Special subvarieties in the Torelli locus

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Introduction

The topic in this talk started with:

(0.1) A conjecture by Coleman. *Given $g \geq 4$ there are only finitely many non-singular projective curves C over $\mathbb{C}$, up to isomorphism, of genus g and the Jacobian J_C is a CM abelian variety.* See [14], Conjecture 6.

This conjecture does not hold for $4 \leq g \leq 7$. The first examples were given by de Jong and Noot, see [29]. See Section 5 below. A modified version of this conjecture does hold, as Chai and Oort showed, see [11], see (4.2). This conjecture has stimulated research, which can be phrased as a study of

$$\mathrm{CM}(\mathcal{A}_{g,1}) \cap \mathcal{T}_g^0 = \mathrm{CM}(\mathcal{T}_g^0),$$

where $\mathrm{CM}(\mathcal{A}_{g,1})$ is the set of CM points on this moduli space, and where $\mathcal{T}_g^0$ is the open Torelli locus. We explain that, using the AO conjecture, see below, the Coleman conjecture can be translated into a question of the existence of *special subvarieties* in the Torelli locus. We present examples and theorems. However the main message is: *there are still more open problems than answers.* The main theme of this note is the Expectation (3.1). Or more fundamentally: *compare natural metrics and coordinate systems on $\mathcal{A}_g$ on the one hand, and on $\mathcal{M}_g$ on the other hand.*

Material for this talk can be found in: [46] and [60]. Please consult those papers for a survey of this topic, and for all details not discussed below. We use results in the following papers: [29], [70], [20], [67], [45], and many more. This note is just a survey. We make no attempt to give explanations; please consult literature cited.

As base field we use $\mathbb{C}$; however, mainly everything can be considered also over $\mathbb{Q}^a = \overline{\mathbb{Q}}$. Rationality questions will play no role.

1 The Torelli locus

The moduli space of (geometrically irreducible, complete, non-singular) curves of genus g will be denoted by $\mathcal{M}_g$. For a curve C we write $J_c = \underline{\mathrm{Pic}}^0_C \cong \underline{\mathrm{Alb}}(C)$ for its Jacobian and

$\mathrm{Jac}(C) = (J_C, \Theta_C)$ for its principally polarized Jacobian. The morphism

$$j : \mathcal{M}_g \longrightarrow \mathcal{A}_{g,1}, \quad [C] \mapsto [\mathrm{Jac}(C)]$$

is called the Torelli morphism.

The Torelli theorem: on geometric points this map is injective.

And in fact, it is an immersion, see [62] (note that we work in characteristic zero). The image

$$\mathcal{T}_g^\circ := j(\mathcal{M}_g)$$

is called the *open Torelli locus.* The inclusion $\mathcal{T}_g^\circ \subset \mathcal{A}_{g,1}$ is a locally closed subvariety. The Zariski closure

$$\mathcal{T}_g := \left(\mathcal{T}_g^\circ\right)^{\mathrm{Zar}} \subset \mathcal{A}_{g,1}$$

is called the *closed Torelli locus.* Points in $\mathcal{T}_g$ correspond with Jacobians of compact type curves.

The moduli spaces $\mathcal{M}_g$ and $\mathcal{A}_{g,1}$ are well studied, and we know many properties. However it is as if they live in different worlds. Canonical structures on one of them seem to have little relations with the same on the other. Often it is hard to tell whether a principally polarized abelian variety is a Jacobian. And, even more it is difficult to translate properties on one of the objects in aspects of the other. Here is a typical example: if we know that (A, λ) is the Jacobian of a curve C, geometrically unique by Torelli's theorem, how do we describe properties of (A, λ) ensuring that C is hyperelliptic? How can we detect from properties of C that J_C admits smCM? Along these lines, analogously, many difficult questions can be posed. (In positive characteristic there are even more intriguing questions of this kind, which cry for study and answers.) – We like to understand this dichotomy.

2 Special subvarieties and the André-Oort conjecture

We refer to the standard literature for properties of Shimura varieties. The moduli space $\mathcal{A}_{g,1}$ is an example of a Shimura variety. A Shimura subvariety of a Shimura variety is called a *special subvariety.* Several characterizations can be given. E.g. see [11], [46].

An abelian variety A of dimension g over a field K is said to *admit sufficiently many complex multiplications* (smCM), or is said to be a CM abelian variety if $\mathrm{End}(A)^0 = \mathrm{End}(A) \otimes \mathbb{Q}$ contains a commutative semi-simple algebra of rank $2g$ over $\mathbb{Q}$. In this case $[(A, \mu)] \in \mathcal{A}_g$ is called a CM point. In $\mathcal{A}_g$ every zero dimensional special subvariety is a CM point, and conversely every CM point is a special subvariety of dimension zero. Note that in a Shimura subvariety the set of special points is dense in the classical topology. Hence the set of special points is dense in the Zariski topology. The converse is the AO conjecture:

(2.1) The André-Oort Conjecture. *Let S be a Shimura variety, let Γ be a set of special points.* In the case $S = \mathcal{A}_g$ this means Γ is a set of CM points. *The Zariski closure of Γ is a finite union of special subvarieties:*

$$\Gamma^{\mathrm{Zar}} \stackrel{?}{=} \bigcup_i^{<\infty} S_i,$$

where $S_i \subset S$ are special subvarieties.

Many special case have been proved. A proof of the general case is announced by Klingler-Yafaev, and Ullmo-Yafaev, conditional under the GRH.

3 Special subvarieties in the Torelli locus

(3.1) Expectation. *For large g* (in any case $g \geq 8$), *there does not exist a special subvariety $Z \subset \mathcal{A}_{g,1}$ with* $\dim(Z) \geq 1$ *such that $Z \subseteq \mathcal{T}_g$ and $Z \cap \mathcal{T}_g^\circ$ is nonempty.* See [57] Section 7, [58], Section 5.

Note that in the boundary $\mathcal{T}_g \setminus \mathcal{T}_g^\circ$ there are many special subvarieties.

Many special cases have been settled, and these confirm this expectation. It seems that in order to prove, or the contradict this expectation we need new techniques.

We expect this is a hard question. Even better, one would like to have a classification of all such special subvarieties in $\mathcal{T}_g$. For $g \leq 3$ this is doable, Because in those cases $\mathcal{T}_g = \mathcal{A}_{g,1}$ and in those cases every Shimura subvariety is of PEL type. However, already for $g = 4$ we do not know such a complete classification. See (7.1).

Here is one reason why we are interested in this expectation:

(3.2) (2.1) + (3.1) *imply the Coleman conjecture for that value of g.*

4 The modified Coleman conjecture

Here we follow [11].

(4.1) Definition. A CM field L of degree $[L : \mathbb{Q}] = g$ is called a Weyl CM field if the normal closure $L^\sim$ has degree $[L^\sim : \mathbb{Q}] = 2^g \cdot (g!)$ (the maximal possible value).

It can be shown that, in a suitable sense, most CM abelian fields are of this type.

Please consult the paper [11] why this definition is made, why it is useful. We say a point is a Weyl CM point if the related abelian variety of dimension g has smCM by a Weyl CM field. We write $\mathrm{WCM}(\mathcal{A}_{g,1})$ for the set of Weyl CM points on this moduli space. We show that a weaker version of the Coleman conjecture does hold:

(4.2) Theorem (modified Coleman conjecture; Chai & Oort). *Let $g \in \mathbb{Z}_{\geq 4}$. Assume* AO. *Then*

$$\#\left(\mathcal{T}_g \cap \mathrm{WCM}(\mathcal{A}_{g,1})\right) < \infty.$$

See [11], 3.7.

(4.3) Remark / Exercise. *Let C be a curve, nonsingular, complete, with $g(C) > 1$. Assume that either C is non-hyperelliptic and $\#(\mathrm{Aut}(C)) > 1$ or C hyperelliptic and $\#(\mathrm{Aut}(C)) > 2$. Then J_C is not a Weyl CM abelian variety.*

Hence curves ($g > 1$) with many automorphisms do not give examples of Weyl CM Jacobians. This makes it hard to construct examples of such.

5 Cyclic covers of the projective line

We follow [29], [70], [67], [45], [46], [60]. In this section we discuss some counterexamples to the Coleman conjecture. These will use $4 \leq g \leq 7$, and they are all obtained by families of cyclic covers of the projective line.

In this section we discuss the examples that are known to us of special subvarieties $S \subset \mathcal{T}_g$ of positive dimension, with $g \geq 4$ and $S \cap \mathcal{T}_g^\circ \neq \emptyset$. The examples all come from families of cyclic covers of $\mathbb{P}^1$.

(5.1) To obtain such examples, fix an integer $m \geq 2$, an integer $N \geq 4$, and monodromy elements $a_1, \ldots, a_N$ in $\mathbb{Z}/m$; then we consider cyclic covers of $\mathbb{P}^1$ with group μ_m, branch points $t_1, \ldots, t_N$ in $\mathbb{P}^1$ and local monodromy $\exp(2\pi i a_j/m) \in \mu_m$ about t_j. If the branch points are all in $\mathbb{A}^1$, this cover is given by the affine equation

$$y^m = (x - t_1)^{a_1}(x - t_2)^{a_2} \cdots (x - t_N)^{a_N}, \qquad (1)$$

with $\zeta \in \mu_m$ acting by $(x, y) \mapsto (x, \zeta \cdot y)$. Varying the branch points t_i gives us a family of curves, and the moduli map gives a subvariety which we denote by $Z(m, N, a)$. It turns out that for certain choices of the data involved, the corresponding family of Jacobians traces out a special subvariety in $\mathcal{A}_{g,1}$.

We call two triples (m, N, a) and (m', N', a') equivalent if $m = m'$ and $N = N'$ and if the classes of a and a' in $(\mathbb{Z}/m)^N$ are in the same orbit under $(\mathbb{Z}/m)^* \times \mathfrak{S}_N$. Here $(\mathbb{Z}/m)^*$ acts diagonally on $(\mathbb{Z}/m)^N$ by multiplication, and the symmetric group $\mathfrak{S}_N$ acts by permutation of the indices.

Such a family is called *special* if the moduli map produces a subvariety $Z(m, N, a)$, of dimension $N - 3$, which is special. See [29], and [46] for examples, explanation and references.

Table 1: Examples of special subvarieties in the Torelli locus

	genus	m	N	a
(1)	1	2	4	(1,1,1,1)
(2)	2	2	6	(1,1,1,1,1,1)
(3)	2	3	4	(1,1,2,2)
(4)	2	4	4	(1,2,2,3)
(5)	2	6	4	(2,3,3,4)
(6)	3	3	5	(1,1,1,1,2)
(7)	3	4	4	(1,1,1,1)
(8)	3	4	5	(1,1,2,2,2)
(9)	3	6	4	(1,3,4,4)
(10)	4	3	6	(1,1,1,1,1,1)
(11)	4	5	4	(1,3,3,3)
(12)	4	6	4	(1,1,1,3)
(13)	4	6	4	(1,1,2,2)
(14)	4	6	5	(2,2,2,3,3)
(15)	5	8	4	(2,4,5,5)
(16)	6	5	5	(2,2,2,2,2)
(17)	6	7	4	(2,4,4,4)
(18)	6	10	4	(3,5,6,6)
(19)	7	9	4	(3,5,5,5)
(20)	7	12	4	(4,6,7,7)

(5.2) Theorem (Rohde, Moonen). *These families $Z(m, N, a) \subset \mathcal{T}_g$ give special subsets of the Torelli locus, and conversely $Z(m, N, a)$ is special if this triple is in the list above.*

(5.3) If $\dim(Z(m,N,a)) < a-3$ it still might be true a priori that $Z(m,N,a)$ is a special subvariety. See [46], 5.16 for explanation, and see [45] for the proof, quite involved, that this however is not the case.

It may happen that $Z(m,N,a)$ is not special, but that it contains a special subvariety, see (7.17).

(5.4) Let us mention the paper [38], in which Möller investigates algebraic curves in $\mathcal{M}_g$ over $\mathbb{C}$ that are totally geodesic with respect to the Teichmüller metric (these are called Teichmüller curves), such that the image of this curve in $\mathcal{A}_g$ is a special subvariety. It is shown in [38] that for $g=2$ and $g \geq 6$ there are no such curves. For $g=3$ and $g=4$ there is precisely one example, corresponding to Examples (7) and (12) in Table 1.

6 PEL Shimura subvarieties with totally real multiplications

(6.1) **Theorem** (de Jong & Zhang). *Let $g \geq 4$, and let $S \subset \mathcal{A}_{g,1}$ be a Hecke translate of a Hilbert modular subvariety of $\mathcal{A}_{g,1}$, i.e., S is a special subvariety of PEL type obtained from a totally real field D of degree g. Then S is not contained in the Torelli locus $\mathcal{T}_g$.* See [30], Corollary 1.2.

(6.2) **Theorem** (Moonen). *Let $g \geq 4$. Consider a Shimura subvariety $S \subset \mathcal{A}_{g,1}$ of PEL type, arising from PEL data arising from a* totally real *Albert algebra D. Then S is not contained in the Torelli locus:*

$$S \not\subset \mathcal{T}_g.$$

See [46], 4.15.

7 Questions and open problems

(7.1) **Question.** *For every g give a complete classification of all special subvarieties contained in $\mathcal{T}_g$.* More precisely:

Consider, for $g \in \mathbb{Z}_{>0}$ the set

$$\mathcal{ST}(g) := \{\text{special subvarieties } Z \subset \mathcal{T}_g \text{ with } \dim(Z) > 0 \text{ and } Z \cap \mathcal{T}_g^\circ \neq \emptyset\}$$

of special subvarieties of positive dimension, contained in the Torelli locus, and not fully contained in the boundary of $\mathcal{T}_g$. The expectation is that for $g \gg 0$ we have $\mathcal{ST}(g) = \emptyset$.

We would like to classify all pairs (g, Z) with $Z \in \mathcal{ST}(g)$. For $g=2$ and $g=3$ we have $\mathcal{T}_g = \mathcal{A}_g$ and in this case every special subvariety of $\mathcal{A}_g$ is of PEL type; hence in this case we can classify all pairs (g, Z), up to Hecke translation, by listing all possible endomorphism algebras. We know that $\mathcal{ST}(g) \neq \emptyset$ for all $g < 8$. However, already for $g=4$ we do not have a good description of $\mathcal{ST}(4)$. It seems very difficult to describe $\mathcal{ST}(g)$ for arbitrary g.

(7.2) **Definition.** Let k be an algebraically closed field. Let C be a complete, irreducible, regular curve over k. Write $G := \mathrm{Aut}(C)$. We say that C has *many automorphisms* if the local deformation functor of (C, G) on schemes over k is representable by a zero-dimensional scheme.

Some examples of curves with many automorphisms give examples of CM Jacobians; e.g. see [60]

(7.3) Question. *Do we know an example of a curve with many automorphisms such that its Jacobian is not a CM abelian variety?*

We can prove the existence of CM Jacobians either by the methods as in Section 5 or using curves with many automorphisms. See [60], Section 5, especially 5.14, for a discussion.

(7.4) Question. *Do we know an example of a curve C with trivial automorphism group, and with* CM *Jacobian?*

(7.5) Question. *De we know any counterexample to Coleman's conjecture not given by or associated with examples in Section* 5 *?*

(7.6) Question. *Does there exist $g > 3$ and a special subvariety $Z \subset \mathcal{A}_g$ contained in the Torelli locus $\mathcal{T}_g$ such that the geometric generic fiber over Z gives an abelian variety with endomorphism ring equal to $\mathbb{Z}$?* We do not know a single example.

(7.7) Question. *How can we find for $g > 3$ CM curves which are "isolated"* (in the sense not contained in the closure of an infinite set of points defined by CM Jacobians) *and not with many automorphisms? It seems plausible that such curves exist.* In other words: *Do we know the existence, or the construction, of a* CM *Jacobian which is not on a positive dimensional special subvariety in the Torelli locus, and where the corresponding curve does not have many automorphisms?*

(7.8) Question. *Choose some triple (m, N, a) as in Section* 5 *which is not in Table 1. Then we know that $Z(m, N, a)$ is not a special subvariety. Can we find a situation where $Z(m, N, a)$ contains a positive dimensional special subvariety?*

(7.9) Non-PEL Shimura curves for $g = 4$. In [51], § 4, Mumford constructs one-dimensional special subvarieties $Z \subset \mathcal{A}_4$ that are not of PEL type. The abelian variety corresponding to the geometric generic fiber of Z has endomorphism algebra $\mathbb{Z}$. Such a curve Z is complete. Note that $\mathcal{T}_4 \subset \mathcal{A}_4$ is a closed subvariety of codimension one, which by a result of Igusa [28] is ample as a divisor. Hence we see that $Z \cap \mathcal{T}_4 \neq \emptyset$.

(7.10) Question. *Is there a "Mumford curve" $Z \subset \mathcal{A}_{4,1}$ that is contained in $\mathcal{T}_4$?*

(7.11) Question. *For which $g \geq 2$ does there exist a positive dimensional subvariety $Z \subset \mathcal{T}_g$ with $Z \cap \mathcal{T}_g^\circ \neq \emptyset$, such that the abelian variety corresponding with the geometric generic point of Z is isogenous to a product of elliptic curves?*

This question was stimulated by results in [33], which say that under more restrictive conditions such a family does not exist for large g.

(7.12) Question. In Section 5 we have seen examples of special subvarieties $Z \subset \mathcal{T}_g$ with $Z \cap \mathcal{T}_g^\circ \neq \emptyset$ arising from families of cyclic covers of $\mathbb{P}^1$. *Can one obtain further such examples by taking non-cyclic covers, or from a family of covers of another base curve?*

To make this more precise, consider a (complete, nonsingular) curve B over $\mathbb{C}$ of genus h, and let G be a finite group. Define a group $\Pi = \Pi(h, N)$ by

$$\Pi := \langle \alpha_1, \ldots, \alpha_h, \beta_1, \ldots, \beta_h, \gamma_1, \ldots, \gamma_N \mid [\alpha_1, \beta_1] \cdots [\alpha_h, \beta_h] \cdot \gamma_1 \cdots \gamma_N = 1 \rangle .$$

Given $t_1, \ldots, t_N$ in B, fix a presentation

$$\pi_1(B \setminus \{t_1, \ldots, t_N\}) \xrightarrow{\sim} \Pi,$$

and fix a surjective homomorphism $\psi\colon \Pi \twoheadrightarrow G$ such that $\psi(\gamma_i) \neq 1$ for all $i = 1, \ldots, N$. Correspondingly, we have a Galois cover $C_t \to B$ with group G, branch points $t_1, \ldots, t_N$ in B, and with local monodromy about t_i given by the element $\psi(\gamma_i)$. Varying the branch points, we get a family of curves $C \to T$, for some open $T \subset B^N$. The corresponding family of Jacobians gives a moduli map $T \to \mathcal{A}_{g,1}$; denote the image by Z°, with Zariski closure $Z \subset \mathcal{T}_g \subset \mathcal{A}_{g,1}$. Having set the scene in this way we can ask for which choices of the data involved Z is a special subvariety of positive dimension.

(a) *Are there examples with non-cyclic Galois group such that $Z \subset \mathcal{A}_{g,1}$ is a special subvariety of positive dimension?*

(b) *Are there examples where B is not a rational curve, and such that $Z \subset \mathcal{A}_{g,1}$ is a special subvariety of positive dimension?* Note that if Z is special, the Jacobian of B is a CM abelian variety.

(7.13) Here is a more general setup. Let B be a curve which has a CM Jacobian (or $B \cong \mathbb{P}^1$). Let G be a finite group. Let $N \in \mathbb{Z}_{\geq 0}$ and let $\Gamma \subset B^N$ be a closed subset not containing any of the big diagonals. For some

$$P = \{P_1, \cdots, P_N\} \in \Gamma$$

we consider a representation

$$\rho : \pi_1(B \setminus \{P_1, \cdots, P_N\}) \twoheadrightarrow G.$$

We study a family of G-covers of B given by these data (B, N, Γ, ρ) and let g be the genus of such a cover. This family of curves, for all $P \in \Gamma$ defines a closed set of Jacobians $Y(B, N, \Gamma, \rho) \subset \mathcal{T}_g \subset \mathcal{A}_{g,1}$. Can we choose these date such that $Y(B, N, \Gamma, \rho)$ is special ? Can we give a complete classification of such examples?

Table 2: Examples of special subvarieties in the Torelli locus (continued)

	genus	group	N	a
(21)	1	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$	4	$((1,0),(1,0),(0,1),(0,1))$
(22)	3	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$	4	$((1,0),(1,1),(0,1),(0,2))$
(23)	3	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$	4	$((1,0),(1,2),(0,1),(0,1))$
(24)	4	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$	4	$((1,0),(1,1),(0,2),(0,3))$
(25)	4	$(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$	4	$((1,0),(1,0),(1,2),(0,1))$
(26)	2	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$	5	$((1,0),(1,0),(1,0),(1,1),(0,1))$
(27)	3	$(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$	6	$((1,0),(1,0),(1,1),(1,1),(0,1),(0,1))$

(7.14) Example. We take (24) and consider a subquotient. Let $B = \mathbb{P}^1$, and $N = 6$. Consider $H = (\mathbb{Z}/2) \times (\mathbb{Z}/2) \subset \mathrm{Aut}(B)$; note that $(\mathbb{Z}/p) \times (\mathbb{Z}/p) \subset \mathrm{Aut}(\mathbb{P}^1)$ is not possible for a prime number $p > 2$ over a field of characteristic non-p; however

$$(x : y) \mapsto (-x : y) \quad \text{and} \quad (x : y) \mapsto (y : x)$$

give the desired example for $p = 2$ in characteristic non-2 (and indeed, it is unique up to isomorphism). Let $\{P_1, P_2\}$ and $\{P_3, P_4, P_5, P_6\}$ be two orbits for this action; the first two points are fixed, and the set of remaining four points moves in a one-dimensional family. Consider the set $\Gamma \subset B^6$ of such sextuples. The family of cyclic covers of B with $G = \mathbb{Z}/3$ branched over sets of points in Γ is a special family of dimension one of genus-4 curves.

(7.15) Example. Let B be the elliptic curve with automorphism group $\mathbb{Z}/3$. This action has 3 fixed points $\{P_1, P_2, P_3\}$ (and these points stay fixed). We choose a variable point P_4 and its $\mathbb{Z}/3$-orbit $\{P_4, P_5, P_6\}$. Total ramification in a $G = \mathbb{Z}/3$ covering in these 6 points gives a one-dimensional family of curves of genus 7. WE recognize this is a sub-quotient of the family (19).

(7.16) Example / Exercise. Show (18) gives rise to a one-dimensional sub-family of (16).

(7.17) Problem. It might very well be that for some $Z(m, N, a)$ considered above, but not appearing in Table 1, hence $Z(m, N, a)$ not special, there does exist $\Gamma \subset B^N$, with $B = \mathbb{P}^1$ such that $Y(B, N, \Gamma, \rho)$ as in (7.13) is special; see (7.8). It seems possible / plausible that such examples exist.

We have some partial answers to these questions, some examples. However, we do not know a systematic treatment or a complete answer to the question formulated in (7.13).

For more questions and details see [46], [60].

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