements corresponding to $\mathcal{G}_1 \times \mathcal{H}$ and $\mathcal{G}_2 \times \mathcal{H}$ are *not* comparable in the Bruhat ordering.

We can vary on the above questions by taking "aditional structures" into account, such as the action of a semi-simple $\mathbb{F}_p$-algebra on $\mathcal{G}$ or a polarization form. See [Mo] for details. For instance, if (X, λ) is a principally polarized abelian variety over k then the p-kernel $\mathcal{G} := X[p]$ is a BT_1 of type $(2g, g)$, and λ induces a "principal quasi-polarization" (abbreviated as pq-polarization) $\mu\colon \mathcal{G} \xrightarrow{\sim} \mathcal{G}^D$. If $g := \dim(X)$ then the group G to consider is the symplectic group $G := \mathrm{Sp}_{2g}$ and for X we take the conjugacy class of stabilizers of maximal isotropic subspaces of k^{2g}. Analogous to $(*)$ above we have a bijection

$$\left\{\begin{matrix}\text{isomorphism classes of pq-polarized} \\ \mathrm{BT}_1\text{'s } (\mathcal{G}, \mu) \text{ over } k\end{matrix}\right\} \xrightarrow{\sim} W_X\backslash W_G\,.$$

As shown in [Oo], this gives rise to a stratification of the moduli space: writing $A = A_{g,1} \otimes k$ we have

$$A = \coprod_{w \in W_X\backslash W_G} S_w\,,$$

where the moduli point of (X, λ) lies in the stratum S_w iff the associated pair $(\mathcal{G}, \mu)$ maps to the element w. It can be shown that each stratum S_w is equidimensional, of dimension equal to $\ell(\dot{w})$, the length of $\dot{w}$ as a Weyl group element.

References

[Mo] B.J.J. Moonen, *Group schemes with additional structures and Weyl group cosets*, to appear in the proceedings of the 1999 Texel conference on "The moduli space of abelian varieties".

[Oo] F. Oort, *A stratification of a moduli space of abelian varieties*, to appear in the proceedings of the 1999 Texel conference on "The moduli space of abelian varieties".

Problem 12. Proposed by T. Katsura.

[A slightly enlarged version of the Artin-Shioda conjecture.] For K3 surfaces S over an algebraically closed field of positive characteristic p, are the following six conditions equivalent to each other?

(i) S is a Zariski surface, i.e., there exists a purely inseparable dominant rational map of degree p from the projective plane $\mathbb{P}^2$ to S.

(ii) S is unirational.

(iii) S is uniruled.

(iv) S is supersingular in the sense of Shioda, i.e., the Picard number $\rho(S) = 22$.

(v) S is supersingular in the sense of Artin, i.e., the height of formal Brauer group of S is equal to infinity.

(vi) S is rationally connected, i.e., every two points x_1, x_2 on S are joined by a connected chain of finitely many rational curves.

Comments: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) holds. Condition (ii) easily implies (vi). If the ground field is uncountable then (vi) implies (iii). (See Miyaoka-Peternell [4], Part I, Lecture II, Proposition 5.4.) If S has an elliptic fibration, then (iv) is equivalent to (v), see Artin [1] and Milne [3]. In case $p = 2$, it is known that the five conditions (i)–(v) are equivalent to each other. (Combine results in Rudakov-Shafarevich [5], [6], Artin [1] and Milne [3].) If S has an elliptic fibration, then (v) is equivalent to (vi) in any positive characteristic (see Artin [1] and Milne [3]). In case $p \geq 3$, for Kummer surfaces, the four conditions (ii)–(v) are equivalent to each other (see Shioda [7], and also see Katsura [2]). Note that any Kummer surface has an elliptic fibration. In Katsura [2], we can find some results on (i) for Kummer surfaces.

References

[1] M. Artin, *Supersingular K3 surfaces*, Ann. Scient. Ec. Norm. Sup. **7** (1974), 543–568.

[2] T. Katsura, *Generalized Kummer surfaces and their unirationality in characteristic p*, J. Fac. Sci. Univ. of Tokyo **34** (1987), 1–41.

[3] J.S. Milne, *Duality in the flat cohomology of a surface*, Ann. Scient. Ec. Norm. Sup. **9** (1976), 171–202.

[4] Y. Miyaoka and T. Peternell, *Geometry of Higher Dimensional Algebraic Varieties*, DMV Seminar **26**, Birkhäuser, Basel, 1997.

[5] A.N. Rudakov and I.R. Shafarevich, *Quasi-elliptic surfaces of type K3*, Russ. Math. Surv. **33** (1978), 215–216.

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[7] T. Shioda, *Some results on unirationality of algebraic surfaces*, Math. Ann. **225** (1977), 153–168.

Problem 13. Proposed by T. Katsura.

(1) Is any elliptic surface in characteristic $p > 0$ liftable to characteristic zero?

(2) Construct non-liftable Calabi-Yau varieties in positive characteristic. Here, a Calabi-Yau variety means a nonsingular complete variety X with a trivial canonical bundle and $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \ldots, \dim(X) - 1$.

Comments: In characteristic 3, an example of non-liftable Calabi-Yau threefold is known (see Hirokado [2]). In the case of dimension 2, any K3 surface is liftable (see P. Deligne [1]).

References

[1] P. Deligne, *Relèvement des surfaces K3 en caractéristique nulle (prepared for publication by Luc Illusie)*, Lecture Notes in Math. **868**, Algebraic surfaces (Orsay, 1976–78), 58–79, Springer, Berlin-New York, 1981.

[2] M. Hirokado, *A non-liftable Calabi-Yau threefold in characteristic 3*, Tohoku Math. J. **51** (1999), 479–487.

Problem 14. Proposed by Bas Edixhoven.

An abelian variety A is said to have complex multiplications if its endomorphism algebra contains a commutative semi-simple algebra whose degree over $\mathbb{Q}$ is twice the dimension of A. Over the complex numbers, an abelian variety has complex multiplications if and only if its Mumford-Tate group is commutative.

For g a positive integer, let A_g be the moduli space of principally polarized abelian varieties of dimension g. A complex valued point of A_g is called CM if the corresponding abelian variety has complex multiplications. Such a point is in fact defined over the algebraic closure of $\mathbb{Q}$, hence the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ acts on the set of CM points of A_g.

The question is then whether there exists, for g fixed, positive real numbers c and d such that for every CM point x in A_g one has:

$$|G_{\mathbb{Q}} \cdot x| \geq c \cdot |\mathrm{discr}(R_x)|^d ,$$

where R_x is the center of the endomorphism ring of the abelian variety corresponding to x, and $\mathrm{discr}(R_x)$ is its discriminant.

A positive answer would be very useful in proving the André-Oort conjecture for subvarieties of A_g. For $g = 1$, the answer is positive by the Brauer-Siegel theorem. The problem for higher g is that the Galois action on CM points is via the so-called reflex type norm, which means that the problem is not only a problem about class numbers, but more about images of morphisms between class groups. More generally, one could ask similar questions about Galois orbits of special points in Shimura varieties.

References

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Problem 15. Proposed by Marc-Hubert Nicole.

Question: What is the arithmetical counterpart of the theory of the Jones polynomial knot invariant (see [K])?

A profound conceptual analogy links algebraic number fields and closed 3-manifolds (for a precise dictionary, see [R]; see also the appendix of [P]): it is a fact that the cohomological dimension of the absolute Galois group of a number field is three (cf. [M2]). For example, we can picture a prime of $\mathbb{Z}$ as a knot in S^3, since the local fundamental group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is equal to $\hat{\pi}_1(S^1) = \hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. Furthermore, class field theory is then seen to be analogous to the study of the homology of ramified coverings of S^3, and Iwasawa theory corresponds to the study of the homology of the infinite cyclic cover of the complement of a knot (i.e., the Alexander polynomial); see [Mc]. From this point of view, it is tempting to speculate on the implications of the interplay between physics, topology (especially in low dimensions) and arithmetic. Earlier works in this direction comprise [HZ], where, for example, the Atiyah-Patodi-Singer index theorem applied to:

(1) 4-manifolds yields classical Dedekind sums, and
(2) on the involution of 3-dimensional lens spaces, its output is the Legendre-Jacobi symbol (see also [GZ]).

Our question points also in the direction of Connes' noncommutative geometry (the Jones polynomial originally stemmed from the study of subfactors in von Neumann algebras) and its resolution could maybe help to understand the connection between the former and arithmetic (see also [D] and [K] for a different perspective on the above interplay).

Some references (for a more exhaustive bibliography, contact Marc-Hubert Nicole):

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Problem 16. Proposed by Arthur Ogus and Frans Oort.

Extending p-divisible groups and abelian schemes.

Situation. Let R be a local ring of dimension at least 2, let $S = \mathrm{Spec}(R)$, let $0 \in S$ be the closed point, and let $U \subset S$ be the complement of 0 in S. Let $X_U \to U$ be either a p-divisible group or an abelian scheme over U.

Remark. In case R is regular of equal characteristic zero, Grothendieck proved that X_U can be extended to S, see [G], Coroll. 4.5. In general an extension does not exist: if R is not regular, or in case R is of positive characteristic examples are easy to give. In case R is regular of mixed characteristics, there is an example by Raynaud-Ogus-Gabber showing that an extension in general does not exist, see [JO], Section 6. Already many years ago Arthur Ogus asked several people whether "extension up to isogeny" is possible:

Question. Suppose moreover that R is regular. Does there exist a p-divisible group or an abelian variety $Y \to S$ and an isogeny $Y_U \sim_U X_U$ of the restriction $Y_U := Y\mid_U$ to X_U?

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Problem 17. Proposed by Ching-Li Chai.

Let $\mathcal{O} = \mathcal{O}_K$ be a complete discrete valuation ring. Let K be the fraction field of $\mathcal{O}$ and let κ be the residue field of $\mathcal{O}$. Let T be a torus over K. It is a fact that T has a Néron model $\underline{T}$ over $\mathcal{O}$, which is smooth and locally of finite type over $\mathcal{O}$ and satisfies the standard universal property for Néron models. Choose a finite Galois extension L/K such that T is split over L. From the universal property of Néron models we get a homomorphism

$$\underline{\mathrm{can}}_{L,K}\colon \underline{T} \underset{\mathrm{Spec}(\mathcal{O}_K)}{\times} \mathrm{Spec}(\mathcal{O}_L) \longrightarrow \underline{T_L}\,,$$

where $\underline{T_L}$ is the Néron model of T_L, whose neutral component is equal to the split form of T over $\mathcal{O}_L$. The differential $d(\underline{\mathrm{can}}_{L,K})$ of the homomorphism $\underline{\mathrm{can}}_{L,K}$ is a homomorphism between free $\mathcal{O}_L$-modules of finite rank. Write

$$\mathrm{Coker}\big(d(\underline{\mathrm{can}}_{L,K})\big) \cong \bigoplus_{i=1}^{d} \mathcal{O}_L/\big(\pi_L^{e\cdot c_i}\mathcal{O}_L\big) \qquad c_1 \le \cdots \le c_d\,,$$

where $e = e(L/K)$ is the ramification index of L/K, where $d = \dim(T)$, and $c_i \in \mathbb{Z}[1/e]_{\ge 0}$. These non-negative rational numbers $c_1, \ldots, c_d$ do not depend on the choice of the field L that splits T; on can think of them as numerical invariants of the integral representation ρ_T of $\mathrm{Gal}(L/K)$ on the character group $X^*(T)$ of T. It is known that $\sum_{i=1}^{d} c_i$ is equal to one-half of the Artin conductor of ρ_T if the residue field κ is perfect. (This is proved in [1].)

Questions:

(i) What can one say about the numerical invariants $c_1(\rho), \ldots, c_d(\rho)$ attached to integral representations ρ: $\mathrm{Gal}(L/K) \to \mathrm{GL}_{d,\mathbb{Z}}$? Can one obtain good estimates for them? Can one relate the invariants for an integral representation ρ to those for the dual representation $\rho^\vee$? How do these invariants vary in a family of representations?

(ii) When the residue field κ is not assumed to be perfect, is the sum $\sum_{i=1}^{d} c_i$ related to "the conductor of ρ_T" for some general theory of conductors?

(iii) For an abelian variety A over K the same procedure gives invariants $c_1(A), \ldots, c_d(A)$, where $d = \dim(A)$. Let $c(A) = \sum_{i=1}^{d} c_i(A)$. Suppose $0 \to A_1 \to A_2 \to A_3 \to 0$ is a short exact sequence of abelian varieties over K. Is $c(A_2)$ equal to $c(A_1) + c(A_3)$? (They are equal if the residue field κ is finite, or if the fraction field K has characteristic 0 and the residue field κ is perfect; see [2].) How are the invariants $c_i(A)$ and $c(A)$ for an abelian variety A related to those for the dual abelian variety A^t?

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Problem 18. Proposed by Ching-Li Chai and Frans Oort.

Questions on Hecke orbits.

Let k be an algebraically closed field of characteristic $p > 0$. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g over k. Let ℓ be a prime number. For any point $x = \big[(A_x, \lambda_x)\big] \in \mathcal{A}_g(k)$, where λ_x is a principal polarization on the abelian variety A_x, denote by $\mathcal{H}_\ell \cdot x$ (resp. $\mathcal{H} \cdot x$) the countable subset of $\mathcal{A}_g(k)$ consisting of all points $\big[(A_y, \lambda_y)\big] \in \mathcal{A}_g(k)$ such that there exists an isogeny $\alpha\colon A_y \to A_x$ with $\alpha^*(\lambda_x) = \ell^n \cdot \lambda_y$ for some $n \in \mathbb{Z}_{\geq 0}$ (resp. $\alpha^*(\lambda_x) = m \cdot \lambda_y$ for some $m \in \mathbb{Z}_{\geq 1}$). It is known that if $\ell \neq p$, then the Zariski closure $(\mathcal{H}_\ell \cdot x)^{\mathrm{cl}}$ of $\mathcal{H}_\ell \cdot x$ is equal to $\mathcal{A}_g$ for every $x = \big[(A_x, \lambda_x)\big] \in \mathcal{A}_g(k)$ such that A_x is an ordinary abelian variety; see [C].

In Section 5 of [FO], Frans Oort gives a conjectural description of a *foliation structure* on $\mathcal{A}_g$, which can be described as follows. For any point $x = \big[(A_x, \lambda_x)\big] \in \mathcal{A}_g(k)$, consider the subset $Z(x)$ consisting of all points $y \in \mathcal{A}_g$ such that the quasi-polarized Barsotti-Tate group $\big(A_y[p^\infty], \lambda_y[p^\infty]\big)$ is geometrically isomorphic to $\big(A_x[p^\infty], \lambda_x[p^\infty]\big)$. In fact, we see that $Z(x)$ is a locally closed subset of $\mathcal{A}_g$ (in the proof we follow a suggestion by T. Zink). A locally closed subset of the form $Z(x)$ is called a *leaf* in $\mathcal{A}_g$. The foliation structure on $\mathcal{A}_g$ is the collection of all leaves, so that $\mathcal{A}_g$ is the disjoint union of all leaves in $\mathcal{A}_g$. See [FO] for properties of the foliation structure.

Questions:

(i) (Conjectured by Oort) Prove that the Zariski closure of $\mathcal{H} \cdot x$ is equal to the subset of all points $\big[(A_y, \lambda_y)\big] \in \mathcal{A}_g$ such that the Newton polygon of A_y is either equal to, or lies above, the Newton polygon of A_x.

(ii) (Conjectured by Oort) Prove that for any prime $\ell \neq p$ and any $x \in \mathcal{A}_g(k)$, the ℓ-power Hecke orbit $\mathcal{H}_\ell \cdot x$ is Zariski dense in the leaf $Z(x)$ that passes through x.

Remarks:

(a) Conjecture (i) is a consequence of Conjecture (ii) and the general properties of the foliation structure on $\mathcal{A}_g$.

(b) If $x = \big[(A_x, \lambda_x)\big]$ and A_x is an ordinary abelian variety then the leaf $Z(x)$ is the ordinary locus in $\mathcal{A}_g$, and the Conjecture (ii) holds by [C]. If the p-rank of A_x is equal to $g-1$, then the leaf $Z(x)$ is equal to the subset of $\mathcal{A}_g$ consisting of all points $\big[(A_y, \lambda_y)\big]$ such that the p-rank of A_y is equal to $g-1$, and the Conjecture (ii) is verified in a preprint by Ching-Li Chai: *Density of Hecke orbits for abelian varieties of p-corank one*, available from `http://www.math.upenn.edu/~chai`.

(c) Conjecture (ii) has been verified when $g \leq 3$ by the authors (unpublished).

(d) One can also formulate Conjecture (ii) for any Shimura variety M of PEL-type over k. For a Hilbert-Blumenthal variety attached to a totally real number field in which p is unramified, the authors have checked Conjecture (ii). Here is another case when

Conjecture (ii) has been verified. Suppose that M is a $U(n-1,1)$-type Shimura variety of PEL-type over k attached to an imaginary quadratic field K such that p splits in K, and $x \in M(k)$ corresponds to an n-dimensional ordinary abelian variety with multiplication by $\mathcal{O}_K$, then the ℓ-power Hecke orbit of x is dense in M, for any $\ell \neq p$. The proof uses a result in Ching-Li Chai: *Local monodromy for deformation of one-dimensional formal groups*, J. reine Angew. Math. **524** (2000), 227–238.

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Problem 19. Proposed by Gerard van der Geer.

Does there exist a supersingular smooth curve of genus g in characteristic p for every g and p? (The answer is yes for $p = 2$.)

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G. van der Geer, M. van der Vlugt, *On the existence of supersingular curves of given genus*, J. Reine Angew. Math. **458** (1995), 53–61.

Problem 20. Proposed by Brian Conrad.

Let k be a field complete with respect to a non-trivial non-archimedean absolute value. Let k'/k be an analytic extension field. A general "extension of the ground field" functor can be defined on affinoid rigid spaces over k. This functor takes open immersions of affinoids to open immersions of affinoids and commutes with fiber products, so it can be naturally defined for quasi-separated rigid spaces over k; see [1], 9.3.6. This is a functor from quasi-separated rigid spaces over k to quasi-separated rigid spaces over k', compatible with formation of fiber products and quasi-compact open immersions. Can this functor be naturally extended to be defined on all rigid spaces over k (in a manner compatible with fiber products and open immersions)? More precisely, let $i\colon U \to X$ be an open immersion of rigid spaces over k, with X (and hence U) separated. Is the naturally induced morphism $i'\colon U' \to X'$ of rigid spaces over k' an open immersion? This is the essential problem. When i is quasi-compact or a Zariski-open map, this is relatively straightfoward to answer in the affirmative. In general, one only knows that i' is an injective local isomorphism, but it is not a priori clear if $i'(U')$ is even an admissible open.

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Problem 21. Proposed by Frans Oort and Bjorn Poonen.

Open Problem: The number of isomorphism classes in an isogeny class.

Fix a prime power q and a Newton Polygon β. For any $\mathbb{F}_q$-isogeny class I of abelian varieties, let N_I denote the number of $\mathbb{F}_q$-isomorphism classes of principally polarized abelian varieties in I. We would like to understand the "average" of N_I, as I ranges over all $\mathbb{F}_q$-isogeny classes with Newton Polygon equal to β.

Moduli spaces. We have the coarse moduli space of principally polarized abelian varieties. Once a prime number p and a Newton Polygon β are fixed, we denote by W^0_β the locally closed subset whose geometric points correspond to principally polarized abelian varieties having Newton polygon equal to β; its dimension is denoted by $\mathrm{sdim}(\beta)$. This is foliated by two foliations; the dimension of the central leaves is denoted by $c(\beta)$; it is conjectured that every central leaf is the Zariski closure inside W^0_β of a Hecke-prime-to-p-orbit, see Problem 18 in this list. The iterated-α_p-Hecke orbit has components of dimension $i(\beta) = \mathrm{sdim}(\beta) - c(\beta)$. See F. Oort, *Newton polygon strata in the moduli space of abelian varieties*, to appear in the Proceedings of the Conference "The moduli space of abelian varieties", Texel 1999.

Let $S_{\beta,q}$ be the set of $\mathbb{F}_q$-isomorphism classes of principally polarized abelian varieties over $\mathbb{F}_q$ with Newton polygon β. Write σ for the supersingular Newton polygon (all slopes are equal to one half). Suppose $\beta \neq \sigma$. We expect that most $(X, \lambda) \in S_{\beta,q}$ have only ± 1 as automorphisms over the algebraic closure; hence we expect the map $S_{\beta,q} \to W^0_\beta(\mathbb{F}_q)$ to be mostly 2-to-1. Moreover, it is conjectured that W^0_β is geometrically irreducible; hence we expect that

$$\# S_{\beta,q} = \big(2 + o(1)\big) \cdot q^{\mathrm{sdim}(\beta)} \qquad \text{as } q \to \infty.$$

Isogeny classes. Let $d(i,\beta)$ be the length of the vertical segment between β and the supersingular Newton polygon σ at x-coordinate equal to i, with $0 < i \leq g$, i.e., $d(i,\beta) = \sigma(i) - \beta(i)$; let $d(\beta) = \sum_{0<i\leq g} d(i,\beta)$. We expect that $c(\beta) = 2 \cdot d(\beta)$. A slight generalization of a result of DiPippo and Howe shows there exists a "volume constant" v_g (depending only on g, not on p or β) such that for every Newton polygon $\beta \neq \sigma$, the number of $\mathbb{F}_q$*-isogeny classes* of abelian varieties of dimension g having Newton polygon equal to β is equal to:

$$\big(v_g + o(1)\big) q^{d(\beta)} \qquad \text{as } q \to \infty.$$

See Stephen A. DiPippo and Everett W. Howe, *Real polynomials with all roots on the unit circle and abelian varieties over finite fields*, J. Number Theory **73** (1998), 426–450; corrigendum: J. Number Theory **80** (2000), 182. (They prove this estimate for β ordinary, with an explicit error term; the same method works for any $\beta \neq \sigma$.)

Question. From the previous we expect that the average $\mathbb{F}_q$-isogeny class with Newton polygon β contains

$$e^{O(1)} q^{\mathrm{sdim}(\beta) - d(\beta)}$$

$\mathbb{F}_q$-isomorphism classes of principally polarized abelian varieties; note that $\mathrm{sdim}(\beta) - d(\beta) = i(\beta) + d(\beta)$. Moreover, the constant $e^{O(1)}$ should be $2/v_g + o(1)$. Can one give a proof of this, or at least a heuristic explanation of the exponent of q, without using the previous considerations? In order to understand the interplay between isomorphism classes and isogeny classes we should realize that not every $\mathbb{F}_q$-isogeny class contains a principally polarized abelian variety (see Everett W. Howe, *Principally polarized ordinary abelian varieties over finite fields*, Trans. Amer. Math. Soc. **347** (1995), 2361–2401); moreover we have to consider (iterated) α_p-isogenies, and degree p isogenies and prime-to-p isogenies.

Understanding these phenomena asymptotically and their interplay in one isogeny class seems to be the key to what we are asking.

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