Moduli of abelian varieties in mixed and in positive characteristic

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Abstract. We start with a discussion of CM abelian varieties in characteristic zero, and in positive characteristic. An abelian variety over a finite field is a CM abelian variety, as Tate proved. Can it be CM lifted to characteristic zero? Here are other questions. Does there exist an abelian variety, say over $\mathbb{Q}^a$, or over $\overline{\mathbb{F}_p}$, of dimension $g > 3$ not isogenous with the Jacobian of an algebraic curve? Can we construct algebraic curves, say over $\mathbb{C}$, where the Jacobian is a CM abelian variety? We give (partial) answers to these questions and discuss stratifications and foliations of moduli spaces of abelian varieties in positive characteristic.

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Introduction

0.1. In 1857, discussing what we now call Riemann surfaces of genus p, Riemann wrote: "*... und die zu ihr behörende Klasse algebraischer Gleichungen von 3p-3 stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen.*" See [125], Section 12. Therefore, we now use the concept "moduli" as the parameters on which deformations of a given geometric object depend.

0.2. Moduli of CM abelian varieties. Most readers, reading this title will have the reaction: "CM abelian varieties have no moduli." Indeed, over $\mathbb{C}$ this is true, see 3.3, and "the moduli" of such objects over $\mathbb{C}$ is not a very interesting topic. The arithmetic of CM points over a number field is fascinating, on which Hilbert stated: "*... the theory of complex multiplication ... was not only the most beautiful part of mathematics but also of all science.*" See [124], page 200. However this will not be our focus.

We will study CM abelian varieties in positive characteristic, and in mixed characteristic. In positive characteristic there are many CM abelian varieties which "do have moduli": there are CM abelian varieties which cannot be defined over a finite field. A theorem by Grothendieck, see 3.2, and see [92], however tells us that after applying an isogeny, we can descend to a finite field. We end Section 3 by discussing a proof by Yu of this theorem.

A theorem by Tate tells us that every abelian variety defined over a finite field is a CM abelian variety, see [132]. Does every abelian variety over a finite field admit a CM lifting? A theorem by Honda says that after extending the base field and moreover applying an isogeny we can arrive at a situation where a CM lifting is possible; see 4.5. Is an isogeny necessary? Is a field extension necessary? These questions have a satisfactory answer, see Section 4. For complete information see [12]

Fix an algebraically closed field k, and an integer $g > 3$. Does there exist an abelian variety of dimension g not isogenous with a Jacobian? We discuss partial answers to this interesting question; see Section 5.

0.3. Moduli of abelian varieties in positive characteristic. In the second part we discuss *stratifications* and *foliations* of our basic hero $\mathcal{A}_g \otimes \mathbb{F}_p$: the moduli spaces of polarized abelian varieties of dimension g in positive characteristic p.

In characteristic zero we have strong tools at our disposal: besides algebraic-geometric theories we can use analytic and topological methods. It seems that we are at a loss in positive characteristic. However the opposite is true. Phenomena, only occurring in positive characteristic, provide us with strong tools to study these moduli spaces.

We describe constructions of various stratifications and foliations which result from p-adic aspects of abelian varieties in characteristic p. The terminology "stratification" and "foliation" will be used in a loose sense.

- A *stratification* will be a way of writing a space as a *finite* disjoint union of locally closed subspaces of that space; in some cases we will also check whether the boundary of one stratum is the union of "lower strata".
- A *foliation* will be a way of writing a space as a disjoint union of locally closed subspaces of that space; in this case we have some extra conditions, specified below.

For an abelian variety $A \to S$ over a base scheme S, and a positive integer m we define the group scheme:

$$A[m] := \mathrm{Ker}(\times m : A \longrightarrow A).$$

Note that $A[m] \to S$ is a finite, flat group scheme. For a prime number p we define the p-divisible group of A by:

$$A[p^\infty] = \cup_{1 \leq i < \infty} \ A[p^i] = \mathrm{lim.ind}_{i \to \infty} \ A[p^i].$$

This ind-group scheme is also called a Barsotti-Tate group scheme. If the prime number p is invertible on the base, the study of $A[p^\infty]$ amounts to the same as the study of $T_p(A)$, the Tate p-group of A. However, the ind-group scheme $A[p^\infty]$ provides us with information very different from aspects of $T_\ell(A) := \mathrm{lim.proj.}_i A[\ell^i]$, where ℓ is a prime number invertible on the base scheme, respectively different from the characteristic p of the base field.

For $g, d \in \mathbb{Z}_{>0}$ we write $\mathcal{A}_{g,d} \to \mathrm{Spec}(\mathbb{Z})$ for the moduli space of abelian schemes of dimension g, with a polarization of degree d^2 over base schemes over $\mathbb{Z}$. See [79]. In the second part of this paper we fix a prime number p and we write $\mathcal{A}_g$ for the scheme

$$\mathcal{A}_g = \cup_d \ \mathcal{A}_{g,d} \otimes \mathbb{F}_p,$$

the moduli scheme of polarized abelian varieties in characteristic p. In some cases we only have coherent results for subvarieties of a given type of $\mathcal{A}_{g,1}$, the principally polarized case; e.g. EO strata, and the Grothendieck conjecture. However, in other cases it is interesting and necessary to study also non-principally polarized abelian varieties, e.g. in the case of NP strata and of leaves.

In §§ 6 – 14 base fields, and base schemes will be in characteristic p, unless otherwise specified. We will write k and Ω for an algebraically closed field. We write K for an arbitrary field.

0.4. Here is a survey of the *strata* and *leaves* we are going to construct. For an abelian variety A over an algebraically closed field and its p-divisible group $X = A[p^\infty]$, we consider three "invariants" of A:

NP $A \mapsto A[p^\infty] \mapsto A[p^\infty]/\sim$;

over an algebraically closed field, by the Dieudonné - Manin theorem, the isogeny class of a p-divisible group can be identified with the Newton polygon of A, see 6.5, 6.6. We obtain the Newton polygon strata. See Section 7.

EO $(A, \lambda) \mapsto (A, \lambda)[p] \mapsto (A, \lambda)[p]/\cong$;

over an algebraically closed field the isomorphism class of $(A, \lambda)[p]$ will be called the EO class of (A, λ); we obtain EO strata; see [104]. Important feature (Kraft, Oort): the number of geometric isomorphism classes of group schemes of a given rank annihilated by p is *finite*. See Section 10 for definitions and more details.

Fol $(A, \lambda) \mapsto (A, \lambda)[p^\infty] \mapsto (A, \lambda)[p^\infty]/\cong$;

we obtain a foliation of an open Newton polygon stratum; see [111]. Note that for $f < g-1$ the number of central leaves is *infinite*; here f is the p-rank, see 6.4. See Section 11 for definitions and more details.

It will turn out that strata and leaves defined in this way are locally closed in $\mathcal{A}_g$. To the p-divisible group $X = A[p^\infty]$ of an abelian variety A we attach various "invariants":

$A[p^\infty]$ up to $\sim$	ξ	NP	W_ξ
$A[p^1] = X[p^1]$ up to $\cong$	φ	EO	S_φ
$(A[p^\infty], \lambda)$ up to $\cong$	(X, λ)	Fol	$C(x)$

We explain these notions and notations below.

0.5. Here are some motivating questions and problems connected with stratifications and foliations considered:

- What is the Hecke orbit of a point in the moduli space of polarized abelian varieties? Over $\mathbb{C}$: such an orbit is dense in the moduli space $\mathcal{A}(\mathbb{C})$. What can we say about this question in positive characteristic? See 13.
- What is the maximal dimension of a complete subvariety of $\mathcal{A}_g(\mathbb{C})$?
- What are the complete subvarieties of maximal dimension in $\mathcal{A}_g \otimes \mathbb{F}_p$?
- Describe NP strata in the moduli space of abelian varieties in characteristic p. Are they irreducible? If not, what is the number of geometrically irreducible components?
- A conjecture by Grothendieck: which Newton polygons occur in the local deformation space of a given p-divisible group, or a given polarized abelian variety? See Section 8. This conjecture pins down the following question.
- What are the boundary points inside $\mathcal{A}_g$ of an open Newton polygon stratum? A similar question for EO strata and for central leaves.
- What kind of strata are given by fixing the isomorphism class of the p-kernel of abelian varieties studied.

- What kind of leaves are given by fixing the isomorphism class of the p-divisible group of the abelian varieties studied.
- In which way do these stratifications and foliations "intersect"?

It will turn out that various stratifications and foliations of $A_g \otimes \mathbb{F}_p$, and a description of these structures give access to most of these questions.

0.6. Hecke correspondences in characteristic zero, or more generally Hecke orbits involving isogenies of degree prime to the characteristic of the base field, are finite-to-finite. However, in characteristic p Hecke correspondences may blow up and down subsets of the moduli space (if we consider "α-Hecke-orbits"). We will understand this phenomenon, by introducing "isogeny leaves", and we will see that on "central leaves" all Hecke correspondences are finite-to-finite.

For some of our results we have to restrict to principally polarized abelian varieties, in order to obtain nice, coherent statements. For example the Grothendieck conjecture, see Section 8, holds for principally polarized abelian varieties, but its analogue for non-principal polarizations admits counterexamples.

In some cases, by some miracle, statements holds more generally for all degrees of polarizations (e.g. the dimensions of the p-rank-strata, e.g. irreducibility of non-supersingular central leaves). However, in other cases the condition that the polarization is principal is essential, e.g. the question whether $(a = 1)$-locus is dense in a NP stratum.

Note: $X \cong Y \Rightarrow \mathcal{N}(X) = \mathcal{N}(Y)$; conclusion: every central leaf in **Fol** is contained in exactly one Newton polygon stratum in **NP**. Here $\mathcal{N}(X)$ stands for the Newton polygon of X, see 6.4.

However, a NP-stratum can contain points in many different EO strata, and an EO stratum may intersect several NP-strata; this phenomenon is only partially understood. If the p-rank is smaller than $g-1$ a NP-stratum contains infinitely many central leaves. Whether an EO stratum equals a central leaf is studied and answered in the theory of minimal p-divisible groups, see Section 12.

We will see that supersingular abelian varieties on the one hand and non-supersingular abelian varieties on the other hand in general behave very differently.

0.7. Supersingular NP-*strata,* EO-*strata and central leaves in general are* **reducible** (Katsura-Oort, Li-Oort, Harashita) (for $p \gg 0$). But

0.8. Non-supersingular NP-*strata*, EO-*strata in the principally polarized case and central leaves are* **geometrically irreducible** (Oort, Ekedahl-Van der Geer, Chai-Oort).

These structures will be studied for (polarized) abelian varieties. They can also be discussed for p-divisible groups and for quasi-polarized p-divisible groups. These questions, usually easier, will be omitted, except for a brief discussion of the papers [155], [120].

Many of results discussed below can be considered for arbitrary Shimura varieties instead of the moduli space of abelian varieties. Especially stratifications and foliations studied below have been described in that language. It would be nice to have a survey of results in that quickly developing field. However for this note that would lead us too far. So we have decided to restrict this survey to results about CM abelian varieties and about stratifications and foliations in the moduli space of abelian varieties.

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1. Notation/Preliminaries.

1.1. We write K and κ for a field. We write $\mathbb{Q}^a$ for an algebraic closure of $\mathbb{Q}$. Also L will be used, but in that case this will usually be a CM field. We write k for an algebraically closed field. Once a positive characteristic p is fixed, we write $\mathbb{F} := \overline{\mathbb{F}_p}$, an algebraic closure of the prime field in characteristic p.

We write $\mathrm{End}(A)$ for the ring of endomorphisms of an abelian variety A over a field K; this ring has no $\mathbb{Z}$-torsion, i.e. if $n \in \mathbb{Z}_{>0}$ and $\varphi \in \mathrm{End}(A)$ with $n{\cdot}\varphi = 0$, then $\varphi = 0$. We write $\mathrm{End}^0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, *the endomorphism algebra* of A.

An abelian variety A over a field K is called *simple*, or K-simple if confusion can occur, if 0 and A are the only abelian subvarieties of A, over K. It may happen, and examples are easy to give, that A is K-simple, although $A \otimes_K K'$ is not K'-simple for some field extension $K \subset K'$.

If an abelian variety A over a field K is simple, if and only if its *endomorphism algebra* $\mathrm{End}^0(A)$ is a division algebra.

Suppose the characteristic of K equals $p > 0$. For a group scheme G over K we write $f(G)$, called the p-rank of G, for the number $f = f(G)$ such that $\mathrm{Hom}(\mu_{p,k}, G_k) \cong (\mathbb{Z}/p)^f$ for an algebraic closure k of K. For an abelian variety $G = A$ this number can also be defined by:

$$\mathrm{Hom}(\mathbb{Z}/p, A(k)) \cong A(k)[p] \cong (\mathbb{Z}/p)^f.$$

For an abelian variety A of dimension g we have $0 \le f \le g$, and all values do appear.

1.2. We say that an abelian variety A over of dimension g a field $K \supset \mathbb{F}_p$ is *ordinary* in case $f(A) = g$; we say that A is *almost ordinary* if $f(A) = g - 1$.

A polarization on an abelian variety A induces an involution $\iota : \mathrm{End}^0(A) \to \mathrm{End}^0(A)$, called the Rosati involution. This is positive definite; see [63], Ch. 1, § 1.

A central simple algebra of finite dimension over $\mathbb{Q}$ with a positive definite involution is called an *Albert algebra*. Such algebras have been classified; for references e.g. see [78], Section 21; [64], Ch. 5, § 5; [118], 18.2, [15], 10.14.

It has been proved that for any Albert algebra (D, ι) and any prime field $\mathbb{P}$ there exists an algebraically closed field $k \supset \mathbb{P}$, and a simple abelian variety A over k such that its endomorphism algebra with Rosati involution equals (D, ι) up to isomorphism. This has been proved by Albert, by Shimura and by Gerritzen. For abelian varieties over $\mathbb{C}$, see [128], Section 4, especially Theorem 5. For abelian varieties in arbitrary characteristic see [37], Th. 12; see [97], Th. 3.3. However, given $\mathbb{P} = \mathbb{F}_p$ and (D, ι) it is in general not so easy to find the minimal g for which an abelian variety of dimension g realizes this Albert algebra in that characteristic; there are cases where the result depends on the characteristic; see [97].

1.3. The group scheme α_p is defined as the kernel of the Frobenius homomorphism $F : \mathbb{G}_a \longrightarrow \mathbb{G}_a$ on the additive group scheme $\mathbb{G}_a$. The rank of α_p equals p. As a scheme, over a base field $K \supset \mathbb{F}_p$ it equals $\mathrm{Spec}(K[\epsilon]/(\epsilon^p))$. If it is clear over which base S in characteristic p we work, we will write α_p in stead of $\alpha_{p,S}$; we will take care this does not lead to confusion; e.g. the meaning of $\mathrm{Hom}(\alpha_p, \alpha_p)$ is unclear if a base scheme is not specified. For a group scheme G over K we define the a-number of G as

$$a(G) = \dim_E \left(\mathrm{Hom}(\alpha_{p,E}, G_E)\right),$$

where E is a perfect field containing K. For a group scheme of dimension g smooth over a base field $K \supset \mathbb{F}_p$ clearly $0 \leq a(G) \leq g$. By the way, in case A is an abelian scheme of dimension g and $a(A) = g$ then over an algebraically closed field A is isomorphic with a product of supersingular curves; for more information see § 6; such an abelian variety is called *superspecial*; see 6.16.

Moduli of CM abelian varieties

2. Complex multiplication on abelian varieties

2.1. Some references: the book [130] is the classic studying this topic; in [132] we find a proof for the Tate conjecture for abelian varieties over a finite field; the Albert classification is described in [78]. We will only discuss one aspect of this topic.

Proposition 2.2. *Let K be a field, and let A be an abelian variety over K of dimension g. Let $\Lambda \subset \mathrm{End}^0(A)$ be a* commutative, semi-simple *$\mathbb{Q}$-subalgebra. Then* $\dim_{\mathbb{Q}}\Lambda \leq 2g$. □

This proposition is well-known. For example see [12], Ch. I. Note that $\dim_{\mathbb{Q}}\Lambda$ need not be a divisor of $2g$.

Definition 2.3. Let A be an abelian variety of dimension g. We say that A admits *sufficiently many complex multiplications*, sometimes abbreviated as smCM, if there exists a *commutative, semi-simple* $\mathbb{Q}$-subalgebra $\Lambda \subset \mathrm{End}^0(A)$ of dimension $2g$ over $\mathbb{Q}$. An abelian variety which admits smCM will be called a "CM abelian variety".

2.4. The terminology "complex multiplication" stems from the theory of elliptic curves. An elliptic curve E over $\mathbb{C}$ either has $\mathrm{End}(E) = \mathbb{Z}$, or $\mathbb{Z} \subsetneqq \mathrm{End}(E)$. This last case is indicated by the phrase "E has complex multiplication" as every element of $\mathrm{End}(E)$ is induced by multiplication by a complex number z on the tangent space of E. An endomorphism on a complex abelian variety is induced by a linear transformation (not necessarily a multiplication) on its tangent space.

We give some comments. Sometimes people consider an "elliptic curve E over $\mathbb{Q}$ with complex multiplication". However, if E is defined over $\mathbb{Q}$, then $\mathrm{End}(E) = \mathbb{Z}$. The case considered concerns the property that $\mathbb{Z} \subsetneqq \mathrm{End}(E \otimes \mathbb{C})$. Indeed, it may happen that an abelian variety A over a field K does not admit smCM (over K), but that there exists an extension $K \subset K'$ such that $A \otimes K'$ admits smCM. For example the elliptic curve defined by $Y^2 = X^3 - 1$ over $\mathbb{Q}$ has "no CM" over $\mathbb{Q}$, but $\mathrm{End}(E \otimes \mathbb{Q}(\sqrt{-3})) = \mathbb{Z}[\zeta_3]$.

For any elliptic curve the property $\mathbb{Z} \subsetneqq \mathrm{End}(E)$ implies that E admits smCM. However there are many cases where an abelian variety A has an endomorphism ring which is bigger than $\mathbb{Z}$, although A does not admit smCM. For these reasons we feel that the expression "A has complex multiplications" is ambiguous.

Furthermore it might happen that an abelian variety A over a field K admits smCM, and that $\mathrm{End}(A) \subsetneqq \mathrm{End}(A \otimes K')$ for some field extension. For example, as Deuring and Tate proved, an elliptic curve E defined over $\mathbb{F}_p$ admits smCM, and $\mathrm{End}^0(E)$ is an imaginary quadratic field. However, if moreover E is "supersingular", i.e. the elliptic curve E has the property $E(\mathbb{F})[p] = 0$, then $\mathrm{End}(E \otimes \mathbb{F})$ is a maximal order in a quaternion algebra.

We will encounter the terminology "of CM type". We will use this only for abelian varieties in characteristic zero. The type specifies the action of the endomorphism algebra on the tangent space: an abelian variety of CM type is a CM abelian variety over a field of characteristic zero with this extra information. An isogeny induces an isomorphism of the endomorphism algebras; in characteristic zero the type of a CM abelian variety is invariant under isogenies.

However in positive characteristic p the action of the endomorphism ring R cannot be extended to an action of the endomorphism algebra D on the tangent space T, because $p.1 \in R$ acts as zero on T and $\mathrm{End}(A)$ has no $\mathbb{Z}$-torsion. Moreover an isogeny might change the endomorphism ring, and it is not so easy to understand in which way an isogeny $A \to B$ changes the action of $\mathrm{End}(A)$ on T_A into the action of $\mathrm{End}(B)$ on T_B. Even if an isogeny leaves the endomorphism ring invariant, the action of this ring on the tangent space may change. This interesting phenomenon lies at the roots of aspects of the theory and the results we are going to describe.

Proposition 2.5. *Let L be a number field. The following conditions are equivalent.*

- *There exists a subfield $L_0 \subset L$ which is totally real, $[L : L_0] = 2$ and L is totally imaginary.*
- *There exists an involution $\iota \in \mathrm{Aut}(L)$ such that for every embedding $\psi : L \to \mathbb{C}$ complex conjugation on $\mathbb{C}$ leaves $\psi(L) \subset \mathbb{C}$ invariant, and its restriction to $\psi(L)$ coincides with ι.* □

Here "totally real" for a field L_0 means that every embedding of L_0 into $\mathbb{C}$ gives an image contained in $\mathbb{R}$. "Totally complex" for a field L means that no embedding of L into $\mathbb{C}$ gives an image contained in $\mathbb{R}$. For details see [63], Ch. 1, § 2, see [12].

Definition 2.6. A finite extension L of $\mathbb{Q}$, i.e. a number field, is called a CM field if it satisfies one of the equivalent conditions of the previous proposition.

Remark 2.7. If A is a CM abelian variety, then there exists a CM field $L \subset \mathrm{End}^0(A)$ with $[L : \mathbb{Q}] = 2{\cdot}\dim(A)$, see [133], Lemme 2 on page 100. However, warning: a subfield of this size inside $\mathrm{End}^0(A)$ need not be a CM field.

2.8. Some properties of CM fields and of CM abelian varieties have been described in: [130], [63], [132], [78], [128], [133], [64], [97].

3. The isogeny class of a CM abelian variety is defined over a finite extension of the prime field

Definition 3.1. Let A be an abelian variety over a field K, and let $K_1 \subset K$ be a subfield. We say that A *can be defined over* K_1 if there exists a field $K \subset K_2$ and an abelian variety B over K_1 such that $A \otimes K_2 \cong B \otimes K_2$.

Note that some authors use a different definition, saying that A can be defined over K_1 if it can be descended down from K to K_1.

We remark, with notation as in the definition, that this does not imply there exists an abelian variety C over K_1 such that $C \otimes K \cong A$. An example is given in [118], 15.2: for $K = \mathbb{F}_{p^2}$ with $p \equiv 3 \pmod 4$, the Weil p^2-number $\pi = p{\cdot}\sqrt{-1}$ defines (the isogeny class of) an abelian variety A over $K = \mathbb{F}_{p^2}$ such that A can be defined over $K_1 = \mathbb{F}_p$, but such that A cannot be descended directly to K_1.

Here is another example. Let $f \in \mathbb{Q}[X]$ be a cubic polynomial with no multiple zeros in $\overline{\mathbb{Q}}$. Let t be a transcendental over $\mathbb{Q}$, and $K = \mathbb{Q}(t)$. Consider the elliptic curve E over K defined by $tY^2 = f$. We see that $E \otimes \mathbb{Q}(\sqrt{t})$ can be descended to $\mathbb{Q}$; hence E can be defined over $\mathbb{Q}$, although E itself cannot be descended to $\mathbb{Q}$. The theory of quadratic twists explains this example, and shows that we can give such examples for every elliptic curve over a field K which admits a separable quadratic extension.

Here is again an example; compare with 5.21. Consider the elliptic curve E (as the complete, nonsingular model of the affine curve)

$$Y^3 = X{\cdot}(X-1){\cdot}(X-t), \quad \text{where} \quad \mathbb{Q} \subset \mathbb{Q}(t)$$

is a purely transcendental extension. We see that the morphism $(x, y) \mapsto x$ is a $3:1$ (Galois) covering ramified in three points. Hence the curve E can be defined over $\mathbb{Q} = \mathbb{Q}^a \cap \mathbb{Q}(t)$. Another argument: the curve E has CM by $\mathbb{Z}[\zeta_3]$ given by $y \mapsto \zeta_3{\cdot}y$; from, this it follows that the elliptic curve E' defined by $\eta^2 = \xi^3 - 1$ has the property that over the algebraic closure $K^a = \mathbb{Q}(t)^a$ we have $E \otimes K^a \cong E' \otimes K^a$. This can all be made explicit by a computation. We see that E cannot be descended to $\mathbb{Q}$, but there exists a cubic extension $K \subset K'$ such that $E \otimes K' \cong E' \otimes K'$.

Theorem 3.2 (Grothendieck, [95], [151])**.** *Let A be an abelian over field K which admits* smCM*; let $\mathbb{P} \subset K$ be the prime field contained in K.* Hence $\mathbb{P} = \mathbb{Q}$ or $\mathbb{P} = \mathbb{F}_p$. *There exists a finite extension $\mathbb{P} \subset K_1$ and an abelian variety A' over K which is K-isogenous to A such that A' can be defined over K_1.* □

A variant inspired by Grothendieck's proof was published, with his permission, in [95]. Another proof, sketched below, was given by Yu, see [151], 1.3 and 1.4. Note that an isogeny $A \sim_K A'$ as in the theorem can be chosen over K, as follows by the proof of Yu, but in general A' cannot be descended to a finite extension of $\mathbb{P}$.

Corollary 3.3 ([130], Proposition 26 on page 109)**.** *In case the characteristic of the base field is zero, an abelian variety A which admits* smCM *can be defined over a number field.* □

This result was proved long before 3.2 was published. As finite group schemes in characteristic zero are reduced, the result of this corollary also follows from the more general theorem above.

In case the characteristic of the base field equals p, and A is ordinary, or almost ordinary, i.e. the p-rank of an abelian variety A satisfies $f(A) \geq \dim(A) - 1$, and A admits smCM, then A can be defined over a finite extension of the prime field. However for lower p-rank an isogeny may be necessary as we shall see.

Example 3.4. *In positive characteristic there exist abelian varieties which admit* smCM, *and which cannot be defined over a finite field* (i.e. the isogeny as in the theorem sometimes is necessary). We give an example. Suppose A is an abelian surface over field K of characteristic $p > 0$. Suppose $f(A) = 0$. Then A is supersingular, i.e. over an algebraic closure $k = \overline{K}$ it is isogenous with a product of two elliptic (supersingular) curves. There are precisely two possibilities:

- either $a(A) = 2$; in this case $A \otimes k \cong E^2$;
- or $a(A) = 1$; in this case there is a unique $\alpha_{p,K} \subset A$, unique up to a K-automorphism of $\alpha_{p,K}$, and $a(A/\alpha_p) = 2$.

Let E be a supersingular elliptic curve over a finite field κ. (For every finite field there exists a supersingular elliptic curve over K, as follows by results by Deuring, or by the Honda-Tate theory). Let t be a transcendental over κ, and $K := \kappa(t)$. Fix $\alpha_p \subset E$, and construct

$$\varphi : \alpha_p \hookrightarrow \alpha_p \times \alpha_p \subset (E \times E) \otimes K, \quad \text{by} \quad \varphi = (1, t).$$

We see that $A := ((E \times E) \otimes K)/\varphi(\alpha_p)$ is defined over K and $\mathrm{End}^0(E^2) = \mathrm{End}^0(A)$. We easily check: $a(A) = 1$, and A *cannot be defined over a finite field.* Moreover E admits smCM over κ, and $E^2_K \sim A$, hence A admits smCM over K. This is a typical example illustrating the theorem; see [93] ; also see [77].

Remark 3.5. If an abelian variety C admits smCM, then there exists a finite extension $K \subset K'$, and a sequence $C \otimes K' =: B_1 \to B_2 \to \cdots \to B_n$ of quotients by α_p over K' such that B_n can be descended down to a finite field.

The example above is a special case of a general phenomenon. For any abelian variety B over a finite field with $f(B) < \dim(B) - 1$ there exists a field K and an abelian variety $A \sim_K B \otimes K$ such that A admits smCM, and such that A cannot be defined over a finite field.

In fact, for every moduli point in $\mathcal{A}_g \otimes \mathbb{F}_p$ one can define the isogeny leaf passing through that point, see [111], see § 11. The dimension of an isogeny leaf depends on the Newton polygon involved, and on the polarization. The dimension of isogeny leaves is determined in [119]; it is positive in case the p-rank is at most $g - 2$; see § 11. A generic point of an isogeny leaf of positive dimension through a CM point gives a CM abelian variety which cannot be defined over a finite field.

3.6. The Serre tensor construction. An explanation can be found in [20], Section 7, and also in [12]. Consider a scheme S, a commutative ring R, an abelian scheme $A \to S$ and a ring homomorphism $R \to \mathrm{End}(A)$. Let M be a projective R-module of finite rank. Using that M is projective, one shows that the functor $T \rightsquigarrow M \otimes_R A(T)$ on S-schemes is represented by an S-scheme. The representing object will be denoted by $M \otimes_R A$, and the operation $A \mapsto M \otimes_R A$ is called the Serre tensor construction.

Working over a general base, the condition that M is R-projective is needed in general; see 4.10. However, working over a base field '"finitely generated" suffices. We will use this in the following situation: A is an abelian variety over a field, L

is a number field, $R = \mathrm{End}(A)$ is an order in L. Clearly the ring of integers $\mathcal{O}_L$ of L is finitely generated over R. We obtain $\mathcal{O}_L \otimes_R A$, an abelian variety isogenous with A, with $\mathcal{O}_L \subset \mathrm{End}(A)$.

Theorem 3.7 (Poincaré-Weil)**.** *Let A be an abelian variety over a field K. There exist simple abelian varieties $B_1, \cdots, B_n$ over K and an isogeny $A \sim \prod B_i$.* □

This theorem is well know. A proof in case K is a perfect field is not difficult. For a proof over an arbitrary field, see [36].

3.8. Sketch of a proof by Yu of 3.2; see [151]. We can choose an isogeny $A \sim \prod B_i$ with every B_i simple over K by the Poincaré-Weil theorem. In case A admits smCM, every B_i admits smCM. Hence it suffices to prove the theorem in case A is simple over K.

For a simple abelian variety $D := \mathrm{End}^0(A)$ is a division algebra, central simple over its center. As A admits smCM we conclude by [133], lemme 2 on page 100 there exists a CM field $L \subset D$ with $[L : \mathbb{Q}] = 2{\cdot}\dim(A)$. By the Serre tensor construction there exists an isogeny $A \sim_K B$ such that the ring of integers $\mathcal{O}_L$ of L is contained in $\mathrm{End}(B)$; we have $\iota : \mathcal{O}_L \to \mathrm{End}(B)$. Moreover B can be chosen in such a way that B admits an $\mathcal{O}_L$-linear polarization λ of degree d^2 prime to the characteristic of K. In [151], Section 3 a certain moduli space is constructed of (C, ι, μ) where $\deg(\mu) = d^2$, with certain properties on the Lie algebra of the abelian schemes considered. This deformation functor of triples (C, ι, μ) where $\deg(\mu) = d^2$, with certain properties, is "rigid", and as a scheme it is represented and finite over K, see [151], 3.7 and [150]. This finishes a sketch of this proof of 3.2. □

Remark 3.9. Finiteness of polarizations of a given degree up to isomorphisms on an abelian varieties already appeared in [82]. Rigidity of a deformation functor appeared in the case of superspecial abelian varieties in [94], 4.5.

4. CM liftings

4.1. In the Honda-Tate theory CM liftings are constructed and used, see [46], [133]. A refined study whether CM liftings exist in all situations is studied in [99] and in [12]. Also see [132], [21], [118].

Definitions 4.2. Let $\kappa \supset \mathbb{F}_p$ be a field, and let A_0 be an abelian variety over κ. A *lifting* of A_0, meaning *a lifting to characteristic zero*, is given by an integral domain Γ of characteristic zero and an abelian scheme $A \to \mathrm{Spec}(\Gamma)$ with a given isomorphism $A \otimes_\Gamma \kappa \cong A_0$.

If moreover A_0 is a CM abelian variety, a CM lifting is a lifting as above with the property that an order Λ in a CM algebra of rank $2{\cdot}\dim(A_0)$ is contained in $\mathrm{End}(A)$. This implies that over the field of fractions $K = \mathrm{frac}(\Gamma)$ we have that $A \otimes_\Gamma K$ is a CM abelian variety (in characteristic zero).

Question 4.3. Suppose given a CM abelian variety A_0 over $\kappa \supset \mathbb{F}_p$. Does there exist a CM lifting of A_0?

Remark 4.4. What does "rigidity" of CM abelian varieties suggest about this question?

It is very easy to give an example where the answer to 4.3 is negative. Take any abelian variety A_0 which cannot be defined over a finite field, but which is isogenous to an abelian variety defined over a finite field; by a result by Tate, see [132], we know A_0 is a CM abelian variety. If a CM lifting would exist then the generic fiber of that lifting could be defined over a number field, see 3.3. This gives a contradiction with the fact that A_0 cannot be defined over a finite field. This idea can also be implemented for certain abelian varieties over finite fields, see [99], and we obtain the result as in 4.7.

If A_0 is an ordinary (i.e. $f(A_0) = \dim(A)$) or an almost ordinary abelian variety (i.e. we require $f(A_0) = \dim(A) - 1$) over a finite field, then a CM lifting exists. For an ordinary abelian variety this follows from the Serre-Tate theory of canonical liftings [67], [56], [58]; for an almost ordinary abelian variety, see [99], Section 2, see [96], 14.6.

Theorem 4.5 (Honda; [46], Th 1 on page 86; [133], Th. 2)**.** *Let A_0 be an abelian variety over a finite field κ. Then there exists a finite extension $\kappa \subset \kappa'$ and an isogeny $A_0 \otimes \kappa' \sim B_0$ such that B_0 admits a* CM *lifting.* □

Questions 4.6. Consider an abelian variety over a finite field κ. In order to be able to perform a CM lifting

is an isogeny necessary ? an answer will be given in 4.7;

is a field extension necessary ? there will be two answers.

- In order to be able to perform a CM lifting to a *normal domain*: yes, a field extension might be necessary; see 4.8.
- However, for any A_0 over a finite field κ there exists a κ-isogeny $A_0 \sim B_0$ such that B_0 admits a CM lifting to a characteristic zero domain (which need not be normal); see 4.9.

Theorem 4.7 ([99] Th. B, and [12]: in general, an isogeny is necessary)**.** *For every prime number p, every integer g and every integer f such that $0 \leq f \leq g-2$ there exists an abelian variety A_0 over $\mathbb{F}$, an algebraic closure of $\mathbb{F}_p$, such that $\dim(A) = g$ and $f(A) = f$ and such that A_0 does not admit a* CM *lifting to characteristic zero.* □

Example 4.8. See [12]. *There is an example of an abelian variety A_0 over a finite field κ such that for any $A_0 \sim_\kappa B_0$ the abelian variety B_0 does not admit a* CM *lifting to a normal domain of characteristic zero.* In fact, consider a prime number $p \equiv 2,\ 3 \pmod 5$; this means that p remains prime in the cyclotomic extension $\mathbb{Q} \subset \mathbb{Q}(\zeta_5) =: L$. Consider $\pi := p{\cdot}\zeta_5$. This is a Weil q-number for

$q = p^2$; this means that π is an algebraic integer and for every complex embedding $\psi : L \to \mathbb{C}$ the complex number $\psi(\pi)$ has absolute value $\sqrt{q}$. We use Honda-Tate theory; see [133], [144], [118]. This tells us that there exists a simple abelian variety A over $\mathbb{F}_q$ whose Weil number equals $\pi = \mathrm{Frob}_{A,q}$ (and A is unique up to $\mathbb{F}_q$-isogeny). From properties of π one can read off the structure of the division algebra $D = \mathrm{End}^0(A)$; see [133], Th. 1 on page 96; see [118], 5.4 and 5.5. From the fact that $\mathbb{Q} \subset \mathbb{Q}(\pi) = L$ is unramified away from p, and the fact that there is a unique prime above p in L, it follows that D/L is split away from p; hence D/L has all Brauer-invariants equal to zero; hence $D = L$. This proves $\dim(A) = 2$.

Next we compute the reflex field of a CM type of L; see [63], 1.5. As $L/\mathbb{Q}$ is Galois, the reflex field is a CM field contained in L. However L itself is the only CM field contained in L. *Hence L is its own reflex field for any* CM type.

Let $A \sim B_0$, and suppose $B \to \mathrm{Spec}(\Gamma)$ is a CM lifting of B_0 to a *normal* domain Γ with field of fractions $K = \mathrm{frac}(\Gamma)$. As $L = \mathrm{End}^0(A) = \mathrm{End}^0(B_0)$ is a field, and B admits smCM, we would conclude $L = \mathrm{End}^0(B) = \mathrm{End}^0(B_K)$. As any field of definition of a CM abelian variety in characteristic zero contains the reflex field, [130], Prop. 30 on pp. 74/75, see [63], 3.2 Th. 1.1, we conclude that K contains a reflex field of L; hence $K \supset L$. Hence the residue class of Γ on the one hand is $\mathbb{F}_q$ (here we use normality of Γ); on the other hand it contains the residue class field of L at p which is $\mathbb{F}_{p^4}$. The contradiction $\mathbb{F}_{p^4} \subset \mathbb{F}_{p^2}$ proves that any abelian variety $\mathbb{F}_q$-isogenous to A cannot be CM lifted to a normal domain in characteristic zero. This method, using the "residual obstruction", is discussed in [12].

Theorem 4.9 (B. Conrad - Chai - Oort; [12])**.** *For any abelian variety A over a finite field κ there exists a κ-isogeny $A \sim B_0$ such that B_0 admits a* CM *lifting to characteristic zero.* □

A proof of this theorem is quite involved. We note that, even if $\mathrm{End}^0(A)$ is a field, in general any CM lifting may have an endomorphism ring which is smaller than $\mathrm{End}(B)$; see 4.10.

Remark 4.10. In [12] it is shown that there exists a CM lifting of $B \to \mathrm{Spec}(\Gamma)$ of any B_0 over $\mathbb{F}_{p^2}$ as in 4.8, with $\mathrm{End}(B_0) = \mathbb{Z}[\zeta_5]$. The generic fiber of such a lifting has the property

$$\mathbb{Z} + 5{\cdot}\mathbb{Z}[\zeta_5] \subset \mathrm{End}(B_\eta) := R \subsetneqq \mathbb{Z}[\zeta_5].$$

In this case the Serre tensor construction $B \otimes_R \mathbb{Z}[\zeta_5]$ over a ring in mixed characteristic is not representable.

5. Abelian varieties isogenous to a Jacobian

5.1. Main reference: [17]; [112]; see [135]. In this section we mostly work over $\mathbb{Q}^a := \overline{\mathbb{Q}}$.

Question 5.2. *Suppose given an algebraically closed field k, and an integer $g > 3$. Does there exist an abelian variety A which is not isogenous to the Jacobian of an algebraic curve over k ?*

Various things have to be explained. Do we also take in consideration a polarization on A, in this case we consider Hecke orbits, or are we considering isogenies which need not respect (the $\mathbb{Q}$-class of) a polarization? Do we consider only irreducible curves, or are also reducible curves considered? It turns out that these details are not of much influence on the statement in results we have. We refer to [17] for precise description of these details.

5.3. Bjorn Poonen suggested not only to consider the Torelli locus, but, more generally, to ask whether for every $g > 0$ and every closed subset $X \subset \mathcal{A}_g \otimes k$ of dimension smaller than $\dim(\mathcal{A}_g \otimes k)$ there exists an abelian variety A whose isogeny orbit $\mathcal{I}(A)$ does not meet X. The case above is the special case of the closed Torelli locus $\mathcal{T}_g = X$, which indeed is lower dimensional if $g > 3$. This more general situation can be phrased as a statement (which might be true or false):

5.4. $\mathrm{I}(k, g)$ *For every closed subset* $X \subsetneq \mathcal{A}_g \otimes k$, *with* $\dim(X) < g(g+1)/2 = \dim(\mathcal{A}_g)$ *there exists* $[(A, \lambda)] = x \in \mathcal{A}_g(k)$ *such that* $\mathcal{I}(x) \cap X = \emptyset$. Writing $\dim(X)$ we implicitly assume that all irreducible components of X have the same dimension.

Remark 5.5. An easy argument shows that for any *uncountable* field k of characteristic zero the statement $\mathrm{I}(k, g)$ is true, and hence in that case Question 5.2 has a positive answer.

Definition 5.6. A moduli point $[(A, \lambda)] = x \in \mathcal{A}_g$ is called a CM *point*, or is called a *special point*, if A admits smCM over an algebraically closed field of definition. A closed subset $S \subset \mathcal{A}_g \otimes \mathbb{Q}^a$ is called a *special subset* if it is a finite union of Shimura subvarieties; we refer to the theory of Shimura varieties for this notion. E.g. see [76].

Note that a CM point and a special subset in characteristic zero is defined over $\mathbb{Q}^a$. Note that a special point is a Shimura subvariety.

Conjecture 5.7 (**(AO)**, the André-Oort conjecture)**.** *Let T be a Shimura variety. Let $\Gamma \subset T(\mathbb{Q}^a)$ be a set of special points. Then it is conjectured that the Zariski closure Γ^{Zar} is a special subset, i.e. a finite union of special subvarieties.*

This was mentioned as Problem 1 on page 215 of [1] for curves in a Shimura variety. Independently this was conjectured for closed subsets of $\mathcal{A}_g$ of arbitrary dimension; see [101], 6A, and [102]. The common generalization is called the *André-Oort conjecture.* Also see [2]; see [109], § 4, § 5.

Special cases were proved by André, Edixhoven, Moonen, Yafaev, Clozel-Ullmo. *The general case of this conjecture is claimed to be true under assumption* *of the Generalized Riemann Hypothesis in papers by Klingler-Yafaev and Ullmo-Yafaev, see* [60], [136].

5.8. Present status of the question $\mathrm{I}(k, g)$.

For $k = \mathbb{C}$ this property holds.

For $k = \mathbb{Q}^a$ the property holds under assumption of (AO); see 5.9.

For $k = \mathbb{F} := \overline{\mathbb{F}_p}$ it seems to be unknown whether this property holds. Also in this case the answer to Question 5.2 is unknown.

Theorem 5.9 (Chai-Oort, [17])**.** *For any g and for the base field $k = \mathbb{Q}^a$, if the conjecture* (AO) *holds, then* $\mathrm{I}(k, g)$ *holds.* □

Remark 5.10. Instead of considering $\mathcal{A}_g$ one can also consider an arbitrary Shimura variety (over $\mathbb{C}$, or over $\mathbb{Q}^a = \overline{\mathbb{Q}}$), and consider Hecke orbits. The analogue of 5.9 also holds, under (AO), in this more general situation. We refer to [17] for details. In this note we will only consider the case of $\mathcal{A}_g$.

5.11. We sketch some of the ideas going into the proof of 5.9. For details see [17].

We work over $k = \mathbb{Q}^a$ and we write $\mathcal{A}_g$ instead of $\mathcal{A}_g \otimes \mathbb{Q}^a$. If L is a CM field, $[L : \mathbb{Q}] = 2g$, then the normal closure $L^{\sim}$ of L has degree at most $2^g{\cdot}(g!)$ over $\mathbb{Q}$. *We say that L is a Weyl* CM *field if $[L^{\sim} : \mathbb{Q}] = 2^g{\cdot}(g!)$. We say that $[(A, \lambda)] = x \in \mathcal{A}_g(\mathbb{Q}^a)$ is a Weyl* CM *point if the related* CM *algebra is a Weyl* CM *field.* It can be shown that for any given $g > 0$ there are "many" Weyl CM fields; e.g. see [18]; in fact:

Proposition 5.11(a). *For any number field E and any given g there is a Weyl* CM *field of degree $2g$ such that L and E are linearly disjoint over $\mathbb{Q}$.* See [17]. □

Note: *if A is a Weyl* CM *abelian variety* (in characteristic zero), *then A is absolutely simple.* Hence a Weyl CM Jacobian automatically is the Jacobian of an irreducible curve. Or: a Weyl CM point in the closed Torelli locus $\mathcal{T}_g$ is already in the open Torelli locus $\mathcal{T}_g^0$.

Proposition 5.11(b). *Let L be a Weyl* CM *field with maximal totally real field $E = L_0 \subset L$. If $Y \subset \mathcal{A}_g$ is a special subvariety with $0 < \dim(Y) < g(g+1)/2$ which contains a Weyl* CM *point associated with L, then Y is a Hilbert modular variety associated with L_0.* See [17]. □

Once these properties are established we are able to prove the Theorem 5.9 as follows. Consider $X \subset \mathcal{A}_g$ of dimension less than $g(g+1)/2$. Consider the set $\Gamma = \mathrm{CM}(X)$ of all CM points in X. Assuming (AO) we know that $\Gamma^{\mathrm{Zar}} =: S \subset X$ is a finite union of special subvarieties. There are three kind of irreducible components:

- those of dimension zero $S_1, \cdots, S_a$, associated with CM algebras $L_1, \cdots, L_a$,
- $S_{a+1}, \cdots, S_b$ which are a Hilbert modular variety associated with a totally real algebras $E_{a+1}, \cdots, E_b$,
- and all other components.

Using 5.11(a) we can choose a Weyl CM field L of degree $2g$ linearly disjoint from the compositum of $L_1, \cdots, L_a, E_{a+1}, \cdots, E_b$. Let x be Weyl CM point associated with this Weyl CM field. As L is not isomorphic with $L_1, \cdots, L_a$ we see that $\mathcal{I}(x)$ does not contain a zero dimensional component S_i with $i \leq a$. As S is contained in X we see that every irreducible component has dimension less than $g(g+1)/2$; note that L is linearly disjoint from each of the E_i; hence by Proposition 5.11(b) we see that $\mathcal{I}(x)$ does not contain a point in a positive dimensional component of S. Hence $\mathcal{I}(x) \cap S = \emptyset$; this proves the theorem. □

Remark 5.12. We say that a totally real field L_0 of degree g over $\mathbb{Q}$ is *of Weyl type* if the normal closure $(L_0)^{\sim}$ has degree $g!$ over $\mathbb{Q}$. Does 5.11(b) hold for CM points where the totally real field is of Weyl type? Suppose L is a CM field of degree $2g$ over $\mathbb{Q}$ and its totally real field is of Weyl type. In this case there are three possibilities for the normal closure $L^{\sim}$:

- $[L^{\sim} : \mathbb{Q}] = 2{\cdot}g!$;
- (this case only can occur in case g is even) $[L^{\sim} : \mathbb{Q}] = 2^{g-1}{\cdot}g!$;
- (L is a Weyl CM field) $[L^{\sim} : \mathbb{Q}] = 2^{g}{\cdot}g!$.

There are many Shimura varieties in $\mathcal{A}_g$ containing points of the first kind, which are not Hilbert modular varieties. For example, we can take a PEL Shimura variety associated with a quadratic complex field. In other words: in order to have a result like 5.11(b) it does not suffice to consider CM fields with totally real field of Weyl type. However (in case g is even), for a CM point as in the second case the analogue of 5.11(b) does hold: a lower dimensional special subvariety of positive dimension containing such a point is a Hilbert modular variety.

Theorem 5.13 (Tsimerman, [135])**.** *For any g and for the base field $k = \mathbb{Q}^a$, then* $\mathrm{I}(k, g)$ *holds.* □

This makes use of 5.9. In his prove Tsimerman constructs an infinite sequence of Weyl CM points, not using GRH, of which only finitely many have an isogeny orbit intersecting a given X.

Remark 5.14. In [19], Conjecture 6, we find the conjecture that for $g > 3$ there should be only finitely many CM Jacobians (of irreducible curves) of dimension g. A.J. de Jong and R. Noot showed this is not correct; see [51]; see 5.20 below. Later we realized that examples by Shimura, see [129], could be used to contradict this conjecture. For $g = 4, 5, 6, 7$ we can find infinitely many irreducible algebraic curves with CM Jacobian; see 5.20. It might be that for large g the conjecture still holds. We modify the conjecture.

Theorem 5.15 (Modified Coleman conjecture; Chai-Oort, [17])**.** *Assume* (AO). *For any $g > 3$ the number of isomorphism classes of algebraic curves of genus g over $\mathbb{Q}^a$ with Weyl* CM *Jacobian is* finite. (Such curves are irreducible and regular.) □

5.15(1). **Proposition** (A.J. de Jong and S. Zhang, [53], Corollary 1.2). *Let L_0 be a totally real field of degree g over $\mathbb{Q}$.*

If $g > 4$ no Hilbert modular variety attached to L_0 is contained in $\mathcal{T}_g \subset \mathcal{A}_{g,1}$.

If $g = 4$ and a Hilbert modular variety attached to L_0 is contained in $\mathcal{T}_4 \subset \mathcal{A}_{4,1}$ then L_0 contains a field quadratic over $\mathbb{Q}$. □

For the case $g = 4$ it seems unknown whether there exists a Hilbert modular variety contained in $\mathcal{T}_4 \subset \mathcal{A}_{4,1}$.

Proof of Theorem 5.15. Note that for a Weyl CM field L its maximal totally real subfield L_0 does not have any $\mathbb{Q} \subsetneq E' \subsetneq L_0$. By 5.15(1) we see that for any Weyl CM field with $g > 3$ a corresponding Hilbert modular variety is not contained in $\mathcal{T}_g$. Suppose the moduli point of a Weil CM abelian variety is contained in $\mathcal{T}_g$. By 5.11(b) any special subvariety *of positive dimension* containing this point is a Hilbert modular variety. Using (AO) this implies that the Zariski closure of all Weyl CM points in the (open or closed) Torelli locus is a finite set. □

Note that a Weyl CM point in $\mathcal{T}_g$ is contained in $\mathcal{T}_g^0$.

Remark 5.16. At present there seems to be no proof of the modified Coleman conjecture avoiding the use of (AO).

Definition 5.17. Let k be an algebraically closed field. Let C be a complete, irreducible, regular curve over k. Write $G := \mathrm{Aut}(C)$. We say that C has *many automorphisms* if the local deformation functor of (C, G) on schemes over k is representable by a zero-dimensional scheme.

Question 5.18. *How can we find irreducible, regular, complete curves of genus g with* CM *Jacobian*?

It seems difficult to give a complete answer. I know two methods which cover special cases.

5.18 **(1). Curves with "many automorphisms".** We can try to find a curve with automorphisms, such that the action of $\mathrm{Aut}(C)$ guarantees that the Jacobian has smCM.

Note that for a given value of g the number of isomorphism classes of curves of genus g with many automorphisms is finite. Hence it is not possible along these ideas to give infinitely many CM curves for a given genus.

Note that a curve with many automorphisms, as defined above, does not give a Weyl CM Jacobian.

Most CM curves I know do not have many automorphisms.

Probably there exist curves with many automorphisms which are not a CM curves.

5.18 **(2). Shimura varieties inside the Torelli locus.** Suppose given g, and a Shimura subvariety $S \subset \mathcal{A}_{g,1}$ contained in the closed Torelli locus:

$$S \subset \mathcal{T}_g \quad \text{with} \quad S \cap \mathcal{T}_g^0 \neq \emptyset.$$

As the set of CM points is dense in S we see that if $\dim(S) > 0$ then the set of CM points in $S \cap \mathcal{T}_g^0$ is infinite. This is the case for $1 \leq g \leq 3$ and $S = \mathcal{A}_g$. For $g = 4, 5, 6, 7$ along these lines infinite series of CM curves of a given genus can be constructed. We will indicate below an idea of such a construction.

Note that this is an existence theorem which a priori does not indicate how to construct explicit examples. In general it is difficult to use properties of $S \subset \mathcal{T}_g$ to derive a description of CM curves giving moduli points in S.

Here is an easy example which explains the difficulty. Every elliptic curve can be given (over an algebraically closed field of characteristic not equal to 2) by an equation

$$Y^2 = X(X-1)(X-\lambda).$$

There are many ways to see that for infinitely many values of λ the corresponding curve is a CM elliptic curve (an existence theorem). However I do not know an explicit formula to give infinitely many values of $\lambda \in \mathbb{C}$ with this property. The same remark applies to all examples used in 5.18 **(2).**

See [76] for a description of the question for the (non-)existence of Shimura varieties in the Torelli locus. Using an answer to that question and (AO) it might be that one can settle the original Coleman conjecture for certain values of g. However it seems a difficult question to determine all positive dimensional Shimura varieties in $\mathcal{T}_g$ for all g.

Examples 5.19. **(1a)** $g = (n-1)(n-2)/2; \quad g = 1, 3, 6, 10, \cdots.$

Consider for any $n \in \mathbb{Z}_{>2}$ the Fermat curve defined by

$$X^n + Y^n = Z^n.$$

The genus of this irreducible, regular curve over $\mathbb{Q}^a$ equals $(n-1)(n-2)/2 > 0$ whose Jacobian has smCM. See [127], VI, 1.2 and 1.5

(1b) $g = (\ell - 1)/2; \quad g = 1, 2, 3, 5, \cdots.$ Consider an odd prime number ℓ, and define $2g + 1 = \ell$. A curve defined (as the normalization of the compactification of the curve defined) by

$$Y^\ell = X^a(X-1)$$

with $1 \leq a \leq g$ has genus g and its Jacobian has smCM; see [148], pp.814/815; see [53], 1.4.

(1c) (Example communicated to me by Yuri Zarhin.) If $p > 3$ is a prime and $r \in \mathbb{Z}_{\geq 1}$, then the curve defined by

$$y^3 = x^{p^r} - 1$$

is a CM curve. Its genus is $g = p^r - 1$. Its Jacobian is isogenous with abelian varieties of dimension $p^i - p^{i-1}$ with smCM by $\mathbb{Q}(\zeta_{3 \cdot p^i})$ for $1 \leq i \leq r$. Hence the curve is a CM curve.

(1d) For every $a \in k$ and $a \neq 0$ and every $d, m \in \mathbb{Z}_{\geq 0}$ consider the curve defined by $y^d = x^m - a$; see [137], 2.4. This is a CM curve.

Probably there are many more examples of curves with smCM given by many automorphisms.

Examples 5.20. The idea to find CM Jacobians by constructing special subvarieties contained in $\mathcal{T}_g$ was formulated and carried out in special cases in [51]. Their method produces an infinite number of CM Jacobians of a given genus g for which a family of a particular kind can be constructed. Since then more examples have been found. We now have 10 examples of positive dimensional special subvarieties constructed in this way inside $\mathcal{T}_g$ meeting $\mathcal{T}_g^0$ for $4 \leq g \leq 7$. They can be found in [51], [129], [22], [126], [76], [109]. Below we mention 4 of these examples in order to indicate the line of ideas, and we present the original idea contained in [51] why this method does work in the cases indicated. For another, better method and for a complete survey see [76].

(2a) $g \leq 3$. In this case the dimension of $\mathcal{M}_g$ equals the dimension of $\mathcal{A}_g$. Because $\mathcal{A}_{g,1} \otimes k$ is irreducible, we obtain $\mathcal{T}_g \otimes k = \mathcal{A}_{g,1} \otimes k$ for $g \leq 3$. Hence in these cases every CM point of $\mathcal{A}_{g,1}$ defines a CM Jacobian (of a possibly reducible curve). In this case there are many CM Jacobians of an irreducible curve; however the existence theorem does not indicate how to construct such curves explicitly. Already for $g = 1$ it is easy to see how to construct a complex torus with CM. In every concrete case one can derive from the analytic presentation an algebraic equation for the corresponding elliptic curve. However I do not know a mechanism to produce all CM Jacobians in this way.

(2b) $g = 4$. See [129], (2) and see [51], 1.3.1; $g = 4$. The family of curves of genus 4 defined by

$$Y^3 = X(X-1)(X-a)(X-b)(X-c)$$

gives an irreducible component of a PEL type Shimura variety of dimension three inside $\mathcal{T}_4$ meeting $\mathcal{T}_4^0$.

(2c) $g = 5$. The family of curves of genus 5 defined by

$$Y^8 = X^2(X-1)(X-a)$$

gives an irreducible component of a PEL type Shimura variety inside $\mathcal{T}_5$ meeting $\mathcal{T}_5^0$.

(2d) $g = 6$. The family of curves of genus 6 defined by

$$Y^8 = X^2(X-1)^2(X-a)$$

gives an irreducible component of a PEL type Shimura variety inside $\mathcal{T}_6$ meeting $\mathcal{T}_6^0$.

(2e) $g = 7$. Consider the curve

$$Y^9 = X(X-1)(X-a).$$

See [109], 7.7. See [126], [76].

5.21. We explain the essential steps in the proof showing that Example 5.20 (2e) indeed gives a positive dimensional special variety inside T_g, in this case for $g = 7$, meeting the interior $\mathcal{T}_g^0$; for many more examples, references and for better proofs we refer to [126] and to [76]; see [75] for all examples obtained by families of cyclic covers known at present.

The equation $Y^9 = X(X-1)(X-a)$ defines an affine plane curve, which is non-singular for $a \neq 0, 1$; the normalization of a complete model of this curve, for a fixed a is a curve C_a of genus $g = 7$, as we see by an application of the Zeuthen-Hurwitz formula. In this way we obtain a morphism $\mathbb{P}^1 - \{0, 1, \infty\} \to \mathcal{M}_7$. We denote by Λ the closure of its image in $\mathcal{A}_{7,1}$. Following arguments in [51] we show below *that* $\Lambda \subset \mathcal{A}_{7,1}$ *is a special subvariety.*

A basis for the regular differentials on every fiber is given by:

$$\{\frac{XdX}{Y^8}, \frac{XdX}{Y^7}, \frac{dX}{Y^8}, \frac{dX}{Y^7}, \frac{dX}{Y^6}, \frac{dX}{Y^5}, \frac{dX}{Y^4}\}.$$

$$\text{Non-primitive:} \quad \frac{dX}{Y^6}. \quad \text{Dual pair:} \quad \{\frac{dX}{Y^5}, \frac{dX}{Y^4}\}.$$

Remark that $T^3 = X(X-1)(X-a)$ defines an elliptic curve, its Jacobian has smCM by $\mathbb{Q}(\zeta_3)$ and we have a dominant morphism $C_a \to E_a$; up to isogeny the Jacobian of C_a is a product of this elliptic curve an abelian variety of dimension 6; this decomposition is given by the primitive, and the non-primitive weights.

From this basis for the regular differentials on C_a we see the weights of the action $Y \mapsto \zeta_9{\cdot}Y$ by the CM field $\mathbb{Q}(\zeta_9)$ on the tangent space of the Jacobian $\mathrm{Jac}(C_a)$. Take any of the fibers $\mathrm{Jac}(C_a)$, and study the PEL Shimura variety given by this action. By [128], Theorem 5 on page 176, see the proof on page 182, we conclude that the dimension of this Shimura variety equals the number of dual pairs. Hence in this case the dimension of this PEL Shimura variety is equal to the number of parameters, which equals one. As this is the dimension of the Zariski closure $\Lambda \subset \mathcal{T}_7$ of the image of the moduli map of the base of $C \to \mathbb{P}^1 - \{0, 1, \infty\}$ into $\mathcal{T}_g^0$ we conclude that Λ is an irreducible component of this PEL Shimura variety; hence the image of $\mathbb{P}^1 - \{0, 1, \infty\}$ is dense open inside a special subvariety contained in $\mathcal{T}_7$. □

In fact the morphism $\mathbb{P}^1 - \{0, 1, \infty\} \to \Lambda \subset \mathcal{T}_7$ extends to a morphism $\mathbb{P}^1 \to \mathcal{T}_7$. The Zariski closure Λ inside $\mathcal{T}_7$ is complete, but this is of no importance for the argument proving there exist infinitely many $g = 7$ CM Jacobians.

Question 5.22. How can we find CM curves of genus $g > 3$ which are "isolated" (in the sense not contained in the closure of an infinite set of points defined by

CM Jacobians) and not with many automorphisms? It seems plausible that such curves exist. How to find such curves?

Here is the central question which, up to now, seems unsettled.

5.23. Expectation. (See [102], § 5 and [76].) *For large g (in any case $g \geq 8$), there does not exist a special subvariety $Z \subset \mathcal{A}_{g,1} \otimes \mathbb{C}$ with* $\dim(Z) \geq 1$ *such that $Z \subseteq \mathcal{T}_g$ and $Z \cap \mathcal{T}_g^0$ is nonempty.*

Note that if this expectation holds for a certain g, and (AO) holds, then the original Coleman conjecture holds for that g.

5.24. It seems we know very little about possible answers to this expectation. We do not know whether there exists a special subvariety Z as in the expectation where the generic point corresponds with a Jacobian with endomorphism ring equal to $\mathbb{Z}$. We do not know whether for $g = 4$ any of the curves inside $\mathcal{A}_4$ described by Mumford in [80] is contained in the closed Torelli locus. We do not know whether for $g = 4$ there exists a Hilbert modular variety contained in the closed Torelli locus. We refer to [76] for a discussion and for a description of all examples know to us at present. Some aspects of our experience seem to indicate the Expectation 5.23 could be right. However I do not see any structural evidence at present, hence I like to call this an expectation, and not a conjecture yet.

Stratifications and foliations of moduli spaces of abelian varieties in positive characteristic

From now on all base fields, all base schemes will be in characteristic p.

6. Supersingular abelian varieties

6.1. The influential paper [24] studied properties of supersingular elliptic curves and their endomorphism algebras. In [77] and [66] families and moduli spaces of supersingular abelian varieties are studied.

6.2. Dieudonné modules. We work over a perfect field $K \supset \mathbb{F}_p$. *Dieudonné modules classify finite group schemes and p-divisible groups over a perfect field.*

Write $W = W_\infty(K)$ for the ring of infinite Witt vectors over K. Write $\sigma : W \to W$ for the (unique) lift of the Frobenius $a \mapsto a^p$ on K. Write D_K for the ring generated over W by $\mathcal{F}$ and $\mathcal{V}$ satisfying the well-know relations $\mathcal{F}\mathcal{V} = p = \mathcal{V}\mathcal{F}$, and $\mathcal{F}a = \sigma(a)\mathcal{F}$ and $a\mathcal{V} = \mathcal{V}\sigma(a)$. There is an *equivalence* of categories between left modules of finite length over D_K and finite commutative group schemes over K of p-power rank; for N we write $\mathbb{D}(N)$ for its covariant Dieudonné module. If N is of rank p^i, then $\mathbb{D}(N)$ is of length i.

As p-divisible groups are ind-limits of such finite group schemes this also classifies p-divisible groups over K. The category of local-local p-divisible groups over K is equivalent with the category of left D_K-modules which are free of finite rank over W and on which $\mathcal{F}$ and $\mathcal{V}$ operate nilpotently. The category of formal p-divisible groups is equivalent with the category of left D_K-modules which are free of finite rank over W and on which $\mathcal{V}$ operates nilpotently. The height of X equals the rank of $\mathbb{D}(X)$ as a free W-module. The dimension of the p-divisible group equals the K-dimension of $\mathbb{D}(X)/\mathcal{V}{\cdot}\mathbb{D}(X)$.

Remark 6.3. In the original version, see [68], the Dieudonnné functor is contravariant. Over a perfect field K duality of Dieudonné modules and Cartier duality of finite groups correspond. Hence the covariant and the contravariant theory for finite group schemes over a perfect field amount to the same under this operation. The same for duality of modules and the theory of Serre duality on p-divisible groups over K.

We note that the *Serre dual* X^t of a p-divisible group is defined as follows. For a p-divisible group X and $i \in \mathbb{Z}_{>0}$ we have an exact sequence

$$0 \to X[p^1] \longrightarrow X[p^i] \longrightarrow X[p^{i-1}] \to 0.$$

We write:

$$X^t = \text{lim.ind.}_i \ \ (X[p^i] \longrightarrow X[p^{i-1}])^D.$$

We have chosen to use the covariant Diedonné module theory as this is compatible with the theory of displays. In the covariant theory the morphism $F : N \to N^{(p)}$ results in multiplying with $\mathcal{V}$ on $\mathbb{D}(N)$ and $V : N^{(p)} \to N$ results in multiplying by $\mathcal{F}$ on $\mathbb{D}(N)$. Therefore we have distinguished the morphisms F and V on group schemes on the one hand and the operations $\mathcal{V}$ and $\mathcal{F}$ on Dieudonné modules on the other hand. For the theory of Dieudonné modules and related concepts, see [68], [23], [32]; see [104], 15.3 – 15.6 and [15], § 6 for further explanation and references. See [145] for a treatment of the Tate conjecture formulated for p-divisible groups.

6.4. Newton polygons. *Newton polygons classify p-divisible groups up to isogeny.* For a p-divisible group X over a field we define its Newton polygon as “the Newton polygon of the Frobenius action on X”. Over $K = \mathbb{F}_p$ this is a correct definition, but over any other field F need not be an endomorphism of X. Therefore a more refined definition has to be given.

For coprime, non-negative integers m and n we define a p-divisible group $X = G_{m,n}$. We write $G_{1,0} = \mathbb{G}_m[p^\infty]$ and

$$G_{0,1} = G_{1,0}^t = \underline{\mathbb{Q}_p/\mathbb{Z}_p}.$$

For $m > 0$ and $n > 0$ we define $G_{m,n}$ by the Dieudonné module

$$\mathbb{D}(G_{m,n}) := D_K/D_K{\cdot}(\mathcal{F}^m - \mathcal{V}^n).$$

We have $(G_{m,n})^t \cong G_{n,m}$ and hence $\dim((G_{m,n})^t) = n$. For $\gcd(m,n) = 1$ we see that $G_{m,n}$ is a simple p-divisible group of dimension m.

Remark on notation: these p-divisible groups are already defined over $\mathbb{F}_p$; for every $K \supset \mathbb{F}_p$ we will use the notation $G_{m,n}$ instead of $G_{m,n} \otimes_{\mathbb{F}_p} K$ if no confusion is possible.

To $G_{m,n}$ we attach the Newton polygon consisting of $m+n$ slopes equal to $m/(m+n)$; indeed this is the Newton polygon of $F : G_{m,n} \to G_{m,n}$ over $\mathbb{F}_p$.

We write $f(X)$ for the number of copies $G_{1,0}$ in the above sum; this is called the p-rank of X. Over an algebraically closed field k we have $\mathrm{Hom}(\mu_p, X) = (\mathbb{Z}/p)^{f(X)}$.

Theorem 6.5 (Dieudonné and Manin, [68])**.** *Let k be an algebraically closed field. For every p-divisible group X over k there are $d_i, c_i \in \mathbb{Z}_{\geq 0}$ and an isogeny*

$$X \quad \sim \quad \sum_i \quad G_{d_i,c_i}.$$

□

A Newton polygon is a lower convex polygon starting at $(0,0)$, ending at (d,h), such that the break points are in $\mathbb{Z} \times \mathbb{Z}$. To $\sum_i G_{d_i,c_i}$ with $\sum_i d_i = d$ and $\sum_i (d_i + c_i) = h$ we associate the Newton polygon obtained by arranging the slopes $d_i/(d_i+c_i)$ with multiplicity (d_i+c_i) in non-decreasing order. We write $\mathcal{N}(Y)$ for the Newton polygon defined by $X = Y \otimes k$ and the isogeny as above. The Newton polygon thus obtained we sometimes indicate by the (formal sum) $\sum_i (d_i, c_i)$.

The isogeny class of a p-divisible group over an algebraically closed field k uniquely determines (and is uniquely determined by) its Newton polygon:

6.6. Corollary [Dieudonné and Manin, see [68], page 35]
"Classification theorem" :

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}.$$

□

6.7. Note that for an abelian variety A its Newton polygon $\mathcal{N}(A) := \mathcal{N}(A[p^\infty])$ is *symmetric* in the sense that β and $1-\beta$ in $\mathcal{N}(A)$ have the same multiplicity. Over a finite field this was proved by Manin, see [68], page 74; in that proof the functional equation of the zeta-function for an abelian variety over a finite field is used. The general case (an abelian variety over an arbitrary field of positive characteristic) follows from [90], Theorem 19.1: that theorem proves $A[p^\infty]^t = A^t[p^\infty]$, and we finish by $(G_{m,n})^t \cong G_{n,m}$.

6.8. Example / Definition. For an elliptic curve E, an abelian variety of dimension one, over a field K of characteristic p, the possible Newton polygons are $(1,0)+(0,1)$ and $(1,1)$. The first case is called ordinary. In the second case we have the following *equivalent* statements

(1) $\mathcal{N}(E) = (1,1)$.

(2) The p-rank of E is zero.

(3) $E(k)[p] = 0$.
(4) $\mathrm{End}(E \otimes k)$ has rank four over $\mathbb{Z}$.
(5) Definition. E is *supersingular*.

6.9 (**Definition 1**)**.** To an abelian variety A over a field $K \supset \mathbb{F}_p$ we can associate its Newton polygon $\mathcal{N}(A)$, see 6.4. We say that A is *supersingular* if every slope in $\mathcal{N}(A)$ is equal to $1/2$, i.e.

$$A \text{ is supersingular} \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathcal{N}(A) = g{\cdot}(1,1).$$

Remark. Equivalently: $A[p^\infty] \otimes k \sim (G_{1,1})^g$.

We write $\sigma = \sigma_g$ for the Newton polygon consisting of $2g$ slopes equal to $1/2$.

Note that "A is supersingular" implies that the p-rank of A equals to zero: $f(A) = 0$. Conversely, every abelian variety of dimension $g = 1$ or $g = 2$ of p-rank zero is supersingular. However for every $g \geq 3$ there exist many abelian varieties of p-rank equal to zero of dimension g which are not supersingular.

Terminology. Of course, a supersingular abelian variety is not a singular variety. Where does the terminology come from? In characteristic zero, the j-invariant of an elliptic curve E with CM is called a *singular j-value*. Deuring in his influential paper [24] studied elliptic curves of p-rank zero over a finite field; such a curve (over a finite extension of the field) has an endomorphism ring which is a maximal order in a quaternion algebra; this ring is bigger than the endomorphism ring of a CM elliptic curve in characteristic zero; hence the terminology "supersingular" was invented. A purist perhaps would like to say "an elliptic curve with supersingular j-value".

Note that for any abelian variety A over any field we have $\mathrm{rk}_{\mathbb{Q}}\, \mathrm{End}^0(A) \leq (2g)^2$. Equality $\mathrm{End}^0(A \otimes k) = (2g)^2$ holds if and only if the base field K has positive characteristic, A is supersingular and all endomorphisms of A are defined over K.

6.10 (**Definition 2**)**.** Let E be a supersingular elliptic curve over a finite field. An abelian variety A over a field K (of characteristic p) *is supersingular if and only if*

there exists an isogeny $A_k \sim_k E^g$, where E is a supersingular elliptic curve.

Theorem 6.11 (See [93], Section 4; also see [132], Th. 2 on page 140)**.** *Definition 1 and Definition 2 are equivalent.*

Sketch of a proof. If E is a supersingular elliptic curve then $\mathcal{N}(E^g) = g{\cdot}(1,1)$. Hence (2) $\Rightarrow$ (1).

Conversely suppose that $\mathcal{N}(A) = g{\cdot}(1,1)$. Then there exists an isogeny $A_k[p^\infty] \sim (G_{1,1})^g$. Hence there exists an isogeny $A_k \sim_k B$ with $a(B) = g$. Let $[(B,\mu)] = y \in \mathcal{A}_g \otimes \mathbb{F}_p$. Let T be an irreducible component of $\mathcal{A}_g \otimes \mathbb{F}_p$ containing y. Deformation theory shows that any infinitesimal deformation of B keeping $a(-) = \dim(B)$ constant is trivial. Hence the set of points $y' \in T$ with

$y' = [(B', \mu')]$ and $a(B') = g$ is zero-dimensional and closed. This proves that B is defined over a finite field.

Next we show: *let B be a supersingular abelian variety defined over $\mathbb{F}_q$. Then there exists an isogeny $B_{\mathbb{F}} \sim_{\mathbb{F}} E^g$, where E is a supersingular elliptic curve.* Suppose n is even (or perform a quadratic extension, and replace B by a simple factor if necessary). Let $\pi = \pi_B$ be the Weil q-number associated with B; i.e. π is the algebraic number defined as the endomorphism $\pi = F^n$, the "geometric Frobenius" on $B/\mathbb{F}_q$. Then $\mid \psi(\pi) \mid = \sqrt{q}$ for every $\psi : \mathbb{Q}(\pi) \to \mathbb{C}$ by the Weil conjecture. Note that $\beta := \pi/\sqrt{q}$ is a unit at all finite places not dividing p. The condition that all Frobenius slopes are equal to $1/2$ shows that β is also a unit at every finite place dividing p. A well-known theorem in number theory, see [3], page 105, Th.2, shows that under these conditions β is a root of unity, say $\beta^s = 1$ for some $s \in \mathbb{Z}_{>0}$. Hence $\pi^s \in \mathbb{Q}$, which shows that $B \otimes \mathbb{F}_r \sim E^g$, with $r = q^s$, where E a supersingular elliptic curve over $\mathbb{F}_r$, see [133], page 97. □

Remark 6.12. In characteristic zero an abelian variety of CM type is defined over a number field, see [130], Proposition 26 on page 109. However in positive characteristic there are examples of an abelian variety admitting *sufficiently many complex multiplications* not defined over a finite field (hence the first part of the proof is not superfluous). See § 3 for a discussion. In retrospect it is not surprising what we see in the proof.

6.13. The number $h = h(p)$ of isomorphism classes of supersingular elliptic curves over $\mathbb{F} := \overline{\mathbb{F}_p}$ was computed by Hasse, Deuring and Igusa. See [24], [48]. It can be given as a class number. Also see [54] page 118. In fact:

$$h(p) = \{1 - (\frac{-3}{p})\}/3 + \{1 - (\frac{-4}{p})\}/4 + \frac{p-1}{12},$$

where $(\frac{a}{p})$ denotes the Legendre symbol. Equivalently:

$$h(2) = 1 = h(3); \quad h(p) = \frac{p-1}{12} \ \text{ if } \ p \equiv 1 \pmod{12};$$

$$h(p) = 1 + \frac{p-5}{12} \ \text{ if } \ p \equiv 5 \pmod{12}; \quad h(p) = 1 + \frac{p-7}{12} \ \text{ if } \ p \equiv 7 \pmod{12};$$

$$h(p) = 2 + \frac{p-11}{12} \ \text{ if } \ p \equiv 11 \pmod{12}.$$

It can also be given by:

$$\sum_{j(E)} \frac{1}{\#(\mathrm{Aut}(E))} = \frac{p-1}{24},$$

the sum taken over all isomorphism classes of supersingular elliptic curves over $\mathbb{F}$.

We see that the number of supersingular curves over $\mathbb{F}$ depends on p, and goes to ∞ for $p \to \infty$, which is a special case of a general phenomenon; see 7.5.

6.14. Deligne and Shioda proved: *Suppose $g \geq 2$. Let $E_1, \cdots, E_{2g}$ be supersingular elliptic curves over $\mathbb{F}$. Then*

$$E_1 \times \cdots \times E_g \quad \cong \quad E_{g+1} \times \cdots \times E_{2g}.$$

See [131], Th. 3.5.

Write $\Lambda_{g,j}$ for the set of isomorphism classes (E^g, μ) over $\mathbb{F} = \overline{\mathbb{F}_p}$, where E is a supersingular elliptic curve, and $\mu : E^g \to (E^g)^t$ is a polarization with $\mathrm{Ker}(\mu) = E^g[F^j]$. This number $\Lambda_{g,j}$ can be expressed as a class number; e.g. see [47], Section 2; see [66], Chapter 4, Chapter 8.

6.15. For a supersingular elliptic curve E in characteristic p we know that $j(E) \in \mathbb{F}_{p^2}$; this is not hard to prove: the morphism $F^2 : E \longrightarrow E^{(p^2)}$ has kernel $E[F^2] = E[p]$. Hence $E^{(p^2)} \cong E/E[p] \cong E$, and this proves $(j(E))^{p^2} = j(E)$. However we know more: any supersingular value $j(E)$ is a fourth power in $\mathbb{F}_{p^2}$; see the appendix of [33].

6.16. Theorem/Definition (see [94], Th. 2). *Let A be an abelian variety of dimension g over a field K. Write k for an algebraic closure of $K \supset \mathbb{F}_p$. The following are equivalent:*

(1) $a(A) = \dim(A) = g$.

(2) *There exists an isomorphism $A_k \cong E^g$, where E is a supersingular elliptic curve.*

(3) Definition. *A is called* superspecial. □

Remark 6.17. Note the following curious fact.

For $g = 1$ a polarization on an elliptic curve is uniquely determined by its degree, e.g. $\#(\Lambda_{1,0}) = 1$, and there are many isomorphism classes of supersingular elliptic curves (many, if $p \gg 0$). However,

for $g \geq 2$ there is only one isomorphism class of a superspecial abelian variety of dimension g, but $\#(\Lambda_{g,0})$ is large for $p \gg 0$.

6.18. In characteristic zero a decomposition of the Lie algebra of a Lie group (of course) does not imply a decomposition of the Lie group. The analogue of the Lie algebra of a Lie group of an abelian variety A in positive characteristic is its p-divisible group $X = A[p^\infty]$. Does a decomposition, say up to isogeny, of $A[p^\infty]$ imply a decomposition of A ?

YES for supersingular abelian varieties: see Definition 2.

NO, in all other cases:

Theorem 6.19 ([65]; also see [13], 5.)**.** *For every symmetric Newton polygon $\xi \neq \sigma$ there exists a simple abelian variety B over $\mathbb{F} := \overline{\mathbb{F}_p}$ with $\mathcal{N}(B) = \xi$.* □

6.20. More information about supersingular abelian varieties can be found in the following papers: [24], [48], [44], [4], [5], [152], [27], [45], [88], [89], [153], [154], [94], [131], [47], [54], [55], [77], [66], [32].

7. NP strata

7.1. Consider the locus of all moduli points corresponding with an abelian variety with a given Newton polygon. Grothendieck and Katz show this is a locally closed set of the moduli space; see [38] for the first idea, and [57] for the final result. We discuss what the dimension of such a stratum is and we consider (ir)reducibility of such strata.

7.2. Let ζ be a Newton polygon. Let $X \to S$ be a p-divisible group over a base scheme S in characteristic p. We write

$$\mathcal{W}^0_\zeta(S) = \{s \in S \mid \mathcal{N}(X_s) = \zeta\}.$$

This is called the (open) Newton polygon stratum in S defined by ζ.

Theorem 7.3 (Grothendieck-Katz. See [57], Th. 2.3.1 on page 143)**.** *The subset* $\mathcal{W}^0_\zeta(S) \subset S$ *is locally closed.* □

7.4. Working over a perfect field, we will endow $\mathcal{W}^0_\zeta(S)$ with the reduced scheme structure. We will write

$$W^0_\xi := \mathcal{W}^0_\xi(\mathcal{A}_{g,1})$$

for the (open) Newton polygon stratum defined by the symmetric Newton polygon ξ in the moduli space of *principally polarized* abelian varieties in characteristic p. See 8.8 for the notation W_ξ.

It would be nice to have a natural scheme structure on NP strata. However, up to now attempts have failed to construct such a theory.

We write

$$\mathcal{S}_{g,1} = \mathcal{W}_\sigma(\mathcal{A}_{g,1}) = W_\sigma,$$

where $\sigma = g{\cdot}(1,1)$ is the Newton polygon defining supersingular abelian varieties.

We have seen that $\mathcal{S}_{1,1}$, the locus of supersingular elliptic curves, has many components for $p \gg 0$. We shall see that the same kind of behavior holds for the supersingular locus for all g.

For a p-divisible group $X \to S$ over an irreducible scheme S we write $a(-/S)$ for the number $a(X_\eta)$, where η is the generic point of S; analogous notation for an abelian scheme $A \to S$.

Theorem 7.5 (Li – Oort)**. (1)** *For every irreducible component S of $\mathcal{S}_{g,1} \otimes \mathbb{F}$ we have $a(-/S) = 1$.*

(2) *The number of irreducible components of $\mathcal{S}_{g,1} \otimes \mathbb{F}$ equals the class number $H_g(p,1)$ if g is odd, respectively $H_g(1,p)$ in case g is even.*

(3) *The subscheme $\mathcal{S}_{g,1}$ is of pure dimension $[g^2/4]$.* □

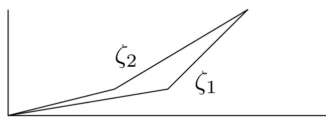
See [66] for the definition of these class numbers, and see [66], 4.9 for these results. The results (1) and (3), and hence (2), were reproved in [105].

Remark 7.6. All three statements do fail for certain values of g and certain components of the supersingular locus $\mathcal{W}_\xi(\mathcal{A}_g)$ *in the non-principally polarized case*. For example see [55], 6.10, see [66], 10.5; for many more examples, and an "explanation", see [119].

Note that $H_g(p,1) > 0$ and $H_g(1,p) > 0$ for $g \gg 0$. In those cases $\mathcal{S}_{g,1} \otimes \mathbb{F}$ is reducible.

Parts (1) and (3) generalize to all Newton polygon strata in $\mathcal{A}_{g,1}$; see Theorem 7.9. However (2) is special for the supersingular locus in $\mathcal{A}_{g,1}$; see Theorem 7.12.

7.7. We introduce a partial ordering in the set of Newton polygons having the same endpoints.



$\boxed{\zeta_1 \succ \zeta_2}$

We write $\zeta_1 \succ \zeta_2$ if ζ_1 is "below" ζ_2, i.e. if no point of ζ_1 is strictly above ζ_2. This might seem strange. However, explanation: if ζ_1 is "below" ζ_2, the locus defined by ζ_1 is "bigger" than the locus defined by ζ_2.

What can be said about (ir)reducibility of other Newton polygon strata and about their dimensions?

7.8. The dimension of Newton polygon strata, polarized case. We fix an integer g. For every *symmetric* NP ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid y < x \leq g, \quad (x,y) \ \text{ on or above } \ \xi\},$$

and we write

$$\boxed{\mathrm{sdim}(\xi) := \#(\triangle(\xi)).}$$

Theorem 7.9 ([105], [103])**.** *For every symmetric Newton polygon ξ the stratum $W_\xi \subset \mathcal{A}_{g,1}$ is pure of dimension* $\mathrm{sdim}(\xi)$*. For every irreducible component S of W_ξ we have $a(-/S) = 1$.* $\square$

Note that 7.5(3) is a special case. Note that without the condition that we work with *principal polarizations* dimensions of components of Newton polygon strata can be quite different from what is said above; see [119] for precise results, see 7.13. Also in the non-principally polarized case there are irreducible components on which $a(-/S)$ can be larger than one.

We discuss some of the techniques used for a proof of 7.9 and an application.

Theorem 7.10 ([103])**.** *Let X_0 be a p-divisible group over a field K. Assume $a(X_0) = 1$. Write $\mathcal{N}(X_0) =: \zeta'$. Write $D = \mathrm{Def}(X_0)$ for the local deformation space. For every $\zeta \succ \zeta'$ the NP stratum $\mathcal{M}_\zeta(D)$ is regular, and these strata are nested (inclusion of their Zariski closures) exactly as given by the ordered graph of Newton polygons below ζ'.*

Let (X_0, λ) *be a principally quasi-polarized* p*-divisible group over a field* K*. Assume* $a(X_0) = 1$*. Write* $\mathcal{N}(X_0) =: \xi'$*. Write* $DP = \mathrm{Def}(X_0, \lambda)$ *for the local deformation space. For every* $\xi \succ \xi'$ *the NP stratum* $\mathcal{M}_\zeta(DP)$ *is regular, of dimension* $\mathrm{sdim}(\xi)$ *and these strata are nested (inclusion of their Zariski closures) exactly as given by the ordered graph of symmetric Newton polygons below* ξ'*.* □

Note that without the condition that we work with *principal polarizations* the conclusion of second half of the theorem does not hold in general.

We have seen that for an abelian variety A its Newton polygon $\mathcal{N}(A) = \xi$ is symmetric, see 6.7. Manin conjectured the converse:

A conjecture by Manin. *Let* p *be a prime number, and let* ξ *be a symmetric Newton polygon. There exists an abelian variety* A *in characteristic* p *with* $\mathcal{N}(A) = \xi$*.* See [68], page 76, Conjecture 2.

Corollary 7.11 (**Manin Conjecture**; Serre-Honda-Tate, Oort)**.** *For every* p *and symmetric* ξ *the Manin conjecture holds.* □

This was proved independently by Serre (unpublished), and by Honda and Tate; see [133], page 98.

For a proof purely in characteristic p see [103], Section 5. We give a sketch in which way the corollary follows from 7.5 and 7.10.

(1) Note that for the supersingular Newton polygon $\sigma = \sigma_g = g{\cdot}(1,1)$ the conjecture holds.

(2) Construct a principally quasi-polarized supersingular p-divisible group (X_0, λ) with $a(X_0) = 1$.

(3) For a given symmetric ξ (automatically below σ), using 7.10, we construct $(X, \lambda) \to S$, a formal deformation, with special fiber (X_0, λ), and generic fiber (X_η, λ) with $\mathcal{N}(X_\eta) = \xi$. Here we use the fact that $a(X_0) = 1$.

(4) Use (1) and (2) and produce (B_0, λ) with $(B_0, \lambda)[p^\infty] \cong (X_0, \lambda)$.

(5) By Serre-Tate theory there exists a formal abelian scheme $(\mathcal{B}, \lambda) \to S$ with $(\mathcal{B}, \lambda)[p^\infty] \cong (X, \lambda)$.

(6) By Chow-Grothendieck the formal polarized abelian scheme $(\mathcal{B}, \lambda) \to S$ can be algebraized to an abelian scheme $(B, \lambda) \to S$. We have produced B_η with $\mathcal{N}(B_\eta) = \mathcal{N}(X_\eta) = \xi$.

For details see [103], Section 5. □

It is interesting to see that in this proof we algebraize twice; first by noting that supersingular p-divisible groups come from an abelian variety, $X_0 \mapsto B_0$; the second time the polarization helps to show the existence of an algebraization $X \mapsto \mathcal{B}$.

Finally, Conjecture 8B of [101] was proved:

Theorem 7.12 (Chai – Oort, [107], [14], Theorem A)**.** *For every symmetric Newton polygon* $\xi \neq \sigma$ *the locus* W_ξ *is geometrically irreducible. Moreover* $a(-, W_\xi) \leq 1$*.* □

Remark 7.13. What can be said about dimensions of strata in the non-principally polarized cases? For every g and every $0 \leq f \leq g$ there is a unique symmetric Newton polygon which is maximal (the lowest) in the set of all Newton polygons having that p-rank. These are: the ordinary Newton polygon $g{\cdot}(1,0)+g{\cdot}(0,1)$, with $g = f$, the almost ordinary Newton polygon $(g-1){\cdot}(1,0)+(1,1)+(g-1){\cdot}(0,1)$ with $g-1 = f$, and for $f < g-1$ we take $f{\cdot}(1,0)+(g-f-1,1)+(1,g-f-1)+f{\cdot}(0,1)$; for these Newton polygons all components of the stratum $\mathcal{W}_\xi(\mathcal{A}_g)$ have the same dimension, equal to sdim(ξ).

For all other ξ the locus $\mathcal{W}_\xi(\mathcal{A}_g)$ is not equi-dimensional; in [119] we find which dimensions do appear for a given Newton polygon.

7.14. Newton polygon strata were discussed in the following papers: [68], [61], [57], [98], [66], [101], [8], [14], [34], [107], [122], [113], [115].

8. A conjecture by Grothendieck

8.1. Grothendieck proved that "Newton polygons go up under specialization" and he conjectured conversely that a pair of comparable Newton polygons for p-divisible groups appears in a deformation/specialization, see [38]. This conjecture was proved for p-divisible groups, for principally polarized p-divisible groups and for principally polarized abelian varieties. For the non-principally polarized case the equivalent of this conjecture does not hold; see [103], [105], [119].

8.2. Grothendieck proved that "Newton polygons go up under specialization". That means that the Newton polygon of a special fiber in a family of p-divisible groups has no points under the Newton polygon of the generic fiber. Does the converse hold? In [38], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: "$\cdots$ *The wishful conjecture I have in mind now is the following: the necessary conditions* $\cdots$ *that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular deformation* $\cdots$ *we want to know if every sequence of rational numbers satisfying* $\cdots$ *these numbers occur as the sequence of slopes of a fiber of G as some point of S."*

This conjecture has been proved for p-divisible groups. The equivalent of this conjecture holds for principally quasi-polarized p-divisible groups, and for principally polarized abelian varieties; however the equivalent of this conjecture in general does not hold for polarized p-divisible groups and for polarized abelian varieties.

Theorem 8.3 (Oort, [52], [103], and [105])**.** (The Grothendieck Conjecture, Montreal 1970.) *Let K be a field of characteristic p, and let X_0 be a p-divisible group over K. We write $\mathcal{N}(\mathcal{X}_0) =: \beta$ for its Newton polygon. Suppose given a Newton polygon γ "below" β, i.e. $\beta \prec \gamma$. There exists a deformation X_η of X_0 such that $\mathcal{N}(\mathcal{X}_\eta) = \gamma$.* □

Theorem 8.4 (Oort, [105])**.** (The principally quasi-polarized analog of the Grothendieck Conjecture.) *Let (X_0, λ_0) be a principally quasi-polarized p-divisible group, and let ζ_2 be a Newton polygon such that* $\zeta_1 = \mathcal{N}(X_0) \prec \zeta_2$ (i.e. no point of ζ_1 is below ζ_2)*. Then there exists an irreducible scheme S, a closed point $0 \in S$ and a principally quasi-polarized p-divisible group $(X, \lambda) \to S$ such that the fiber above 0 is (X_0, λ_0) and the generic fiber has Newton polygon equal to ζ_2.*

An analogous statement holds for principally polarized abelian varieties. □

Remark 8.5. The statement can be rephrased by saying that for $\xi_1 \prec \xi_2$ the locus $W^0_{\xi_1}$ is in the Zariski closure of $W^0_{\xi_2}$. Note that the notation W^0_{ξ} is only used in the case of principal polarizations. See [15], 1.17 – 1.22, and see §8, and §11 for more explanation and comments.

Remark 8.6. The analog of the Grothendieck conjecture for quasi-polarized p-divisible groups, respectively polarized abelian varieties (in case these are not principal polarizations) in general does not hold. An example can be found in [55], 6.10: inside $\mathcal{A}_{3,p^3}$ there is a component of the supersingular locus which has dimension equal to three; hence this is not contained in the closure of $\mathcal{W}^0_{(2,1)+(1,2)}(\mathcal{A}_3)$, because the p-rank zero locus inside $\mathcal{A}_3$ has pure dimension $(3(3-1)/2) - 3 = 3$ by [87], Theorem 4.1. More supersingular examples to be found in [66], § 10. A complete answer to the question whether an irreducible component of a Newton polygon stratum is contained in the closure of another Newton polygon stratum is described in [119], thus giving many counterexamples to the (non-principally) polarized analogue of the Grothendieck conjecture.

Remark 8.7. As usual, deformations of the p-divisible group of a polarized abelian variety give, by the Serre-Tate theory, polarized formal abelian schemes, and by the Chow-Grothendieck theorem this gives deformations of polarized abelian schemes. For an explanation and references see [103], 5.5. Hence the first statement in 8.4 implies the analogous result for principally polarized abelian varieties.

Remark 8.8. There are two ways to produce a natural definition of a *closed* Newton polygon stratum. One can consider the Zariski closure

$$\mathcal{W}_\xi(\mathcal{A}_g) \subset (\mathcal{W}_\xi(\mathcal{A}_g))^{\mathrm{Zar}} \subset \mathcal{A}_g.$$

Also we can consider

$$\{[(A,\mu)] \mid \mathcal{N}(A) \prec \xi\} \subset \mathcal{A}_g.$$

As this last subset is closed, if follows that it contains $(\mathcal{W}_\xi(\mathcal{A}_g))^{\mathrm{Zar}}$:

$$(\mathcal{W}_\xi(\mathcal{A}_g))^{\mathrm{Zar}} \subset \{[(A,\lambda)] \in \mathcal{A}_g \mid \mathcal{N}(A) \prec \xi\}.$$

There are many examples showing that these two closed sets do not coincide, see 8.6, and see [119], in the non-principally polarized case. However, the Grothendieck conjecture says that inside $\mathcal{A}_{g,1}$ these two coincide, and we write and conclude

$$W_\xi := (\mathcal{W}_\xi(\mathcal{A}_{g,1}))^{\mathrm{Zar}} = \{[(A,\lambda)] \in \mathcal{A}_{g,1} \mid \mathcal{N}(A) \prec \xi\} \subset \mathcal{A}_{g,1}.$$

9. Purity

9.1. In [52] it was proved that if a Newton polygon jumps in a family, then already it jumps in codimension one (and hence the name "purity").

Note that this property seems unlikely as we see that in the proof of 7.3 a priori several equations are needed to describe the locus where the Newton polygon jumps. Therefore the purity theorem came as a surprise.

In this section we explain basic ingredients which are used in proofs of 7.9, 8.3, 8.4. We have at our disposal local deformation theory, for abelian varieties, and for polarized abelian varieties, for p-divisible groups, as initiated by Schlessinger. So why not write down a universal deformation, study Newton polygon strata in such a deformation space, compute dimensions, and see which strata are in the boundary of other strata? It seems that such a direct approach could be successful. However it turns out, that it is difficult in general to describe Newton polygon strata in this way. In general the number of equations obtained is larger than the codimension of the stratum we are looking for, and perhaps the stratum we are looking for (as in the Grothendieck conjecture) can very well be empty; indeed, in the non-principally polarized case we can construct (many) examples where this is the case.

This is very much in analogy with the problem of lifting polarized abelian varieties from characteristic p to characteristic zero. There the principally polarized case can be handled directly, as Grothendieck showed, see [91]. But deformation theory did not give the possibility of a direct approach in the non-principally polarized case. Mumford showed us how to proceed. For ordinary abelian varieties the theory of Serre-Tate canonical liftings works well. For the arbitrary case we first deform in characteristic p to a polarized ordinary abelian variety (a non-canonical choice, why would it exist?), and then finish by performing the canonical lifting of the generic fiber of that deformation. This program was carried out in [87]. For another approach see [86].

This two-step procedure, in a different form, also gives access to the Grothendiek conjecture considered here. In [103] we see that deforming p-divisible groups, and deforming principally polarized abelian varieties with $a(X) \leq 1$, respectively $a(A) \leq 1$, gives a deformation space in which the Newton polygon strata are regular, and nested exactly as requested by the partial ordering of Newton polygons; in these cases all results 7.9, 8.3, 8.4 follow directly by what I call the "method of Cayley-Hamilton". Hence we are done if *we can deform to a situation with* $a(-) \leq 1$ *keeping the Newton polygon fixed,* in the unpolarized case, and in the (quasi) principally polarized case. This turned out to be a difficult problem. It took me many years, some failures, and drastically new ideas to perform this step. Here is the central idea.

Theorem 9.2 (Deformation to $a \leq 1$; see [52], 5.12 and [105], 2.8)**.** *Let X_0 be a p-divisible group over a field K. There exists an integral scheme S, a point $0 \in S(K)$ and a p-divisible group $\mathcal{X} \to S$ such that the fiber $\mathcal{X}_0$ is isomorphic with X_0, and for the generic point $\eta \in S$ we have:*

$$\mathcal{N}(X_0) = \mathcal{N}(X_\eta) \quad \textit{and} \quad a(X_\eta) \leq 1.$$

□

Theorem 9.3 (Deformation to $a \leq 1$ in the principally quasi-polarized case; see [52], 5.12 and [105], 3.10)**.** *Let X_0 be a p-divisible group over a field K with a principal quasi-polarization $\lambda_0 : X_0 \to X_0^t$. There exists an integral scheme S, a point $0 \in S(K)$ and a principally quasi-polarized p-divisible group $(\mathcal{X}, \lambda) \to S$ such that there is an isomorphism $(X_0, \lambda_0) \cong (\mathcal{X}, \lambda)_0$, and for the generic point $\eta \in S$ we have:*

$$\mathcal{N}(X_0) = \mathcal{N}(X_\eta) \quad \textit{and} \quad a(X_\eta) \leq 1.$$

□

Once 9.2 is proved, the Grothendieck conjecture is easily derived, see [105], 3.10. Here is the new idea which can be used to prove 9.2 and 9.3.

Theorem 9.4 (Purity of the Newton polygon stratification; de Jong - Oort)**.** *Let S be an integral scheme, and let $X \to S$ be a p-divisible group. Let $\gamma = \mathcal{N}(X_\eta)$ be the Newton polygon of the generic fiber. Let $S \supset D = S_{\neq\gamma} := \{s \in S \mid \mathcal{N}(A_s) \precneqq \gamma\}$* (Note that D is closed in S by Grothendieck-Katz.) *Then either D is empty or* $\mathrm{codim}(D \subset S) = 1$, i.e. every irreducible component of D has codimension one in S. □

9.5. We know two proofs of this theorem, and both proofs are non-trivial. See [52], Th. 4.1. Also see [138], Th. 6.1; this second proof of Purity was analyzed and reproved [138], [106], [139], [156]. Purity was further discussed in [85], [156], [106], [15]. Also see [149]. In fact the version proposed and proved by Vasiu gives a stronger result: it shows that $D \hookrightarrow S$ is an affine morphism.

The purity theorem gives, via the theory of catalogs, a proof of 9.2 for simple p-divisible groups, see [52], 5.11 and 5.12. I do not know a direct, easy proof of this useful theorem 9.2. Once the case X_0 is simple in Theorem 9.2 is settled, the general case, and the polarized versions are easily proved; see [105], 3.10 and 3.11.

Once we have this result, using dimensions of Newton polygon strata we obtain a proof of 9.2, see [52], 5.12. For a discussion see [15], Section 7. It would be nice to have a more direct proof of 9.2; what is the essential structure underlying this fact?

10. EO strata

10.1. Consider a set inside the moduli space characterized by the property that any two points $x = [(A, \lambda)]$ and $y = [(B, \mu)]$ in one stratum give geometrically isomorphic $A_k[p] \cong B_k[p]$. Such a locus is called an EO stratum, see [104]. We will give a survey of some properties of these strata. In this section we will be brief, as a more extensive survey can be found in [35].

10.2. A theorem by Kraft, see [62], later, independently found by Oort, says that over an algebraically closed field k in characteristic p the number of isomorphism classes of groups schemes of a given rank, annihilated by p is finite. This motivated Ekedahl and Oort to study a stratification of the moduli space $\mathcal{A}_{g,1}$ where the strata, *finite in number*, are given by the geometric isomorphism classes of possible $A[p]$. In [104] we find the first definitions and results on this idea. Here we give a short survey of some of these results. Later Moonen, Van der Geer, and many others found another description of the index set, and other interpretations of this stratification; e.g. see [28] for a detailed description.

10.3. Truncated Barsotti-Tate groups. For a p-divisible group X we can consider its truncation at level one $X[p]$. A p-divisible group is also called a Barsotti-Tate group; the truncation at level one is what we call a BT_1 group scheme. Here is the precise definition.

We say that a finite, locally free group scheme $N \to S$ is a BT_1, or a BT_1 group scheme, or *a Barsotti-Tate group truncated at level one*, if $F(N) = N[V]$; see [49], 1.1 for a more general definition. Note that this implies that N is annihilated by $VF = p$; moreover for a BT_1 also the property $V(N) = N[F]$ holds. Note that if $N \to S$ is a BT_1, then all fibers are BT_1 groups schemes. Note that a BT_1 group scheme, or more, generally a BT_n group scheme, over a perfect field is the truncation of a p-divisible group, see [49], 1.7

Note there exist examples of a finite, locally free group scheme $N \to S$, say with S irreducible, where the generic fiber is a BT_1, but where N/S is not a BT_1; e.g. over $S = \mathrm{Spec}(k[[t]])$ there exists a group scheme, an extension of α_p by α_p, such that the generic fiber is a BT_1, and the closed fiber is isomorphic with $\alpha_p \times \alpha_p$, see [90], 15.5 for a discussion of such examples; or, consider a deformation of α_p to (a twist of) μ_p, and note that the first is not a BT_1, and the second is; see [134] for a discussion of such examples.

There exist examples of a finite, locally free group scheme $N \to S$, say with S irreducible, where a special fiber is a BT_1, but where the generic fiber is not; hence N/S is not a BT_1. For example take an elliptic curve $E \to S = \mathrm{Spec}(k[[t]])$ with special fiber supersingular, and generic fiber ordinary; the group scheme $N := E[F^2]$ has the property mentioned.

10.4. We produce a stratification on $\mathcal{A}_{g,1}$. One could also study such a stratification on all components of $\mathcal{A}_g$. However, results are complicated and not very neat if we consider polarizations of degree divisible by p; also we do not know whether such stratifications could be interesting, while the EO stratification on $\mathcal{A}_{g,1}$ turns out to be very useful.

We give a description of a classification of isomorphism classes of possible $A[p]$, for $[(A, \lambda)] = x \in \mathcal{A}_{g,1}$. Most material is easily constructed and described; in some cases some combinatorics is needed. However there is one detail in the constructions and definitions which is non-obvious. We classify on the one hand BT_1 group schemes N which are symmetric, meaning there exists some isomorphism $N \cong N^D$; such a group scheme can arise as $N = A[p]$, where a principal polarization $A \to A^t$ induces

$$N = A[p] \xrightarrow{\sim} A^t[p] \cong (A[p])^D.$$

On the other hand we can consider pairs $(N, <>)$, where $N = A[p]$ and where $<>$ is a bilinear, non-degenerate, alternating on the Dieudonné module of N induced by a principal polarization on A. Over an algebraically closed field these two concepts are the same; see [104], Section 9. This fact is not so hard to prove in characteristic not equal to two, and one can give a more natural approach to the bilinear form needed on N; however, in characteristic $p = 2$ I do not know an easy proof of this essential tool; it seems desirable to characterize the pairing $<>$ on $\mathbb{D}(N)$ only in terms of group schemes.

As we seem to be forced to give the bilinear form on the Dieudonné module we have to be careful doing deformation theory (as "Dieudonné modules" of finite group scheme over an arbitrary base are not very easy to handle); see [104], 12.1 – 12.5 where we use displays for quasi-polarized p-divisible groups and the property "$\mathrm{V} = \mathrm{p}/\mathrm{F}$" in order to reconstruct V on a finite group scheme $\mathcal{B}[p]$ from the display defining the deformation $\mathcal{B}$.

10.5. On a BT_1 finite group scheme N over a field K we construct the *canonical filtration*. Consider $V : N^{(p)} \to N$, and take all images under iteration of V. From all finite subgroup schemes obtained we take the inverse images under $F : N \to N^{(p)}$; on the set of subgroup schemes we repeat these processes, each time by V possibly constructing new group scheme in between $0 \subset N[F] = \mathrm{Im}(N)$, and by F^{-1} in between $N[F] = \mathrm{Im}(N) \subset N$; each step respects the filtration we already have; after a finite number of steps the process does not give anymore new subgroup schemes; we arrive at the coarsest filtration stable under V and F^{-1}; this is called the *canonical filtration*.

10.6. The canonical filtration in group schemes of given rank can have various lengths. Let N be a symmetric BT_1 finite group scheme; here the word symmetric means there is some isomorphism $N \cong N^D$. In this case the canonical filtration is symplectic and V and F^{-1} stable; a filtration with these properties is called a good filtration, see [104], 5.3.

For a symmetric group scheme of rank p^{2g} over an algebraically closed field we construct a filtration

$$0 \subset N_1 \subset \cdots \subset N_g = N[F] = \mathrm{Im}(N) \subset \cdots N_{2g}$$

which is symplectic, stable under V and F^{-1}, i.e. a good filtration, with all subquotients N_{i+1}/N_i of rank p (of length one). This refines the canonical filtration. This is called a *final filtration*. In general a final filtration is not unique, see 10.12. However for a symmetric N and some final filtration $\{N_j \mid 1 \leq j \leq 2g\}$ there is a sequence of non-negative integers

$$\psi = \{\psi(1), \cdots, \psi(2g)\} \quad \text{defined by} \quad V(N_j) = N_{\psi(j)};$$

this is called the final sequence attached to the symmetric N. I.e. this sequence tells us where the steps in the filtration are mapped to which steps under V. It turns out that, although the final filtration need not be unique, the final sequence does not depend on the final filtration chosen. It can be computed combinatorially from the data describing the canonical filtration. The sequence

$$\varphi = \{\varphi(i) = \psi(i) \mid 1 \leq i \leq g\} = \{\varphi(1), \cdots, \varphi(g)\}$$

is called the *elementary sequence* attached to the symmetric N, notation $\mathrm{ES}(N) = \varphi$. For properties of elementary sequences and final sequences we refer to [104], §§ 2, 5. The concepts elementary sequence, final sequence and canonical type correspond one-to-one.

In [28] and in [72] we see another combinatorial way to encode the elementary sequence of a symmetric BT_1 group scheme.

In particular an elementary sequence φ has the property (and this notion is defined by):

$$\varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1;$$

we write $\varphi(0) = 0$. Hence the number of elementary sequences of length g equals 2^g. We will see that every elementary sequence of length g does occur on $\mathcal{A}_g$; see Theorem 10.11.

Moreover $\psi(2g - i) = \psi(i) + g - i$. Hence $\psi = \{\psi(1), \cdots, \psi(2g)\}$ can be computed knowing only $\varphi = \{\psi(1), \cdots, \psi(g)\}$.

10.7. Let N be a symmetric BT_1 finite group scheme over a perfect field K. Let $M = \mathbb{D}(N)$ be its Dieudonné module. we consider

$$<,>: M \times M \longrightarrow K,$$

a *non-degenerate, alternating pairing*; "alternating" means: $< x, x >= 0$ for every $x \in M$. The pair $(N, <,>)$ will be called a polarized BT_1; we see that this is a special case of a symmetric BT_1.

Suppose (A, λ) is a principally polarized abelian variety over K. The principal polarization induces a pairing on the Dieudonné module of $A[p^\infty]$ and hence it induces a non-degenerate, alternating pairing on $M = \mathbb{D}(A[p]$. A principally polarized abelian variety (A, λ) induces this pair $(N, <>)$, see [104], 12.2(2). We write

$$(N, <,>) = (A, \lambda)[p].$$

Given an elementary sequence $\varphi = \{\varphi(1), \cdots, \varphi(g)\}$, with related final sequence ψ we construct $(N_\varphi, <,>_{\rm st})$, called the *standard type* given by φ, see [104], 9.1 for details. The result is given by an ordered basis $\{Z_1, \cdots Z_g; Z_{g+1}, \cdots, Z_{2g}\}$ for $M = \mathbb{D}(N_\varphi)$, where the sets

$$\{Y_g, \cdots, Y_1\} \cup \{X_1, \cdots, X_g\} \quad \text{and} \quad \{Z_1, \cdots Z_g; Z_{g+1}, \cdots, Z_{2g}\}$$

are equal, the pairing is given by

$$< Y_i, X_j >= \delta_{i,j}, \quad < Y_i, Y_j >= 0 =< X_i, X_j > \quad \forall\ 0 \leq i, j \leq g$$

and

$$\mathcal{F}(X_i) = Z_i, \quad \mathcal{F}(Y_i) = 0 \quad \forall\ 0 \leq i \leq g.$$

This determines the action by $\mathcal{V}$ on M; in fact

$$\mathcal{V}(Z_i) = 0, \quad \mathcal{V}(Z_{2g-i+1}) = \pm Y_i \quad \forall\ 0 \leq i \leq g;$$

here

$$\mathcal{V}(Z_{2g-i+1}) = +Y_i \quad \text{if} \quad Z_{2g-i+1} \in \{Y_g, \cdots, Y^1\}$$

and

$$\mathcal{V}(Z_{2g-i+1}) = -Y_i \quad \text{if} \quad Z_{2g-i+1} \in \{X_1, \cdots, X^g\}.$$

Theorem 10.8. *Let $(N, <,>)$ be a polarized BT_1 group scheme of rank $2g$ defined over a field K. Let φ be the elementary sequence determined by N. Let k be an algebraic closure of K. There exists an isomorphism*

$$(N, <,>) \otimes k \cong (N_\varphi, <,>_{\rm st}) \otimes k.$$

□

See [104], Theorem 9.4. A proof for this theorem is not difficult in characteristic $p > 2$. However, I do not know a simple proof in characteristic 2.

10.9. Conclusion. *If a BT_1 group scheme admits a polarization, then over an algebraically closed field the pairing $<,>$ is uniquely determined up to isomorphism. A symmetric BT_1 group scheme over an algebraically closed field is uniquely determined by its elementary sequence.*

After these combinatorial preparations we list the definition of EO strata and some of the properties.

Definition 10.10. Suppose given an elementary sequence φ of length g. We write:

$$S_\varphi = \{[(A, \lambda)] = x \in \mathcal{A}_{g,1} \mid {\rm ES}(A[p]) = \varphi\};$$

this is called an AO stratum. We write

$$\mid \varphi \mid := \sum_{i=1}^{i=g} \varphi(i).$$

We define

$$\varphi' \prec \varphi \quad \Longleftrightarrow \quad \varphi'(i) \leq \varphi(i) \ \ \forall i.$$

For a subset $T \subset \mathcal{A}_{g,1}$ we write $T^{\rm Zar}$ for the Zariski closure of T inside $\mathcal{A}_{g,1}$. We write

$$\varphi' \subset \varphi \Longleftrightarrow S_{\varphi'} \subset S_{\varphi}.$$

We have $\varphi' \prec \varphi \Rightarrow \varphi' \subset \varphi$, [104], 11.1, but the converse does not hold, [104], 14.3.

It turns out that $S_\varphi \subset \mathcal{A}_{g,1}$ is *locally closed*. Hence we have written $\mathcal{A}_{g,1}$ as a disjoint union of a finite number of locally closed subsets. For these strata we have the following properties.

Theorem 10.11 (Ekedahl - Oort, [104], 4.1, 1.2, 1.3, § 13)**. (1)** *S_φ is quasi-affine* ("The Raynaud trick").

(2) $\dim(S_\varphi) =\mid \varphi \mid$.

(3) $S_{\varphi'} \cap S_\varphi^{\rm Zar} \neq \emptyset \Rightarrow S_{\varphi'} \subset S_\varphi^{\rm Zar}$*; hence*

(4) *the boundary of S_φ consists of the union of all $S_{\varphi'}$ with $\varphi' \subsetneqq \varphi$.* □

Most of these results were reproved in [28]; e.g. the dimension formula we find in their Section 8 and in 9.3.

Example 10.12. Here is an (easy) example showing that a final filtration is not unique, although the final sequence is determined by a symmetric BT_1. Suppose N is a superspecial BT_1 of rank $2g$. This means that $0 \subset N[F] = \mathrm{Im}(N) \subset N$, where $N[F]$ is also annihilated by V. This is the case if and only if $F^2 = 0 = V^2$ on the symmetric N. For $g > 1$ a related final filtration is not unique. We work out the example for $g = 2$. A final filtration has the property that it is symplectic and $V(N_3) = N_1 = F(N_3)$. Study the Dieudonné module $M = \mathbb{D}(N)$. As this is superspecial we can write $M = K{\cdot}e + K{\cdot}\mathcal{F}e + K{\cdot}f + K{\cdot}\mathcal{F}f$ with the pairing e.g. given by $< e, \mathcal{F}e >= \epsilon =< f, \mathcal{F}f >$, with $\epsilon \neq 0$ (having other properties) and other pairs giving zero. We see that any choice M_3, generated by M_2 and a vector $X = b{\cdot}e + c{\cdot}f \in M$, with $b, c \in K$ has the property that X is pairing to zero with $M_1 = K{\cdot}\mathcal{F}(X)$. Hence $< X, \mathcal{F}X >= (bb^p + cc^p)\epsilon$. As M_3 only depends on b/c we see that there are exactly $p+1$ final filtrations refining the canonical filtration on a superspecial BT_1 for $g = 2$.

Lemma 10.13. *Let $g \geq 1$; write $r = \lceil g/2 \rceil$; i.e. either $2r = g$ or $2r = g + 1$. Let $\varphi = \{\varphi(1), \cdots, \varphi(g)\}$ be an elementary sequence. The* EO *stratum $S_\varphi \subset \mathcal{A}_{g,1}$ is contained in the supersingular locus $\mathcal{S}_{g,1} = W_\sigma = \mathcal{W}_\sigma(\mathcal{A}_{g,1})$ if and only if $\varphi(r) = 0$:*

$$\varphi(r) = 0 \Longleftrightarrow S_\varphi \subset W_\sigma.$$

Also see Step 2 in the proof of Theorem 4.8 of [14].

Proof ($\Longrightarrow$). Suppose $\varphi(r) = 0$. Consider the related final sequence

$$\psi = \{\varphi(1), \cdots, \varphi(g); \psi(g+1), \cdots, \psi(2g)\}, \quad \text{with} \quad \varphi(i) = \psi(i) \ \text{for} 1 \leq i \leq g.$$

Consider $[(A,\lambda)] = x \in S_\varphi$. We write $N = A[p]$. On $\mathbb{D}(N) = M = M_{2g}$ we have a filtration given by φ

$$0 \subset M_1 \subset \cdots \subset M_g = \mathrm{Ker}(\mathcal{V}) = \mathrm{Im}(\mathcal{F}) \subset \cdots \subset M_{2g}.$$

Claim:

$$\textit{if } \varphi(r) = 0 \textit{ then } \mathcal{F}(M_{g+r}) \subset M_r \textit{ and } \mathcal{V}(M_{g+r}) \subset M_r.$$

Note that $N_{r+g}/N_r \subset A/N_r$. If we prove the claim, we conclude that $a(A/N_r) = g$, where $\mathbb{D}(N_r \subset A[p]) = (M_r \subset M_{2g})$; hence A/N_r is superspecial. Hence this shows A is supersingular.

Proof of the claim. We start by giving two examples; then we give the general argument.

For $g = 4, r = 2$ the property $\varphi(r) = 0$ implies that the related final sequence is of the form

$$\psi = \{0, 0, \cdot, \cdot;\ \cdot, 2, 3, 4\}.$$

We see, using notation as in [104], 9.1, that $\mathbb{D}(A[p]) = M_{2g}$ has a basis

$$\{Y_4, Y_3, Z_3, Z_4; Z_5, Z_6 = Z_{g+r}, X_3, X_4\},$$

where $\{Y_2, Y_1, X_1, X_2\} = \{Z_3, Z_4, Z_5, Z_6\}$, the identification depending on the further structure of φ. We see that M_{g+r}/M_r has a basis given by $\{Z_3, Z_4, Z_5, Z_6\}$. As $\psi(g+r) = 2$ we see that $\mathcal{F}(M_{g+r}) \subset M_r$. Note that Y_4, Y_3, Z_3, Z_4 span the image of $\mathcal{F}$, hence the kernel of $\mathcal{V}$. The elements Z_5, Z_6, X_3, X_4 under $\mathcal{V}$ map to $\pm Y_1, \pm Y_2, \pm Y_3, \pm Y_4$ in this ordering. Thus $\mathcal{V}(M_{g+r}) \subset M_r$. Hence the claim for $g = 4$.

For $g = 5, r = 3$ and $\varphi(r) = 0$ we obtain

$$\psi = \{0, 0, 0, \cdot, \cdot;\ \cdot, 2, 3, 4, 5\},$$

and

$$\{Y_5, Y_4, Y_3, Z_4, Z_5; Z_6, Z_7, Z_8 = X_3, X_4, X_5\},$$

and we prove by an analogous computation that $\mathcal{F}$ and $\mathcal{V}$ map $Z_4, \cdots, Z_7, X_3$ into $N_r = N_3$.

Here is the proof in the general case. By $\varphi(r) = 0$ we see

$$\psi = \{0, \cdots, \psi(r) = 0, \cdots; \cdots, \psi(2g-r) = g-r, \cdots, \psi(2g) = g\}$$

and a basis for N_{2g} is:

$$\{Y_g, \cdots Y_{g-r+1}, Z_{r+1}, \cdots, X_{g-r+1}, \cdots, X_g\}.$$

Here we use that $g \leq 2r$. As $\{Z_{g+r+1}, \cdots, Z_{2g}\} = \{X_{g-r+1}, \cdots, X_g\}$ we conclude $\psi(Z_{g+r}) = g - (g-r) = r$. This shows that $\mathcal{F}(M_{g+r}) \subset M_r$. As $Z_{g+r+1}, \cdots, Z_{2g}$ under $\mathcal{V}$ map to $\pm Y_i$, $\;g - r \geq i \geq 1$, we see that all elements in M_{g+r} under $\mathcal{V}$ map into the span of $Y_g, \cdots, Y_{g-r+1}$. Hence $\mathcal{V}(M_{g+r}) \subset M_r$. $\Longrightarrow \square$

Proof ($\Longleftarrow$). Choose g. Let μ be the "almost supersingular symmetric Newton polygon". This is the (unique) highest symmetric Newton polygon which is not supersingular. To be concrete, if $g = 2r-1$, then $\mu = (r, r-1) + (r-1, r)$. If $g = 2r$ then $\mu = (r, r-1) + (1,1) + (r-1, r)$. We compute the elementary sequence $\tau = \varphi_\mu$ related to the minimal p-divisible group $H(\mu)$, see Section 12 below. Note that $a(H(\mu)) = g-1$. Using [104], 8.3 we compute that

$$\varphi_\mu = \{1, \cdots \tau(r-1) = 0, \tau(r) = 1, \cdots 1\}.$$

Using material in 12, in particular 12.5, we conclude that $S_\tau \subset W_\mu$; in particular no point of S_τ is supersingular. Consider any φ with $\varphi(r) \neq 1$. Then $\tau \prec \varphi$, in the notation of [104], see 10.10. By [104], Section 11, we conclude that the Zariski closure $(S_\varphi)^{\mathrm{Zar}}$ contains S_τ. Hence $S_\varphi \not\subset W_\sigma$. $\square$

This lemma was used in several papers by Harashita, and later generalized in [39], 4.1 and 4.2. Using this lemma we deduce:

Theorem 10.14 (Ekedahl - Van der Geer)**.** *Let φ be an elementary sequence such that $S_\varphi \not\subset W_\sigma$. Then S_φ is geometrically irreducible.* $\square$

In fact, 10.13 shows how to translate the property $S_\varphi \not\subset W_\sigma$ into $\varphi(r) \neq 0$. This is equivalent to the condition given in [28], Th. 11.5, and hence that theorem gives this result.

The result of this theorem was conjectured in [104], 14.1. On the other hand I also conjectured that any S_φ contained in the supersingular locus is reducible for $p \gg 0$. This was proved to be true by Harashita, and see [40], 3.5 for an even more general result.

Theorem 10.15 (Faltings, Chai, Ekedahl – Oort; [29], Korollar on page 364; see [6] for the case $p > 2$; also see [30], Chap. IV, Coroll. 6.8; see [104], Coroll. 1.4)**.** *For every prime number p the moduli space $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is geometrically irreducible.*
$\square$

10.16. Using the theory of EO strata we have given a new proof of this theorem (purely in characteristic p). We sketch this proof. There is a unique 1-dimensional EO stratum: $S_{\varphi'}$ with $\varphi' = \{0, \cdots 0, 1\}$. We show that $L := (S_{\varphi'})^{\mathrm{Zar}}$ (a union of rational curves) is geometrically connected, see [104], §7; this proof is purely algebraic, it treats lattices in the supersingular locus; I like to mention that this result and its proof are entirely due to Ekedahl, and it was one of our first bits of evidence that this technique might have applications. Using 10.11 (1) and studying EO strata at the boundary of a toroidal compactification of $\mathcal{A}_{g,1}$ we prove that for every elementary sequence φ we have $S_{\{0,\cdots 0,1\}} \subset L \subset S_\varphi^{\mathrm{Zar}}$. Hence $\mathcal{A}_{g,1}$ is geometrically connected. As $\mathcal{A}_{g,1,n}$ for any prime-to-p level structure $n \geq 3$ is regular, this proves the theorem.

10.17. More information and more material about EO strata can be found in: [62], [104], [14], [49], [77], [34], [28], [39], [40], [41], [43], [141], [72], [74], [146], [39], [40], [41], [43].

11. Foliations

11.1. Consider a set inside the moduli space characterized by the property that any two points $x = [(A, \lambda)]$and $y = [(B, \mu)]$ give geometrically isomorphic quasi-polarized p-divisible groups $(A, \lambda)_\Omega[p^\infty] \cong (B, \mu)_\Omega[p^\infty]$. Such a locus is called *a leaf, or a central leaf*, see [111]. These parts of the moduli space have properties very similar to characteristic zero moduli spaces; for example, Hecke correspondences are finite-to-finite on central leaves. Newton polygon strata are, up to a finite morphism, isomorphic to a product of a central leaf and of an isogeny leaf. The theory of leaves is crucial in the Hecke orbit conjecture, see § 13.

Definition 11.2. Central leaves. For a polarized abelian variety (A, μ) we define $\mathrm{type}^{(p)}(\mu)$ as the isomorphism class of μ restricted to $\prod_{\ell \neq p} T_\ell(A) \otimes \Omega$, where Ω is an algebraically closed field. We see that $\mathrm{type}^{(p)}(\mu)$ is given by elementary divisors of the form

$$(d_1, \ldots, d_g), \qquad \text{where} \qquad d_1, \ldots, d_g$$

are positive integers such that

$$d_1 \,|d_2| \cdots |\, d_g \quad \text{and} \quad d = \prod_{i=1}^{g} d_i = \sqrt{\deg(\mu)^{(p)}},$$

where $\deg(\mu)^{(p)}$ is the largest factor of $\deg(\mu)$ prime to p.

Work over a perfect field $K \supset \mathbb{F}_p$. For a given $[(A, \mu)] = x$ we write

$$\mathcal{C}(x) = \{[(B, \nu)] \in \mathcal{A}_g \mid (B, \mu)[p^\infty] \otimes \Omega \cong (A, \mu)[p^\infty] \otimes \Omega, \ \mathrm{type}^{(p)}(\mu) = \mathrm{type}^{(p)}(\nu)\}.$$

This is called the *central leaf passing through* $[(A, \mu)] = x$. We see that on a leaf the prime-to-p and the p-adic invariants are constant, by definition. The prime-to-p part is discrete, but, as we shall soon see, the p-adic invariant, i.e. the isomorphism class of the p-divisible group, is far from discrete in general. This definition is a refinement given in [14], 2.2 of definitions to be found in [111]. We will shorten "central leaf" to "leaf" if confusion (with "isogeny leaf") is unlikely. One can also take into account in the definition the notion of a level structure.

Note that for $\xi := \mathcal{N}(A)$ we have $\mathcal{C}(x) \subset \mathcal{W}_\xi(\mathcal{A}_g)$, as all points on $\mathcal{C}(x)$ define the same p-divisible group (over an algebraically closed field), and hence the same Newton polygon.

Definition 11.3. Leaves also can be defined for families of p-divisible groups. Let $\mathcal{Y} \to S$ over a base scheme $K \supset \mathbb{F}_p$ be a p-divisible group and let X be a p-divisible group over a field. We write

$$\mathcal{C}_X(S) := \{x \in S \mid \mathcal{Y}_c \otimes \Omega \cong X \otimes \Omega\}.$$

Theorem 11.4 ([111], 3.3)**.** *For every $x \in \mathcal{A}_g(K)$ the central leaf $\mathcal{C}(x) \subset \mathcal{A}_g \otimes K$ is a locally closed subset. Moreover $\mathcal{C}(x) \subset \mathcal{W}^0_\xi(\mathcal{A}_g \otimes K)$, with $\xi = \mathcal{N}(A)$, is closed in this open Newton polygon stratum.* □

In a proof a this theorem we make use of the notion of a *slope filtration* of a p-divisible group over any field, as initiated by Grothendieck, written up by Zink, see [155]. For the case of a constant Newton polygon over a normal base, see [120]. This is an important tool for the study of p-divisible groups over an arbitrary base scheme. Here are some details:

Definition 11.5 ([111], 1.1)**.** *Let S be a scheme, and let $X \to S$ be a p-divisible group. We say that X/S is* geometrically fiberwise constant, *abbreviated* gfc *, if there exist a field K, a p-divisible group X_0 over K, a morphism $S \to \mathrm{Spec}(K)$, and for every $s \in S$ an algebraically closed field $k \supset \kappa(s) \supset K$ containing the residue class field of s and an isomorphism $X_0 \otimes k \cong_k X_s \otimes k$.* An analogous terminology will be used for quasi-polarized p-divisible groups and for (polarized) abelian schemes.

Theorem 11.6 (Zink & Oort, [155], [120], 2.1, and [111], 1.8)**.** *Let S be an integral,* normal *noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group with* constant Newton polygon. *Then there exists a p-divisible $\mathcal{Y} \to S$ and an S-isogeny $\varphi : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}/S$ is* gfc. □

Theorem 11.7 ([111], 1.3)**.** *Let S be a scheme which is integral, and such that the normalization $S' \to S$ gives a noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group; let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $\mathcal{X} \to S$ is* gfc. *Then there exists a finite surjective morphism $T_n = T \to S$, such that $\mathcal{X}[p^n] \times_S T$ is constant over T.* □

11.8. Note that we gave a "point-wise" definition of $\mathcal{C}_X(S)$; we can consider $\mathcal{C}_X(S) \subset S$ as a closed set, or as a subscheme with induced reduced structure; however is this last definition "invariant under base change"? It would be much better to have a "functorial definition" and a nature-given scheme structure on $\mathcal{C}_X(S)$.

A proof of this theorem is quite involved. One of the ingredients is the notion of "completely slope divisible p-divisible groups" introduced by T. Zink, and theorems on p-divisible groups over a normal base, see [155] and [120].

Considering the situation in the moduli space with enough level structure in order to obtain a fine moduli scheme, we see that $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p)$ is regular.

Theorem 11.9 ([111], 3.14, 3.16)**.** *Suppose $[(A, \mu)] = x$ and $[(C, \nu)] = z$ admit an isogeny $A \to C$ which respects the polarizations up to a rational multiple. Then there exists a finite-to-finite correspondence*

$$\mathcal{C}(x) \longleftarrow T \longrightarrow \mathcal{C}(z).$$

□

Using this theorem we can prove: central leaves belonging to the same Newton polygon have the same dimension.

Notation: we write $c(\xi)$ for the dimension of a central leaf in $\mathcal{W}^0_\xi(\mathcal{A}_g)$

Isogeny correspondences in characteristic p may blow up and down subschemes, while in characteristic zero isogeny correspondences are finite-to-finite. However we see that on central leaves such correspondences behave as in characteristic zero. In this way we sort out where this blowing up and down does happen, or does not happen. We pin down the cause of this.

Definition 11.10 (Hecke-α-orbits; [111], 4.1)**.** We study isogeny correspondences where all isogenies involved geometrically are a sequence of isogenies with α_p – kernel; we call such α – correspondences. Consider $[(A, \lambda)] = x \in \mathcal{A}_g(K)$. Consider all diagrams

$$(A, \lambda) \otimes \Omega \xleftarrow{\varphi} (C, \zeta) \xrightarrow{\psi} (B, \mu),$$

where:

Ω is an algebraically closed field containing K,

(C, ζ) is a polarized abelian variety over Ω,

$\varphi : C \to A \otimes \Omega$ is an isogeny such that $\varphi^*(\lambda) = \zeta$,

(B, μ) is a principally polarized abelian variety over Ω, and

$\psi : C \to B$ is an isogeny such that $\psi^*(\mu) = \zeta$;

$\mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\psi)$ are geometrically successive extensions of the finite group scheme α_p;

in this case we write $[(B, \mu)] \in \mathcal{H}_\alpha(x)$. Note that $\mathcal{H}_\alpha(x)$ need not be a closed set; however we can show it is a union of closed sets. We define $I(x)$ the union of all irreducible components of $\mathcal{H}_\alpha(x)$ containing x.

Theorem 11.11 ([111], 4.11)**.** *For any $[(A, \mu)] = x \in \mathcal{A}_g$ the isogeny leaf $I(x)$ exists. It is a closed subset of $\mathcal{A}_g$.* □

This notion was earlier studied in the theory of Rapoport-Zink spaces in mixed characteristic, see [123]. Consider $x \in \mathcal{A}_g$ defined over a perfect field. Any component of $I(x)$ is the reduced scheme underlying a component of the reduction mod p of a Rapoport-Zink space passing through x.

Note that for different points in the same Newton polygon stratum their isogeny leaves can look very different. Here is an easy, though instructive example.

Example 11.12. Consider $[(A, \lambda)] = x \in \mathcal{A}_{3,1}$, where $\mathcal{N}(A) = \xi = (2, 1) + (1, 2)$. We shall see that $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{3,1})$ is irreducible, of dimension equal to three. All central leaves have dimension two, every leaf is regular, and (a general property of central leaves:) any two different leaves do not meet inside W_ξ. By the way, their closures inside $\mathcal{A}_{3,1}$ do meet, a phenomenon not yet understood in the general case. In case $a(A) = 1$, the isogeny leaf $I(x)$ is an irreducible rational curve. In case $a(A) = 2$, the isogeny leaf $I(x)$ has two irreducible components, both a rational curve. For $a(A) = 2$ the set $H_\alpha(x)$ is closed in case $x \in \mathcal{A}_{3,1}(\mathbb{F}_{p^2})$, and it has an infinite number of components in the general case.

Theorem 11.13 (Construction: the product structure; [111], Th. 5.3)**.** ("Central leaves and isogeny leaves almost give a product structure on an irreducible component of a Newton polygon stratum.") *Work over an algebraically closed field k. Choose a symmetric Newton polygon ξ, an irreducible component W of $\mathcal{W}^0_\xi(\mathcal{A}_g)$, an irreducible component C of a central leaf, and an irreducible component I of an isogeny leaf. There exist finite surjective morphisms $f\colon T \twoheadrightarrow C$, $g\colon J \twoheadrightarrow I$, a finite surjective morphism*

$$\Phi : T \times J \quad \twoheadrightarrow \quad W$$

and a polarization-preserving quasi-isogeny

$$\Theta\colon\ f^*(A,\lambda) \cdots \to \Phi^*(B,\mu)\,.$$

such that for every $u \in J$,

$$\Phi(T \times \{u\}) \textit{ is a component of a central leaf,}$$

and for every $t \in T$,

$$\Phi(\{t\} \times J) \textit{ is a component of an isogeny leaf.}$$

□

11.14. Isogenous polarized abelian varieties define leaves between which there is a finite-to-finite correspondence, see 11.9. Using moreover 11.13 we conclude that any two leaves in the same Newton polygon stratum have the same dimension. Note that the degree of the respective polarizations plays no role for such questions in the theory of central leaves, while in the study of EO strata, Newton polygon strata and isogeny leaves this aspect is very different. Hence for $x \in \mathcal{W}^0_\xi(\mathcal{A}_g)$ and for W an irreducible component of this Newton polygon passing through x we see that a component of $I(x)$ inside W its dimension equals $\dim(W) - c(\xi)$.

How can we compute the dimension of a central leaf?

Definition 11.15. Let ξ be a symmetric Newton polygon. Write $\xi = \sum_i (m_i, n_i)$ with $i < j \Rightarrow m_i/(m_i + n_i) \geq m_j/(m_j + n_j)$. To this lower convex polygon we associate the upper convex polygon ξ^* where the slopes of ξ are used in reversed order, the largest slope first, etc., the smallest slope last. We define

$$\triangle(\xi;\xi^*) := \{(x,y) \in \mathbb{Z} \mid (x,y) \prec \xi, \quad (x,y) \gneqq \xi^*, \quad x \leq g\},$$

$$\boxed{\mathrm{cdp}(\xi) := \#\left(\triangle(\xi;\zeta^*)\right);}$$

"cdp" = <u>d</u>imension of <u>c</u>entral leaf, <u>p</u>olarized case. See [119] for explanation, and also for definitions in the unpolarized case.

Theorem 11.16. *For every* $[(A,\mu)] = x \in \mathcal{A}_g$ *with* $\mathcal{N}(A) = \xi$ *we have*

$$\dim(\mathcal{C}(x)) = \mathrm{cdp}(\xi).$$

□

See [119] for a proof, for other properties, also in the unpolarized case, and for applications. We gave three different proofs, one using combinatorics, and [74], [72], [117] and a result by Wedhorn, see [146] and also [147]; one proof uses a beautiful generalization by Chai of Serre-Tate coordinates, see [10]; one proof uses Rapoport-Zink spaces, and a result by Eva Viehmann, see [141], computing this way the dimension of an isogeny leaf inside $\mathcal{A}_{g,1}$, and hence by 11.13 and 11.9 the dimension of any central leaf follows. See [140], 5.4.

By this theorem we see that the dimension of a central leaf can be easily computed once we know the Newton polygon.
Originally the structure of foliations of Newton polygon strata was developed in order to understand the Hecke orbit problem. But soon other applications appeared, as e.g. the PhD-thesis by Elena Mantovan, see [69].

Theorem 11.17 (Chai – Oort, [14], Th. 4.4)**.** *Let* $[(A,\mu)] = x \in \mathcal{A}_g$*. Suppose that* A *is not supersingular. In this case* $\mathcal{C}(x)$ *is geometrically irreducible.* □

11.18. As an illustration we record for $g = 4$ the various data considered:

NP	ξ	f	sdim(ξ)	$c(\xi)$	$i(\xi)$	ES($H(\xi)$)
ρ	$(4,0)+(0,4)$	4	10	10	0	$(1,2,3,4)$
$f=3$	$(3,0)+(1,1)+(0,3)$	3	9	9	0	$(1,2,3,3)$
$f=2$	$(2,0)+(2,2)+(0,2)$	2	8	7	1	$(1,2,2,2)$
β	$(1,0)+(2,1)+(1,2)+(0,1)$	1	7	6	1	$(1,1,2,2)$
γ	$(1,0)+(3,3)+(0,1)$	1	6	4	2	$(1,1,1,1)$
δ	$(3,1)+(1,3)$	0	6	5	1	$(0,1,2,2)$
ν	$(2,1)+(1,1)+(1,2)$	0	5	3	2	$(0,1,1,1)$
σ	$(4,4)$	0	4	0	4	$(0,0,0,0)$

Here $\rho \succ (f=3) \succ (f=2) \succ \beta \succ \gamma \succ \nu \succ \sigma$ and $\beta \succ \delta \succ \nu$. The notation ES, encoding the isomorphism type of a BT_1 group scheme, is as in [104]; the number f indicates the p-rank. We write $c(\xi)$ for the dimension of a central leaf related to the Newton polygon ξ. We write $i(\xi)$ for the dimension of an isogeny leaf inside W_ξ.

11.19. For $g = 5$ and $f = 0$ we obtain 5 possible Newton polygons (a totally ordered set in this case):

ξ	sdim(ξ)	$c(\xi)$	$i(\xi)$	ES($H(\xi)$)
$(4,1) + (1,4)$	10	9	1	$(0,1,2,3,3)$
$(3,1) + (1,1) + (1,3)$	9	7	2	$(0,1,2,2,2)$
$(2,1) + (2,2) + (1,2)$	8	4	4	$(0,1,1,1,1)$
$(3,2) + (2,3)$	7	3	4	$(0,0,1,1,1)$
$(5,5)$	6	0	6	$(0,0,0,0,0)$

Theorem 11.20. *A geometrically fiberwise constant p-divisible group X/S over an (excellent) integral* normal *base has a natural slope filtration.* □

Corollary 11.21. *Let $X \to C$ be a p-divisible group over a central leaf. Then X/S admits a slope filtration.* □

This is the starting point for a generalization of Serre-Tate coordinates, see [10].

For the proof of the theorem one uses first [120], in order to have an isogeny $\psi : Z \to X/S$ from a completely slope divisible p-divisible group , called Z, to the given one X/S. Let $\mathrm{Ker}(\psi) = N \to S$. By induction on the isoclinic parts of the filtration $Z_1 \subset Z_2 \subset \cdots \subset Z$, it suffices to show that $N \cap Z_1$ is flat over S.

Going to a finite cover $T \to S$, using [111], Theorem 1.3, further using that completely slope divisible implies geometrically fiberwise constant (or using directly [111], Lemma 1.4), and using [111], Lemma 1.10, and [111], Lemma 1.9 we see that the fibers of $N_T \cap Z_{1,T}$ have constant rank, hence the same for the fibers of $N \cap Z_1$, hence this is flat. This proves the layer $Z_1 \subset Z$ descends to a sub-p-divisible $X_1 \subset X$, which is the lowest isoclinic part; finish by induction. □

Example 11.22. Here is an example where there does not exist a slope filtration (over a non-normal base scheme). Choose $g = 3$, and $\xi = (2,1) + (1,2)$. Choose a principally polarized abelian variety (A, λ) with $\mathcal{N}(A) = \xi$ and $a(A) = 2$. For $[(A, \lambda)] = x$ we see that $I(x)$ is a curve, which locally at x consists of two branches; over one branch there does exist a slope filtration of the deformed p-divisible group; over the other branch the natural slope filtration does not exist.

In most cases: over an open Newton polygon stratum the natural slope filtration does not exist.

11.23. Historical remark on the dimension of leaves. In summer 2000 I gave a talk in Oberwolfach on foliations in moduli spaces of abelian varieties. After my talk, in the evening of Friday 4-VIII-2000 Bjorn Poonen asked me several questions, especially related to the problem I raised to determine the dimensions of central leaves. Our discussion resulted in Problem 21 in [26]. His expectations coincided with computations I had made of these dimensions for small values of g. Then I jumped to the conclusion what those dimensions for general g could be; that is what was proved later, and reported on here, see 11.16. I thank Bjorn Poonen for his interesting questions; our discussion was valuable for me.

11.24. Discussions and more material on the concept of central leaves can be found in: [111], [14], [73]. [119], [123], [107], [116], [117], [155], [120], [42].

12. Minimal p-divisible groups

12.1. One might wonder whether $X[p] \cong Y[p]$ implies $X \cong Y$; this question is answered by the theory of "minimal p-divisible groups", see [114].

We have seen central leaves in a foliation, and EO strata. It is an interesting question in which way these subspaces intersect. In particular, does it happen that a central leaf and an EO stratum are equal? This question has been completely settled, and in this section we record the result.

In a letter on January 5, 1970 Grothendieck asked Mumford: "*I wonder if the following might be true: assume k algebraically closed. Let G and H be BT groups, and assume $G(1)$ and $H(1)$ are isomorphic. Are G and H isomorphic?*" See [81]. Grothendieck observes this is the case according to Lazard, if these are formal p-divisible groups on one parameter. Mumford replied that already for formal groups on two parameters there are examples where this property does not hold, as we see using results in [68]. See 12.6.

We shall see in 12.3 this question whether $X[p] \cong Y[p]$ implies that X and Y are isomorphic has a positive answer if (and in fact only if) X, or Y, is minimal; one parameter formal groups are automatically minimal; for more parameters there do exist non-minimal ones; for precise statements see 12.6.

Historical remark. The correspondence between Grothendieck and Mumford only became known and available to me in 2010; I did not know this question by Grothendieck and the answer by Mumford when I worked many years ago on the result to be found in [114].

12.2. Definition of minimal p-divisible groups. Given a prime number p, and coprime, non-negative integers m, n we define the p-divisible group $H_{m,n}$ over $\mathbb{F}_p$ by the covariant Dieudonné module

$$\mathbb{D}(H_{m,n}) = \sum_{i=0}^{i=m+n-1} W{\cdot}e_i$$

with the structure of a Dieudonné module defined by:

$$\mathcal{V}{\cdot}e_j = e_{j+m}, \quad \mathcal{F}{\cdot}e_j = e_{j+n}, \; p{\cdot}e_j = e_{j+m+n}, \quad \forall j \geq 0;$$

in particular $\mathcal{F}^m{\cdot}e_j = \mathcal{V}^n{\cdot}e_j$ for all $j \geq 0$. We write $H_{1,0} = G_{1,0} = \mathbb{G}_m[p^\infty]$ and $H_{0,1} = G_{0,1} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

We see that $H_{m,n}$ is isogenous with $G_{m,n}$. In particular $\mathcal{N}(H_{m,n})$ is isoclinic of slope equal to $m/(m+n)$ and heigth $h = m+n$.

Here is another description of $\mathbb{D}(H_{m,n})$ over $\mathbb{F}_p$. Choose integers $a, b \in \mathbb{Z}$ with $am + bn = 1$. Write $\pi = \mathcal{V}^a\mathcal{F}^b$, as an element of $\mathrm{End}^0(G_{m,n}) = \mathrm{End}^0(H_{m,n})$; in fact $\pi \in \mathrm{End}(H_{m,n})$. We see that $e_j = \pi^j \cdot e_0$ and the elements $\{e_j \mid 0 \leq j < m+n\}$ form a W-bases for $\mathbb{D}(H_{m,n})$.

We see: $H_{m,n} \cong G_{m,n}$ if and only if $m \leq 1$ or $n \leq 1$. Taking into account Theorem 12.3 below, this explains the correspondence on this point between Grothendieck and Mumford cited above.

Here is a property which characterizes $H_{m,n} \otimes k$ for any algebraically closed field $k \supset \mathbb{F}_p$:

A p-divisible group $X \sim H_{m,n} \otimes k$ over k is isomorphic with $H_{m,n} \otimes k$ if and only if $\mathrm{End}(X)$ *is the maximal order in the division algebra* $\mathrm{End}^0(G_{m,n} \otimes k)$*.*

Note that $\mathrm{End}_{\mathbb{F}_p}(H_{m,n})$ is commutative and much smaller than $\mathrm{End}(H_{m,n} \otimes k)$ if $m > 0$ and $n > 0$. From now on we write $H_{m,n}$ instead of $H_{m,n} \times S$ for any scheme S over $\mathbb{F}_p$ if no confusion can arrive.

Let $\zeta = \sum_i (m_i, n_i)$ be a Newton polygon, with $\gcd(m_i, n_i) = 1$ for all i. We define

$$H(\zeta) := \sum_i H_{m_i,n_i}.$$

Definition. *We say that $H(\zeta)$ is the minimal p-divisible group with Newton polygon equal to ζ.*

Theorem 12.3 ([114], 1.2)**.** *Let k be an algebraically closed field, and let X be a p-divisible group over k such that*

$$X[p] \cong H(\zeta)[p]; \quad \textit{then} \quad X \cong H(\zeta).$$

□

Note that a priori we have no information about the Newton polygon of X.

12.4. Here is another way of phrasing the results. Starting from a p-divisible group X we obtain by truncating at level one a BT_1 group scheme:

$$[p] : \{X \mid \text{a } \ p-\text{divisible group}\}/\cong_k \quad \longrightarrow \{G \mid \text{a } \ \mathrm{BT}_1\}/\cong_k; \quad X \mapsto G := X[p].$$

This map is known to be surjective; see [49], 1.7, see [104], 9.10; it is not difficult to construct a section for this map, e.g. see [117], 2.5. It is the main theorem of [114] that the fiber of this map over (G up to $\cong_k$) is precisely one p-divisible group X if G is minimal.

For a minimal p-divisible group $H(\xi)$ with symmetric Newton polygon over an algebraically closed field k there is, up to isomorphism, precisely one principal quasi polarization; see [111], 3.7. We define the *central stream*

$$\mathcal{Z}_\xi \subset W^0_\xi \subset \mathcal{A}_{g,1}$$

as the central leaf passing through any (A, λ) with $A[p^\infty] \cong H(\xi)$.

Corollary 12.5. *For any symmetric ξ the central stream $\mathcal{Z}_\xi$ is an* EO *stratum.* $\square$

Example 12.6. For $m > 1$ and $n > 1$ and $\gcd(m,n) = 1$ there exist non-minimal p-divisible groups isogenous with $G_{m,n}$. For such groups we can find X and Y with $X[p] \cong Y[p]$ and $X \not\cong Y$.

We treat the example $m = 2$ and $n = 3$. We choose a perfect field K; in order to get infinitely many examples one can take for example $K = k$, an algebraically closed field. Denote the Dieudonné module $\mathbb{D}(H_{2,3}) = M$. For any $a \in W = W_\infty(K)$ we define the Dieudonné submodule $M^{(a)}$ as the one generated over the Dieudonné ring by $x^{(a)} := e_0 + a{\cdot}e_1$ in the notation as in 12.2. We define $X^{(a)}$ by $\mathbb{D}(X^{(a)}) = M^{(a)}$. We see that $M^{(a)}$ contains $x^{(a)}$ and all e_j for $j \geq 2$. It follows that $X^{(a)} \cong X^{(b)}$ if and only if $a - b \in p{\cdot}W$. Moreover we see that $\mathcal{F}^2 x^{(a)} - \mathcal{V}^3 x^{(a)} \in p{\cdot}M^{(a)}$. This shows that $X^{(a)}[p] \cong X^{(b)}[p]$ for all choices of $a, b \in W$. This finishes the construction of examples with $X[p] \cong Y[p]$ and $X \not\cong Y$. Note that $X^{(a)} \sim H_{2,3}$ but $X^{(a)} \not\cong H_{2,3}$ for $a \notin pW$.

The structure of the catalog constructed in 5.6 – 5.11 in [52] explains the general nature of this example.

We say that a finite group scheme is BT_1-*simple* if it is a BT_1 group scheme and there is no smaller non-zero BT_1 group scheme contained in it. Note that that in general a BT_1-*simple* is not a simple group scheme.

12.7. Work over an algebraically closed field $k \supset \mathbb{F}_p$. Consider BT_1 group schemes. Can we classify the simple ones (in the category of BT_1 group schemes)? In [117] we find:

$$G \textit{ is } \mathrm{BT}_1\textit{-simple} \quad \Longleftrightarrow \quad G \textit{ is indecomposable and minimal.}$$

12.8. Some references: [114], [104], [52], [111], [15], [143].

Instead of truncation at level one we can study truncations at an arbitrary level. Fixing the heigth h there exists an integer $N = N(h)$ such that for for p-divisible groups X_1 and X_2 and $n \leq N$ the condition $X_1[p^n] \cong X_2[p^n]$ implies $X_1 \cong X_2$, see [111], 1.7. For more information what $N(h)$ can be, see [83], [84], [139]

13. Hecke orbits

13.1. In this section we define Hecke orbits, and we discuss the Zariski closure of one Hecke orbit. In characteristic zero any Hecke orbit is everywhere dense in the moduli space (in the classical topology and in the Zariski topology). Chai proved that the Hecke orbit of an ordinary point $x \in \mathcal{A}_g^{\mathrm{ord}}$ is everywhere dense in $\mathcal{A}_g$; see [7]. However this is certainly not true for every moduli point in positive characteristic. E.g. the supersingular locus for any $g \geq 1$ is properly contained in $\mathcal{A}_g$, and hence the Zariski closure of a supersingular point is lower dimensional.

So what is the Zariski closure of a Hecke orbit? In [101], 15.A we find a conjecture what the Zariski closure of a Hecke orbit in positive characteristic should be; see Theorem 13.3; in [16] a proof will appear.

13.2. Consider $[(A,\mu)] = x \in \mathcal{A}_g$. We write $\mathcal{H}(x)$ for the set of points $[(B,\nu)] = y \in \mathcal{A}_g$ such that there exists an isogeny $\gamma : B \otimes \Omega \to A \otimes \Omega$, where Ω is some field, and a positive integer $n \in \mathbb{Z}_{>0}$ such that $\gamma^*(\mu) = n{\cdot}\nu$. Notation $y \in \mathcal{H}(x)$. See [15], § 1 for other definitions and a discussion.

We write $y \in \mathcal{H}_\ell(x)$, where $\ell \neq p$ is a prime number, if moreover the degree of γ and n are a power of ℓ.

Theorem 13.3 (Chai-Oort)**.** *Consider* $[(A,\mu)] = x \in \mathcal{A}_g$. *Write* $\xi = \mathcal{N}(A)$, *for the Newton polygon of* A. *Then*

$$\mathcal{H}(x) \text{ is dense in } \mathcal{W}^0_\xi(\mathcal{A}_g).$$

□

In case A is ordinary this was proved in [7]; also see [15], §9. The general case was conjectured in [101], 15.A. A proof will be published in [16]. For a survey see [9].

13.4. We discuss one aspect of the line of thought. For a supersingular $[(A,\mu)] = x$ the result of the theorem is easy to show; hence suppose A is not supersingular. Using the product structure in the theory of foliations, see 11.13, it suffices to show that $\mathcal{H}(x) \cap \mathcal{C}(x)$ is dense in $\mathcal{C}(x)$. This problem has a "discrete part" and a "continuous part". The discrete part is solved by showing that $\mathcal{C}(x)$ is geometrically irreducible, see 11.17. Choose any prime number $\ell \neq p$. The continuous part follows by showing that $\mathcal{H}_\ell(x) \cap \mathcal{C}(x)$ is dense in $\mathcal{C}(x)$. This is the heart, the difficult, non-trivial part, of the proof.

13.5. Some details about Newton polygon strata, Hecke orbits, and the main theorem of this section can be found in: [56], [25], [121], [7] , [101], [50], [9], [10], [13], [15], [108], [110], [16], [11].

14. Complete subvarieties of $\mathcal{A}_g$

14.1. Although the moduli space of polarized abelian varieties is not complete, it may contain complete subvarieties. E.g. any component of $\mathcal{A}_g$ in any characteristic contains a complete subvariety of dimension $g-1$. There is a complete subvariety of codimension g inside $\mathcal{A}_g \otimes k$ for $g \geq 2$ and for any field k of positive characteristic.

Van der Geer proved that in any characteristic the codimension of a complete subvariety of $\mathcal{A}_g$ is at least g; see [34], 2.7. In [101], 15.B we find the conjecture that over $\mathbb{C}$ a complete subvariety of codimension g should not exist if $g \geq 3$. This was proved in [59]. For a further discussion see [100].

In positive characteristic the stratum of abelian varieties of p-rank equal to zero is complete and has codimension g, see [93]; actually for $g > 2$ this stratum inside $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is also irreducible, as follows from 7.12. Are these the only possibilities for a complete subvariety of codimension g in case $g \geq 3$, as conjectured in [105], 5.2 ? We show this conjecture is not correct for $g = 3$.

Theorem 14.2 (Example)**.** *For $g = 3$ and any prime number p there are infinitely many* complete *subvarieties $S \subset \mathcal{A}_3 \otimes \mathbb{F}$ of dimension 3, where $\mathbb{F} := \overline{\mathbb{F}_p}$.*

Actually, for the examples we are going to construct, the generic point corresponds with an abelian variety of p-rank equal to one.

Proof. Choose the Newton polygon $\xi = (1,0) + 2(1,1) + (0,1)$. The closure of $W_\xi \subset \mathcal{A} := \mathcal{A}_{3,1} \otimes \mathbb{F}$ equals $V_{3,1}$, the p-rank one locus. Both are of dimension equal to four. We are going to construct *a complete subvariety of $V_{3,1}$ of dimension three.*

The central stream $\mathcal{Z}_\xi$ inside W_ξ equals $Z^0 := \mathcal{Z}_\xi = W_\xi(a = 2)$; it is of dimension 3. Note, however, that $\mathcal{Z}_\xi$ is not complete. Write $Z := \mathcal{Z}_\xi^{\mathrm{Zar}}$ for its Zariski closure inside $\mathcal{A}$ and Z^* for the Zariski closure inside the minimal compactification $\mathcal{A}^*$.

Claim. *The set $Z^* - Z$ is of pure dimension zero.* Indeed, under specialization the a-number remains the same, or gets bigger. Hence a boundary point of Z, i.e. a point in $Z^* - Z$, corresponds with a semi-abelian variety, with abelian part a superspecial abelian variety of dimension two. Hence there are only finitely many boundary points in $\mathcal{A}^*$. This proves the claim.

In any projective embedding of $Z \subset Z^*$ we choose a hyperplane H defined over $\mathbb{F}$ meeting $\mathcal{Z}_\xi$ and not containing any of the points in $Z^* - Z$. Write $U := H \cap Z$; write $U^0 := H \cap \mathcal{Z}_\xi$; note that the dimension of U and of U^0 equals 2. In 11.13 we constructed a finite, surjective morphism

$$\Phi : T \times J \longrightarrow W;$$

here we consider this for $W = W_\xi$ (which we know is irreducible, see 11.17, but we do not need that); $\dim(T) = \dim(Z) = 3$ and $\dim(J) = 1$. Choose the projection $q : T \times J \to T$ on the first factor. There is a point $j \in J(\mathbb{F})$ such that Φ gives a finite map

$$T \times \{j\} \longrightarrow Z^0 = \mathcal{Z}_\xi.$$

Let $U' \subset T \times \{j\}$ be an irreducible closed subset mapping onto $U^0 \subset Z^0 = \mathcal{Z}_\xi$. Define

$$T^0 := \Phi(U' \times J) \subset W_\xi, \quad \text{and} \quad T := (T^0)^{\mathrm{Zar}} \subset W_\xi \subset \mathcal{A}_{3,1},$$

the Zariski closure inside $\mathcal{A}$. Clearly $U' \times J$, and T^0, and T are of dimension 3. Write T^* for the closure of T inside $\mathcal{A}^*$.

Claim. *T is complete, i.e. $T = T^*$.* Consider a regular curve $\Lambda \subset T^*$ with $0 \in \Lambda(\mathbb{F})$ and $\Lambda^0 = \Lambda - \{0\} \subset T^0$. We show that $0 \in \mathcal{A}$. After taking a finite cover there is an abelian scheme $A \to \Lambda^0$; there is $\mathcal{N}^0 \subset A \to \Lambda^0$ such that the abelian scheme $A/\mathcal{N}^0$, with an appropriate choice of polarization, gives a moduli map landing into $U^0 \subset Z^0$. Specialization of $A/\mathcal{N}^0$ to any point in U gives an abelian variety (because $U \subset \mathcal{A}$ is complete). Hence $A \to \Lambda^0$ extends to an abelian scheme over Λ. This proves the claim. Hence this finishes the proof of the theorem. □

Question 14.3. Suppose $g \geq 4$. Let $T \subset \mathcal{A}_g \otimes \mathbb{F}_p$ be a complete subvariety with $\dim(T) = (g(g+1)/2) - g$. Does this imply that T is inside the p-rank zero locus?

This was formulated as a conjecture in [105], 5.2 for $g \geq 3$. We have seen here that this is not correct for $g = 3$. But it may still be correct for $g > 3$. Using the method of the proof above we were not able to construct any counterexamples for $g > 3$.

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