Purity reconsidered

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Version 10 December 2002

Talk in the conference:
The Cohomology of Moduli Spaces
Organized by Gerard van der Geer and Carel Faber
Amsterdam, 16 $\sim$ 20 / XII / 2002

Convention: all base schemes will be in characteristic p. Writing k or Ω we will assume $k = \overline{k} \supset \mathbb{F}_p$, respectively $\Omega = \overline{\Omega} \supset \mathbb{F}_p$. We will write $\mathcal{X} \to S$ for a p-divisible group over a base S (and sometimes will say "a family of p-divisible groups"), while we tend to write X when we consider a p-divisible group over a base field (but we are not strict in this convention). Instead of "p-divisible group" one can also use the equivalent notion "Barsotti-Tate group".

1 The Purity Theorem

In this talk we prove the *Purity Theorem*, as in [4], Th. 4.1, in the following form:

(1.1) Theorem (A. J. de Jong and F. Oort). *Let S be an integral, excellent scheme. Let $\mathcal{X} \to S$ be a p-divisible group over S. Let $U \subset S$ be the largest dense set on which the Newton polygon is constant. Then, either $U = S$, or* $\mathrm{codim}((S - U) \hookrightarrow S) = 1$.

Comments. This result and the next theorem are valid under more general conditions (for F-crystals), but we restrict ourselves to this case. Instead of "excellent" we could have required that S is a K-algebraic scheme, or that S has a normalization which is noetherian.

The set $U \subset S$ as in the theorem is open and dense in S as was proved by Grothendieck and Katz, see [5], Th. 2.3.1 on page 143.

(1.2) The Purity Theorem was announced and proved for the first time in [4]. For a further discussion, see [9]. In [10], §5, A. Vasiu announces a new proof of the Purity Theorem, which is different from earlier proofs; we have tried to understand his reasoning. We come to the conclusion that *the basic idea by Vasiu of proving this theorem is sound*; however we have not been able to check all arguments as written up in [10], pp. 34 - 37 . We present the idea by Vasiu combined with results by Manin [6], de Jong [3], Zink [11], and [8], [7]; in this way we derive a new proof of Theorem (1.1).

In fact, Vasiu states in [10], Theorem 5.1 a stronger theorem:

(1.3) Theorem (A. Vasiu). *Let S be an integral, excellent scheme. Let $\mathcal{X} \to S$ be a p-divisible group over S. Let $U \subset S$ be the largest dense set on which the Newton polygon is constant. Then the morphism $U \hookrightarrow S$ is an affine morphism.*

2 Some tools we are going to use

Here are some basic tools we are going to use in the proof.

(2.1) Let S be a scheme and let $G \to S$ and $H \to S$ be locally free, finite group schemes. Define a functor on S-schemes by:

$$T \mapsto H(T) := \mathrm{Hom}(G_T, H_T).$$

(H) *This functor is representable, and its representing object $\underline{\mathrm{Hom}}(G, H) \to S$ is an affine S-scheme.*
This is well-known and not very difficult to prove.

Let $q \in \mathbb{Z}_{>0}$. Let I_q be the functor defined by

$$T \mapsto I_q(T) = \{(\varphi : G_T \to H_T) \in \mathrm{Hom}(G_T, H_T) \mid \mathrm{Ker}(\varphi) \to T \quad \text{is locally free of rank} \quad q\}.$$

(I) *This functor is representable. Its representing object will be denoted by $\underline{\mathrm{I}}_q(G, H) \to S$*
Also this is not difficult to prove.

(2.2) Definition. *Let S be a scheme, and let $X \to S$ be a p-divisible group. We say that X/S is* geometrically fiberwise constant *if there exist a field K, a p-divisible group X_0 over K, a morphism $S \to \mathrm{Spec}(K)$, and for every $s \in S$ an algebraically closed field $k \supset \kappa(s) \supset K$ containing the residue class field of s and an isomorphism $X_0 \otimes k \cong_k X_s \otimes k$.*

(2.3) Completely slope divisible. We take the definition, due to T. Zink, of a completely slope divisible p-divisible group from [11], Def. 10, [8], 1.1; note that in this paper slopes are arranged to ascending order. This means:
Definition. *Let $X \to S$ be a p-divisible group. Let $0 \leq t_1 < t_2 < \ldots < t_m \leq s$ be integers and equalities. A p-divisible group Y over a scheme S is said to be* completely slope divisible with respect to *these integers if Y has a filtration by p-divisible subgroups:*

$$0 = Y_0 \subset Y_1 \subset \ldots \subset Y_m = Y$$

such that the following properties hold:
(1) *The quasi-isogenies*

$$\frac{F^s}{p^{s-t_i}} : Y_i \to Y_i^{(p^s)}$$

are isogenies for $i = 1, \ldots, m$.
(2) *The induced morphisms:*

$$\frac{F^s}{p^{s-t_i}} : Y_i/Y_{i-1} \to (Y_i/Y_{i-1})^{(p^s)}$$

are isomorphisms. This condition implies that the slope of $Y^{(i)} := Y_i/Y_{i-1}$ equals $\lambda_i := t_i/s$.

(2.4) Remark. A completely slope divisible p-divisible group over a noetherian and integral K-scheme S with $S(K) \neq \emptyset$ is geometrically fiberwise constant, see [7], Coroll. 1.8; it is easy to construct, over a field, even over a finite field, a geometrically fiberwise constant p-divisible group which is not completely slope divisible.

(2.5) Theorem (F. Oort and Th. Zink). *Let U be an integral, normal noetherian scheme. Let $\mathcal{Z} \to U$ be a p-divisible group over U with constant Newton polygon. Then there exist a p-divisible group $\mathcal{Y} \to U$, and an isogeny:*

$$\varphi : \mathcal{Y} \to \mathcal{Z} \quad over \quad S$$

such that $\mathcal{Y} \to U$ is completely slope divisible.

This theorem follows from [11], Th. 17 (over a regular base), respectively is contained in [8], Th. 2.1 (over a normal base). The proof of the theorem uses the concept of completely slope divisible p-divisible groups due to Th. Zink, see [11], Def. 10 respectively [8], Def. 1.2; note that every fiber of a completely slope divisible p-divisible group splits into isoclinic summands over a perfect field; use [8], Prop 1.4 in order to see that the geometric fibers as above can be defined over a finite field. Note that a completely slope divisible p-divisible group over an integral base is geometrically fiberwise constant, see [7], Coroll. 1.8.

(2.6) Lemma. *Let Z_1 and Z_2 be p-divisible groups over an algebraically closed field k, and let $i \in \mathbb{Z}_{\geq 0}$. We write*

$$\Psi = \Psi(Z_1, Z_2; i) := \{G \subset Z_1 \mid \exists \psi : Z_1 \to Z_2, \quad \deg(\psi) = p^i, \quad \mathrm{Ker}(\psi) = G\};$$

then

$$\#(\Psi) \quad < \quad \infty.$$

See [7], Lemma 1.10.

(2.7) Proposition. *Let A be a noetherian complete, local, normal domain of characteristic p; let K be its field of fractions and $A/\mathfrak{m}_A =: m$ its residue field. Let $\mathcal{Y} \to \mathrm{Spec}(A)$ be an* isoclinic, completely slope divisible *p-divisible group over A. Then for the p-divisible group $Q = \mathcal{Y} \otimes_A m$ there exists an isomorphism $Q \otimes_k R \cong \mathcal{Y}$.*

Proof. Using [5], Th. 2.7.1 on page 158, in case $A = m[[r]]$, or [4], Coroll. 2.17 on page 217, for the general case, we see there exists a p-divisible group P over k and an isogeny $\psi : P \otimes_k R \to \mathcal{Y}$. As $\mathcal{Y} \to \mathrm{Spec}(A)$ is completely slope divisible, hence geometrically fiberwise constant, we see that the fiberwise maps of ψ land into a finite set, see the lemma above, see [7], Lemma 1.10. Using [7], Lemma 1.9 we see that $\mathrm{Ker}(\psi)$ defines a constant subgroup scheme $\mathrm{Ker}(\psi) = \mathrm{Ker}(\psi \otimes m) \otimes_m A$; as $Q = P/\mathrm{Ker}(\psi \otimes m)$ we obtain $Q \otimes_m A = P/(\mathrm{Ker}(\psi \otimes m) \otimes_m A) \cong \mathcal{Y}$, which proves the proposition. □

(2.8) Theorem. *Let S be a excellent, integral scheme; let $\mathcal{X} \to S$ be a p-divisible group; let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $\mathcal{X} \to S$ is* geometrically fiberwise constant. *Then there exists a finite surjective morphism $T_n = T \to S$, such that $\mathcal{X}[p^n] \times_S T$ is constant over T.*
See [7], Th.1.3.

Let X and Y be p-divisible groups over a field; for integers $N \geq n \geq 0$ we denote by

$$\Phi_n : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X[p^n], Y[p^n])$$

and

$$\Phi_n^N : \mathrm{Hom}(X[p^N], Y[p^N]) \to \mathrm{Hom}(X[p^n], Y[p^n])$$

the natural restriction maps.

(2.9) Proposition. *For every $h', h'', n \in \mathbb{Z}_{>0}$ there exists an integer $N(h', h'', n)$ such that for p-divisible groups X and Y over an* algebraically closed field k *of height* h' =height(X) *respectively* h'' =height(Y) *and for every $N \geq N(h', h'', n)$ the following images are equal:*

$$\mathrm{Im}\,(\Phi_n) = \mathrm{Im}\left(\Phi_n^N\right).$$

The ingredients for proving this proposition are well-known: basically this follows from [6], Chapter III. See [7], Proposition 1.6

3 Proof of (1.3) and of (1.1)

In order to prove (1.3) and (1.1) it suffices to prove the case that moreover there is a morphism $S \to k$, where k is an algebraically closed field of characteristic p; from now on we make this assumption.

(3.1) Construction. We will consider a finite cover of S; the new situation again will be denoted by $\mathcal{X} \to S$, with the notation U for the inverse image of the old U. Then we are going to construct the following diagram:

$$\begin{array}{ccccc} I_{N,q} & \twoheadrightarrow & T & \hookrightarrow & H_n \\ & & \downarrow & & \downarrow \\ & & U & \hookrightarrow & S; \end{array}$$

Here $I_{N,q}$ and H_n are schemes; in this section we are going to show:

(1) We define the set T as the set-theoretic image of the morphism $I_{N,q} \to H_n$. The map $T \to U$ has finite geometric fibers.

(2) The morphism $I_{N,q} \to H_n \to S$ factors through $U \hookrightarrow S$.

(3) The induced map $T \to U$ is *surjective*.

(4) The constructible subset $T \hookrightarrow H_n$ is a *closed subset*; we consider T as a scheme with the induced, reduced scheme structure.

(5) $H_n \to S$ is an affine morphism. Hence $T \to S$ is affine.

(6) The morphism $T \to U$ is quasi-finite.

(7) The morphism $T \to U$ is proper; hence this morphism is finite.

(8) *Conclusion:* The morphism $U \hookrightarrow S$ is affine.

Here is the construction. We start with $X \to S \supset U$ as in (1.1). In order to prove the theorems it suffices to prove the theorem for the situation pulled back to the normalization of S. Hence from now on we assume that S is normal.

We write $\mathcal{Z} = \mathcal{X}_{|U}$, the restriction to $U \subset S$. By (2.5), Th. 2.1, we can choose an isogeny $\varphi : \mathcal{Y} \to \mathcal{Z}$, with $\mathcal{Y} \to U$ completely slope divisible. In this case $\mathcal{Y} \to U$ is geometrically fiberwise constant, see [7], Coroll. 1.9. Choose $a \in \mathbb{Z}_{>0}$ such that $\mathrm{Ker}(\varphi) \subset \mathcal{Y}[p^a]$. Define $\varphi' : \mathcal{Z} \to \mathcal{Y}$ such that

$$\left(\mathcal{Z} \xrightarrow{\varphi'} \mathcal{Y} \xrightarrow{\varphi} \mathcal{Z}\right) = \varphi \cdot \varphi' = [p^a]_{\mathcal{Y}}; \quad \text{write} \quad \deg(\varphi') =: q = p^i.$$

Choose an integer n with $n \geq i+1$. With the number n given and with the height h of $\mathcal{X}/S$, which is also the height of $\mathcal{Y}/U$, we choose $N \geq N(h,h,n)$ as in (2.9).

By (2.8) there exists a finite covering $U' \to U$ over which the group scheme $\mathcal{Y}[p^N] \times_U U'$ is constant; we can assume U' is integral; we replace, if necessary, U' by its normalization, and we choose $S' \to S$ as the normalization in the function field of U'. We pull back all groups in consideration to this new situation. For simplicity we use the letters U, S, $\mathcal{X} \to S$, $\mathcal{Y} \to U$ instead, and we have arrived at the situation where there exists moreover a p-divisible group Y_0/k and an isomorphism $\mathcal{Y}[p^N] \cong Y_0[p^N] \otimes_{\mathrm{Spec}(k)} U$. In order to prove our theorems it suffices to give proofs in the situation satisfying these extra conditions.

We define

$$H_n := \underline{\mathrm{Hom}}(\mathcal{X}[p^n], Y_0[p^n] \times S)$$

and

$$I_{N,q} := \underline{\mathrm{I}}_q(\mathcal{X}[p^N], Y_0[p^N] \times S),$$

using the notation as in (2.1). This constructs the schemes H_n and $I_{N,q}$. The restriction map induces a morphism $\Phi := \Phi_n^N : I_{N,q} \to H_n$. Its *set-theoretic image* is denoted by $\mathrm{Im}(\Phi) =: T \subset H_n$.

Note that any geometric point $t \in T(\Omega)$, by definition, lifts to a geometric point in $I_{N,q}(\Omega)$. Consider the corresponding homomorphism

$$t \leftrightarrow \psi_t(n) : \mathcal{X}[p^n]_s \to Y_0[p^n] \times \Omega; \quad \text{here} \quad (s : \mathrm{Spec}(\Omega) \to S) = (T \to S)\cdot(\mathrm{Spec}(\Omega) \xrightarrow{t} T);$$

hence $t \in T(\Omega)$ by (2.9) lifts to a homomorphism $\psi_t(\infty) : Y_0 \times \Omega \to \mathcal{X}_s$ such that $\psi_t(\infty)[p^n] = \psi_t(n)$; by the choice of n we see that the kernel of this homomorphism has rank p^i, hence it is an isogeny. This shows that $s \in U$; this proves that $T \to H_n \to S$ factors through $U \subset S$. This constructs the diagram as promised (but note that the U and S used in the diagram are finite covers of the U and S in the theorem).

At first we check that all properties except (4) and (7) follow from the construction. The fact that $T \to U$ has finite geometric fibers is (2.6), which shows (1). We already checked (2). Note that for any $s \in U$ we have $\varphi'_s : \mathcal{X}_s \to \mathcal{Y}_s$; this shows (3). In (2.1) we have seen that $H_n \to S$ is affine, hence (4) $\Rightarrow$ (5) & (6). It remains to check (4) and (7).

In order to prove (4) we are going to use that a constructible subset is *closed if it stable under specialization*, see [2], Exc. II.3.19b.
In order to prove (7) that $T \to U$ is proper we are going to check the *valuative criterion of properness*, see [2], Theorem I.4.7.

(3.2) Proposition. *Let A be a noetherian, complete, local, normal domain of characteristic p; let K be its field of fractions and $A/\mathfrak{m}_A = k = \overline{k}$ its residue field; let Ω be a algebraic closure of K; write $Y_0 = \mathcal{Y} \otimes_A k$. Let $\mathcal{Y} \to \mathrm{Spec}(A)$ be a* completely slope divisible *p-divisible group over A. Let $\rho_\Omega \in \mathrm{End}(\mathcal{Y} \otimes \Omega)$ be an isogeny. Then there exists a finite extension $K \subset L$, let B be the integral closure of A in L, and a finite, flat group scheme $G \subset \mathcal{Y} \otimes_A B$ over B, such that $(G \subset \mathcal{Y} \otimes_A B) \otimes \Omega = \mathrm{Ker}(\rho_\Omega)$.*

In fact, let N be an integer such that $\mathrm{Ker}(\rho_\Omega) \subset \mathcal{Y}[p^N] \otimes \Omega$. It suffices to take a cover $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, see (2.8), *such that $\mathcal{Y}[p^N] \cong Y_0[p^N] \otimes_k B$.*

Remarks. Note that we are not claiming that ρ_Ω comes from an endomorphism of $\mathcal{Y}/B$; see (4.1). We are going to use the proposition in case $A = k[[r]]$.

Proof. We write the slope filtration $0 \subset \mathcal{Y}_1 \subset \cdots \subset \mathcal{Y}_m = \mathcal{Y}$. We write $\mathcal{Y}^{(i)} := \mathcal{Y}_i/\mathcal{Y}_{i-1}$ for $0 < i \leq m$. These subquotients are isoclinic of mutually different slopes. Write $Q := \mathcal{Y} \otimes_A k$ and $Q^{(i)} := \mathcal{Y}^{(i)} \otimes_A k$ for their special fibers. Note that $Q = \oplus_i Q^{(i)}$. Note that the endomorphism ρ_Ω is in diagonal form: $\rho_\Omega = \sum_i \rho_\Omega^{(i)}$ with $\rho_\Omega^{(i)} \in \mathrm{End}(\mathcal{Y}^{(i)} \otimes_A \Omega)$.

By (2.7) we know that $\mathcal{Y}^{(i)} \cong Q^{(i)} \otimes_k A$. Hence we obtain an isomorphism

$$Q \otimes_k \Omega = \oplus\, Q^{(i)} \otimes_k \Omega \cong \oplus\, \mathcal{Y}^{(i)} \otimes_A \Omega = \mathcal{Y} \otimes_A \Omega =: \mathcal{Y}_\Omega.$$

Using the fact that k is algebraically closed we deduce:

$$\mathrm{End}(\sum_i Q^{((i)}) = \mathrm{End}(Q) \xrightarrow{\sim} \mathrm{End}(\mathcal{Y} \otimes_A \Omega).$$

We write ρ_k for the element corresponding with ρ_Ω, and $\rho_k = \oplus\, \rho_k^{(i)}$ for the decomposition into summands.

Note that $\mathcal{Y} \to A$ is geometrically fiberwise constant. Hence we can use (2.8) in order to construct a finite covering $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ corresponding with a finite extension $K \subset L$ such that the natural maps $\mathcal{Y}_B^{(i)}[p^N] \cong Q^{(i)}[p^N] \otimes_k B$ give an isomorphism $\mathcal{Y}_B[p^N] \cong Q[p^N] \otimes_k B$. Define

$$G := \mathrm{Ker}(\rho_k) \otimes_k B \subset Q[p^N] \otimes_k B \cong \mathcal{Y}_B[p^N] = \oplus\, \mathcal{Y}_B^{(i)}[p^N].$$

As $\mathrm{Ker}(\rho_k) \otimes_k \Omega = \mathrm{Ker}(\rho_\Omega)$ we see that $G \subset \mathcal{Y}_B$; this sub group scheme has the desired property, and the proposition is proved. $\square$

(3.3) Proof of (4) and (7). Let $A = k[[r]]$, and $K = k((r))$. Suppose given $g : \mathrm{Spec}(A) \to H_n$ inducing a point $g_K = \eta \in T(K) \subset H_n(K)$; in order to check (4) we are going to prove that in fact $\mathrm{Im}(g) \subset H_n$ is contained in T:

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\eta} & T \\ \downarrow & \stackrel{?}{\nearrow} & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{g} & H_n. \end{array}$$

Suppose given $h : \mathrm{Spec}(A) \to S$ lifting to a point $\eta \in T(K) \subset H_n(K)$; in order to check (7) we are going to prove that h can be lifted to $h' : \mathrm{Spec}(A) \to T$ such that $h'_K = \eta$:

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\eta} & T \subset H_n \\ \downarrow & \stackrel{?}{\nearrow} & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{h} & U \subset S \end{array}$$

We treat both cases at the same time with $h = (\mathrm{Spec}(A) \to H_n \to S)$. Let Ω be an algebraic closure of K. As $\eta \in T(K)$ there exists an isogeny $\psi_\Omega : \mathcal{Y}_0 \otimes \Omega \to \mathcal{X}_s \otimes \Omega$ inducing $\eta \leftrightarrow \psi_{\eta,n} : \mathcal{X}_s[p^n] \to Y_0[p^n]$; hence ψ_Ω is an isogeny; we are going to show that $\psi_{\eta,n} \leftrightarrow \eta : \mathrm{Spec}(K) \to T$ extends to a morphism $\mathrm{Spec}(A) \to T$.

Note that we have a our disposal $\varphi : \mathcal{Y} \to \mathcal{Z}$, and also $\varphi' : \mathcal{Z} \to \mathcal{Y}$, isogenies such that $\varphi{\cdot}\varphi' = p^a$. We define

$$(\rho_\Omega : \mathcal{Y}_\Omega \to \mathcal{Y}_\Omega) := \left(\mathcal{Y}_\Omega \quad \stackrel{\varphi\otimes_L\Omega}{\longrightarrow} \quad \mathcal{Z}_\Omega \quad \stackrel{\psi_\Omega}{\longrightarrow} \quad \mathcal{Y}_\Omega\right) \quad \text{with} \quad \varphi : \mathcal{Y} \to \mathcal{Z}.$$

We arrive at a situation where we can apply (3.2). In our construction we already took U in such a way that $\mathcal{Y}[p^N] \cong Y_0[p^N] \otimes_k U$. This shows that $\mathrm{Ker}(\rho_k)$ can be extended to A as in that proposition:

there exists $\mathrm{Ker}(\rho_k) \otimes_k A \cong G \subset \mathcal{Y}_A$ *such that* $(G \subset \mathcal{Y}_A) \otimes \Omega = \mathrm{Ker}(\rho_\Omega)$.

Note that $\mathrm{Ker}(\varphi_A) \subset \mathcal{Y}_A$ and $G_A \subset \mathcal{Y}_A$ with the property that $\mathrm{Ker}(\varphi_A \otimes \Omega) \subset G_\Omega$; hence $\mathrm{Ker}(\varphi_A) \subset G_A \subset \mathcal{Y}_A$. Note that $(\mathcal{Y}/G)[p^N] = \mathcal{Y}[p^N]$ and $(\mathcal{Y}/\mathrm{Ker}(\varphi))[p^N] = \mathcal{Z}[p^N]$. This constructs

$$\psi_{A,N} : \quad (\mathcal{Y}_A/\mathrm{Ker}(\varphi_A))[p^N] = \mathcal{Z}_A[p^N] \longrightarrow \mathcal{Y}_A[p^N],$$

a homomorphism which extends and lifts $\psi_{\eta,n}$; this constructs $\mathrm{Spec}(A) \to I_{N,q}$; this shows that g extends to, respectively h lifts to $\mathrm{Spec}(A) \to I_{N,q} \to T \subset H_n$. Hence $T \subset H_n$ is closed under specialization, which proves (4); the valuative criterion for $T \to U$ is proved, hence (7) follows. This finishes the construction and the proof of the properties stated in (3.1).

$\square$(4),(7)

We see that the proof of Vasiu's Theorem (1.3) is concluded. $\square$(1.3)

(3.4) Lemma. *Let S be an excellent scheme, and let $U \subset S$ be an open, dense immersion which is an affine morphism. Then either $U = S$ or $S - U$ is of pure codimension one in S.*
Proof. It suffices to show this in case S is integral and normal. Moreover it suffices to show this in case we take an affine neighborhood of any point in the boundary $S - U$: we can suppose S is affine (and hence also U is affine), integral and normal. Suppose that every irreducible component of $S - U$ is of codimension at least two in S. Any regular function on $S - U$ extends to S: a regular function on $S - U$ is a rational function on S, and its poles have support in codimension one. Hence the injective homomorphism $\Gamma(S, \mathcal{O}_S) \hookrightarrow \Gamma(U, \mathcal{O}_S)$ is an isomorphism; for U and S affine this implies $U = S$. $\square$

We see that the proof of Theorem (1.1) follows: the properties $U = S$ or $\mathrm{codim}((S - U) \hookrightarrow S) = 1$ follow if known these for a finite cover. Hence it suffices to prove the theorem for a finite cover, as we did in the beginning of this section; we have shown that under the conditions of the theorem, the morphism $U \to S$ is affine, and (3.4) finishes the proof. $\square$(1.1)

4 Some questions and remarks.

(4.1) Example. Let Y be a p-divisible group over a field K, and let $\rho_\Omega \in \mathrm{End}(Y \otimes \Omega)$ for some extension field Ω. *In general there is no finite extension $L \supset K$ over which ρ_Ω is defined.* - Indeed, construct $\mathcal{Y}$, with a Newton polygon which has two different slopes, and a slope filtration $0 \subset \mathcal{Y}_1 \subset \mathcal{Y}$ such that the the extension $\mathcal{Y}_1 \to \mathcal{Y} \to \mathcal{Y}/\mathcal{Y}_1 =: \mathcal{Y}''$ does not split over any finite extension of K; e.g. we can take an ordinary elliptic curve E_0 over k, its universal deformation E over $A = k[[r]]$, with $K = k((r))$, and $\mathcal{Y} = E[p^\infty]$, and $Y = E[p^\infty] \otimes K$. Let Ω be a perfect closure of K; then $\mathcal{Y} \otimes \Omega \cong (\mathcal{Y}_1 \oplus \mathcal{Y}'') \otimes_R \Omega$. The endomorphism $\pi_\Omega := (1,0) \in \mathrm{End}(\mathcal{Y}_1 \otimes_R \Omega) \oplus \mathrm{End}(\mathcal{Y}'' \otimes_R \Omega)$ has the desidered property; hence $\rho_\Omega = 1 + p{\cdot}\pi_\Omega$ is an isogeny having the desired property.

(4.2) Example. In general the main result of [3] does generalize to arbitrary extensions. Let $A = k[[r]]$, with $K = k((r))$; let $K \subset \Gamma$ be a field extension, and let B be the integral closure in Γ. Let $\mathcal{X} \to \mathrm{Spec}(A)$ be a p-divisible group. In general $\rho \in \mathrm{End}(\mathcal{X} \otimes \Gamma)$ does not extend to an endomorphism of $\mathcal{X} \otimes_A B$. Just take for X_0 a supersingular p-divisible group of dimension one, for $\mathcal{X} \to \mathrm{Spec}(A)$ a non-trivial deformation (hence the generic fiber is ordinary), and for Γ the perfection of $K = k((r))$. As in the example above we see there exist an endomorphisms of $\mathcal{X}_\Gamma$ not extending to B.

Remark. The question studied in this example is better if $\mathcal{X} \to \mathrm{Spec}(R)$ has constant Newton polygon, and that is what we used above.

(4.3) Remark. In the proof of (1.1) we use (2.7) and (3.2) for $A = k[[r]]$; we see that it suffices to use [5], Th. 2.7.1 on page 158; in particular, the proof we give here of (1.3) and of (1.1) does not rely on any result of [4].

(4.4) Question. Consider $I_q \to S$ as in (2.1). *Is this an affine morphism? Is this an affine morphism in case $G \to S$ and $H \to S$ are truncated Barsotti-Tate groups schemes?*

(4.5) Question: *Purity for BT_1-group schemes ?* I expect indeed this does hold, and by that I mean the following : *Let S be an integral base scheme; let $G \to S$ be a finite locally free group scheme; let $U \subset S$ be the maximal, dense set on which the isomorphism type of geometric fibers is constant; suppose that $G_{|U} \to U$ is BT_1.* Then I expect: *Either $U = S$ or* $\mathrm{codim}((S - U) \hookrightarrow S) = 1$ (?). (A proof has to be written out.)

(4.6) Example: *No purity for finite group schemes.* We construct a base scheme S, in fact $S = \mathbb{A}^2$, a finite, locally free group scheme $\mathcal{N} \to S$, a closed subset $C \subset S$ of codimension at least two, in fact $C = \{P\}$, such that for any two geometric points $s, t \in S - C$ there exists an isomorphism $\mathcal{N}_s \cong \mathcal{N}_t$, and for any geometric point $s \in S - C$ and any geometric point $c \in C$ we have $\mathcal{N}_s \not\cong \mathcal{N}_c$. Here is the construction of such an example.

Let $R = k[X,Y]$ and $S = \mathrm{Spec}(R)$. Let $M = R^2 = R{\cdot}e \oplus R{\cdot}h$, a free R-module of rank 2. Define a p-linear R-map $\varphi : M \to M$ by:

$$\varphi(e) = XY^p{\cdot}e + Y^{p+1}{\cdot}h, \quad \varphi(h) = -X^{p+1}{\cdot}e - X^pY{\cdot}h.$$

We see that $\varphi^2 = 0$. Further note that for any homomorphism $s : R \to L$, where L is a field, we have $\varphi_s = \varphi \otimes L = 0$ if and only if s factors through $R \to k$, $X \mapsto 0, Y \mapsto 0$. Using [1], Th. II.7.3.5 on page 277 we see that (M, φ) defines a finite group scheme $G \to S$, with (M, φ) as p-Lie algebra, annihilated by F, and its V-action "given by φ". We see that $V^2 = 0$ and for every $s \neq 0$ we have $V_s \neq 0$. This shows that for every geometric point $s \in (S - \{0\})(\Omega)$ we have $G_s \cong W_2[F] \otimes \Omega$; moreover for $s = 0$ we have $G_0 \cong (\alpha_p)^2$. This ends the construction of the example.

References

[1] M. Demazure & P. Gabriel – *Groupes algébriques.* Masson & Cie, North - Holland Publ. Cy 1970.

[2] R. Hartshorne – *Algebraic geometry.* Grad. Texts Math. 52, Springer - Verlag 1977.

[3] A. J. de Jong - *Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic.* Invent. Math. **134** (1998), pp 301 - 333.

[4] A. J. de Jong & F. Oort - *Purity of the stratification by Newton Polygons.* Journ. A.M.S. **13** (2000), 209 - 241.
See: http://www.ams.org/jams/2000-13-01/

[5] N. M. Katz - *Slope filtrations of F-crystals.* Journ. Géom. Alg. Rennes, Vol. I, Astérisque 63, Soc.Math. France, 1979, 113 - 164.

[6] Yu. I. Manin - *The theory of commutative formal groups over fields of finite characteristic.* Usp. Math. **18** (1963), 3 - 90; Russ. Math. Surveys **18** (1963), 1-80.

[7] F. Oort – *Foliations in moduli spaces of abelian varieties.* [To appear]
See: http://www.math.uu.nl/people/oort/

[8] F. Oort & Th. Zink – *Families of p-divisible groups with constant Newton polygon.* Documenta Mathematica **7** (2002), 183 - 201.
See: http://www.mathematik.uni-bielefeld.de/documenta

[9] M. Rapoport – *On the Newton stratification.* Sém Bourbaki, **54** (2001-2002), no. 903, III-2002.

[10] A. Vasiu – *The crystalline boundedness principle.* arXiv: math.NT/02051199v1 17-V-2002.

[11] Th. Zink – *On the slope filtration.* Duke Math. J. Vol. **109** (2001), 79 - 95.

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