Monodromy and irreducibility of leaves

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§1. Introduction

Let p be a prime number, fixed throughout this article. Let n, d be positive integers with $\gcd(n, dp) = 1$. Denote by $\mathcal{A}_{g,d,n}$ the moduli space over $\mathbb{F}_p$ of g-dimensional abelian varieties with a polarization of degree d^2 and with a symplectic level-n structure in characteristic p. The moduli space $\mathcal{A}_{g,d,n}$ has a stratification

$$\{\, \mathcal{W}^0_\xi(\mathcal{A}_{g,d,n}) \mid \xi \,\}$$

indexed by symmetric Newton polygons ξ of height $2g$. Every stratum $\mathcal{W}^0_\xi(\mathcal{A}_{g,d,n})$ is a reduced locally closed subscheme of $\mathcal{A}_{g,1,n}$ whose geometric points correspond to g-dimensional principally polarized abelian varieties with Newton polygon equal to ξ; see 2.5. The smallest among the Newton polygon strata is the supersingular stratum $\mathcal{W}_\sigma(\mathcal{A}_{g,d,n}) = \mathcal{W}^0_\sigma(\mathcal{A}_{g,d,n})$, corresponding to supersingular polarized abelian varieties, i.e. those with Newton polygon isoclinic of slope 1/2; or equivalently, geometrically isogenous to a product of supersinglar elliptic curves.

In this paper we prove three global properties of non-supersingular Newton polygon strata and leaves in $\mathcal{A}_{g,d,n}$.

Theorem A. *For $\xi \neq \sigma$, the Newton stratum $W^0_\xi := \mathcal{W}^0_\xi(\mathcal{A}_{g,1,n})$ is* geometrically irreducible. See Theorem 3.1.

Theorem B. *For any geometric point $x \in \mathcal{A}_{g,d,n}(k)$ not contained in the supersingular stratum $W_\sigma(\mathcal{A}_{g,d,n})$, the leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,d,n}$ is* irreducible. See Theorem 4.1.

Theorem C. *For any geometric point $x \in \mathcal{A}_{g,d,n}(k)$ not contained in the supersingular stratum $W_\sigma(\mathcal{A}_{g,d,n})$, the p-adic monodromy for the leaf $\mathcal{C}(x)$ is* maximal. See Theorem 5.6.

The definition of the leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,d,n}$ passing through x is recalled in 2.2. In Theorem C the p-adic monodromy for $\mathcal{C}(x)$ refers to the action of the Galois group of the function field of $\mathcal{C}(x)$ on the p-divisible group of the abelian variety over the generic point of $\mathcal{C}(x)$; see 5.1 and 5.4 for the definition. The assertion in C is that the image of the monodromy action, regarded as a subgroup of $\mathrm{Aut}((A_x, \lambda_x)[p^\infty])$, is equal to $\mathrm{Aut}((A_x, \lambda_x)[p^\infty])$. Note that the number of irreducible components of W_σ is a suitable class number, which is large when p is large; every leaf in the supersingular locus is 0-dimensional whose cardinality is a class number. So the non-supersingularity assumptions in A, B, C are necessary.

Note that the conclusion of Theorem A is not correct for $\mathcal{W}^0_\xi(\mathcal{A}_{g,d})$ for arbitrary d. However in Theorem B and in Theorem C we need not restrict to the principally polarized case. Note that *all* p-adic invariants are constant on $\mathcal{C}(x)$, in the sense that the geometric isomorphism class of polarized p-divisible groups $(A_y, \lambda_y)[p^\infty]$ attached to points y on $\mathcal{C}(x)$ is fixed.

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Questions on global topological properties of subvarieties of moduli spaces in characteristic p are often challenging. Subvarieties such as $W^0_\xi \subset \mathcal{A}_{g,1,n}$ and $\mathcal{C}(x) \subset \mathcal{A}_{g,d,n}$ are stable under all prime-to-pd Hecke correspondences. One knows from [4] that such a subvariety $Z \subset \mathcal{A}_{g,d,n}$ is irreducible if Z is not contained in the supersingular locus and the prime-to-pd Hecke correspondences operate transitively on the set of geometrically irreducible components of Z; this is the first "reduction step" in the proof of A and B. Although this reduction step offers a general strategy for proving irreducibility statements of subvarieties of modular varieties of PEL-type, in each example that has been worked out some special properties are used.

The reader will find another reduction method that allows us to deduce B from the irreducibility statement in A. That method is based on the notion of *hypersymmetric points* introduced in [3] and [7]. These points in $\mathcal{A}_{g,d,n}$ correspond to abelian varieties whose endomorphism rings are "as large as allowed by their geometric p-divisible groups", or equivalently, "maximal under the given slope constraint"; for hypersymmetric abelian varieties, see 2.7 . The idea of the proof of B is to use the irreducibility in A to produce hypersymmetric points on every irreducible components of the leaf $C(x)$, and then connect these points by suitable prime-to-p Hecke correspondences with the help from group theory.

The existence of hypersymmetric points on the leaf $C(x)$ allows us to apply the third method in [6] and deduce the maximality statement in C. For technical reasons it is more convenient to deduce Theorem B and C by proving first the case when d is a power of p. To make the logical structure of the proof as clear as possible, we first prove 5.6 in the special case when the p-divisible group $A_x[p^\infty]$ is *completely slope divisible*; see 5.9. The general case is proved by the same method in the proof of 5.9, but with some additional technical components; see 5.13. In 5.14 we indicate how 5.6 can be deduced from its special case 5.9.

Methods in this article are developed for the Hecke orbit conjecture, first formulated in [29] in the case of Siegel modular varieties; see also [3] for a sketch of a proof. Implicit in the proof sketched in [3] is a proof of the irreducibility statement in Thm. B. That proof depends on the irreducibility of central leaves in Hilbert modular varieties proved by C.-F. Yu in [40], as well as the "Hilbert trick" and the trick of "splitting at supersingular points". That proof is more involved in terms of the complexity of the logical structure than the proof here.

§2. Prerequisites

(2.1) Some notation to be used below. Let p be a prime number, fixed in this article. All abelian varieties and p-divisible groups are defined over a field or a base scheme of characteristic p. We write $\mathbb{F}$ for an algebraic closure of $\mathbb{F}_p$. For an abelian variety A we write $X = A[p^\infty]$ for its p-divisible group.

We write k and Ω for algebraically closed field of characteristic p. All base fields and base schemes considered will be in characteristic p.

For a group scheme G over a field K we write $a(G)$ for the dimension of the L-vector space $\mathrm{Hom}(\alpha_{p,L}, G_L)$ where $L \supset K$ is a perfect field. Note that $\mathrm{End}(\alpha_{p,L}) = L$, hence the right-$L$-module $\mathrm{Hom}(\alpha_{p,L}, G_L)$ is a vector space over L.

For a scheme W over a field K we write $\Pi_0(W)$ for the set of geometrically irreducible components of W, i.e. we choose $K \subset k$, and we consider the set of irreducible components of $W \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$.

We write $\sigma = \sigma_g$ for the supersingular Newton polygon (i.e. all slopes are equal to 1/2).

We write $\mathcal{A}_{g,d,n}$ for the moduli space of polarized abelian varieties in characteristic p, with degree of polarization equal to d^2, and with symplectic level-n structure; here $g, d, n \in \mathbb{Z}_{>0}$ and $\gcd((n, dp) = 1$; this moduli space should be denoted by $\mathcal{A}_{g,d,n} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)$, but we abbreviate notation. In most cases we assume $n \geq 3$. We write $\mathcal{A}_{g,d}$ as an abbreviation for $\mathcal{A}_{g,d,1}$. When working over a field $K \supset \mathbb{F}_p$ we still write $\mathcal{A}_{g,d}$ instead of $\mathcal{A}_{g,d} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(K)$.

We write ℓ for a prime number different from p, sometimes also not dividing d.

(2.2) Central leaves. Main reference: [29]. Work over an algebraically closed field $k \supset \mathbb{F}_p$. Recall that for each geometric point $x = [(A, \lambda, \iota_A)] \in \mathcal{A}_{g,d,n}(k)$, there is a locally closed, reduced k-subscheme $\mathcal{C}(x)$ of $\mathcal{A}_{g,d,n}$, called the *central leaf* passing through x, characterized by the property that

$$\mathcal{C}(x)(\Omega) = \left\{(B, \mu, \iota_B) \in \mathcal{A}_{g,d,n}(\Omega) \mid (B, \mu)[p^\infty] \cong (A, \lambda)[p^\infty] \otimes \Omega,\ \mathrm{type}^{(p)}(\mu) = \mathrm{type}^{(p)}(\lambda)\right\}$$

for every algebraically closed field $\Omega \supset k$; see [29]. Here ι_A is a symplectic level-n structure of the polarized abelian variety (A, λ), where $(A, \lambda)[p^\infty]$ is the quasi-polarized p-divisible group attached to (A, λ), and $\mathrm{type}^{(p)}(\lambda)$ is the sequence of elementary divisors of the alternating pairing on $\prod_{\ell \neq p} \mathrm{H}_1(A, \mathbb{Z}_\ell)$ induced by the polarization λ; similarly for (B, μ, ι_B). For $n \geq 3$ we have that $\mathcal{C}(x)$ is smooth over k. The sequence $\mathrm{type}^{(p)}(\lambda)$ of elementary divisors has the form $(d_1, \ldots, d_g)$, where $d_1, \ldots, d_g$ are positive integers such that $d_1 \,|d_2| \cdots | \, d_g$ and $d = \prod_{i=1}^{g} d_i$. It is known that the leaf $\mathcal{C}(x)$ is closed in the open Newton stratum $\mathcal{W}^0_\xi(\mathcal{A}_{g,1,n})$ containing $\mathcal{C}(x)$; see [29], Th. 3.3. See 2.5 for the notation and properties of Newton strata. We will shorten "central leaf" to "leaf" if confusion (with "isogeny leaf") is unlikely.

Remark. The above definition of the leaf $\mathcal{C}(x)$ in $\mathcal{A}_{g,d,n}$ is slightly different from the definition in [29] because of the extra condition that we require here that the discrete invariant $\mathrm{type}^{(p)}(\mu)$ is constant on $\mathcal{C}(x)$. This condition enables us to announce Theorem B and Theorem C in the present short form.

(2.3) Newton polygons. Newton polygons with slopes between 0 and 1 will be denoted by a symbol like ζ or ξ. When we write $\zeta = \sum_i (m_i, n_i)$ we intend to say that the (lower convex) Newton polygon starting from the origin of the plane, such that the multiplicity of a slope ν in ζ is equal to $\sum_{\nu = m_i/(m_i+n_i)} (m_i + n_i)$. In the above, it is understood that $m_i, n_i \in \mathbb{Z}_{\geq 0}$ and $\gcd(m_i, n_i) = 1$ for all i; "lower convex" means that ζ is the graph of a convex piecewise linear function defined on $[0, h]$, where $h = \sum_i (m_i + n_i)$. A Newton polygon $\zeta = \sum_i (m_i, n_i)$ can be specified by its *slope sequence* $(\nu_1, \ldots, \nu_h)$, where $0 \leq \nu_1 \leq \cdots \nu_h \leq 1$, and for every $\nu \in \mathbb{Q} \cap [0, 1]$, the multiplicity of ν in the slope sequence above, defined as $\mathrm{Card}\{j \mid \mu_j = \nu,\ 1 \leq j \leq h\}$, is equal to the multiplicity of ν in ζ.

A Newton polygon ξ is said to be *symmetric* if the multiplicity of ν in ξ is equal to the multiplicity of $1 - \nu$ in ξ for every slope ν. We say that two Newton polygons are *disjoint* if they have no slopes in common. Every symmetric Newton polygon ξ can be written as a sum of disjoint symmetric Newton polygons, each having at most two slopes.

Every symmetric Newton polygon ξ can be written in a unique way in the following *standard form*

$$\xi = (\gamma_0 \cdot ((1,0)+(0,1))) + \left(\sum_{1\leq i\leq t} \gamma_i \cdot ((m_i,n_i)+(n_i,m_i)) \right) + (\gamma_{t+1} \cdot (1,1)) .$$

where $f = \gamma_0 \in \mathbb{Z}_{\geq 0}$, $t \in \mathbb{Z}_{\geq 0}$, $\gamma_1, \ldots, \gamma_t, \gamma_{t+1} \in \mathbb{Z}_{>0}$ and $m_i > n_i \geq 0$ for $1 \leq i \leq t$, and $(m_i,n_i) \neq (m_j,n_j)$ if $1 \leq i \neq j \leq t$. The coefficients $\gamma_0, \gamma_1, \ldots, \gamma_{t+1}$ are called the *multiplicities of the simple parts* of ξ. Define $g(\xi) = \gamma_0 + \sum_{1\leq i\leq t} \gamma_i \cdot (m_i+n_i) + \gamma_{t+1}$.

(2.4) According to the Dieudonné-Manin classification of p-divisible groups over an algebraically closed field, see [19], page 35, every p-divisible group X over an algebraically closed field $k \supset \mathbb{F}_p$ is isogenous to a direct product of isoclinic p-divisible groups $G_{m,n}$, with $m,n \in \mathbb{Z}_{\geq 0}$ and $\gcd(m,n)=1$, with $\dim(G_{m,n}) = m$; in this case $G_{m,n}$ has height $m+n$ and is isoclinic of slope $m/(m+n)$. The Newton polygon of a p-divisible group X isogenous to $\prod_i G_{m_i,n_i}$ is

$$\sum_i (m_i,n_i) =: \mathcal{N}(X) .$$

For an abelian variety A over a field $K \supset \mathbb{F}_p$, the Newton polygon attached to $A[p^\infty]$ is a symmetric Newton polygon $\mathcal{N}(A)$, and it can be written in standard form as above. Then we have $\dim(A) = g(\xi)$. We hope there will be no confusion caused by the formal sum expressing ξ and the summation as in the formula for g.

Working over a field $K \supset \mathbb{F}_p$, if no confusion can arise, we write $G_{m,n}$ instead of $G_{m,n} \otimes_{\mathbb{F}_p} K$.

The results is that there is a bijection between the set of k-isogeny classes of p-divisible groups over k and the set of Newton polygons:

Theorem (Dieudonné and Manin), see [19], "Classification theorem " on page 35.

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}.$$

(2.5) Newton polygon strata. The set of Newton polygons is partially ordered; *we write $\gamma \prec \beta$ if no point of γ is below β*:

$$\gamma \prec \beta \quad \Longleftrightarrow \quad \gamma \text{ is ``above'' } \beta.$$

If $(\nu_1, \ldots, \nu_h)$ and $(\mu_1, \ldots, \mu_{h'})$ are the slope sequence of γ and β respectively, with $0 \leq \nu_1 \leq \cdots \leq \nu_h \leq 1$ and $0 \leq \mu_1 \leq \cdots \leq \mu_{h'} \leq 1$, then $\gamma \prec \beta$ if and only if

$$h = h', \quad \sum_{j=1}^{m} \nu_j \geq \sum_{j=1}^{m} \mu_j \quad \text{for} \quad m = 1, \ldots, h-1, \quad \text{and} \quad \sum_{j=1}^{h} \nu_j = \sum_{j=1}^{h} \mu_j .$$

An abelian variety A is isogenous with its dual A^t; using the duality theorem, see [21], 19.1 we conclude that $X \sim X^t$; hence $\mathcal{N}(A) =: \xi$ is symmetric.

For a symmetric Newton polygon ξ we write:

$$\mathcal{W}_\xi(\mathcal{A}_{g,d,n}) \quad := \quad \{[(A,\lambda)] \mid \mathcal{N}(A) \prec \xi\},$$

$$\mathcal{W}^0_\xi(\mathcal{A}_{g,d,n}) \quad := \quad \{[(A,\lambda)] \mid \mathcal{N}(A) = \xi\}.$$

Grothendieck-Katz:

$$\mathcal{W}_\xi(\mathcal{A}_{g,d,n}) \quad \subset \quad \mathcal{A}_{g,d,n} \textit{ is closed,}$$

see [13] page 149–150, [17] Th. 2.3.1 on page 143; hence

$$\mathcal{W}^0_\xi(\mathcal{A}_{g,d,n}) \quad \subset \quad \mathcal{A}_{g,d,n} \textit{ is locally closed.}$$

These are called the **Newton polygon strata.** We write

$$W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1}), \qquad W^0_\xi = \mathcal{W}^0_\xi(\mathcal{A}_{g,1}).$$

(2.6) Theorem (Oort), see [29], Th. 3.3. *Every leaf*

$$\mathcal{C}(y) \subset \mathcal{W}^0_\xi(\mathcal{A}_{g,d,n})$$

is a closed subset of the open Newton polygon stratum $\mathcal{W}^0_\xi(\mathcal{A}_{g,d,n})$ *.*

(2.7) Hypersymmetric abelian varieties. Main reference: [7].
Definition. Let B be an abelian variety over a field $K \supset \mathbb{F}_p$. We say that B is *hypersymmetric* if one (or all) of the following equivalent conditions are satisfied:

(1) The natural map

$$\mathrm{End}\left(B \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{End}\left(B[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right)$$

is an isomorphism.

(2) The natural map

$$\mathrm{End}\left(B \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right) \otimes_{\mathbb{Z}} \mathbb{Q}_p \quad \xrightarrow{\sim} \quad \mathrm{End}\left(B[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism.

(3) There exist an abelian variety A defined over $\mathbb{F}$ and an isogeny

$$B \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K}) \sim A \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(\overline{K}),$$

such that the natural map

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{End}(A[p^\infty])$$

is an isomorphism.

Remark. Tate proved that for an abelian variety A *defined over a finite field* $\mathbb{F}_q$, the natural homomorphism

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{End}(A[p^\infty])$$

is an isomorphism, see [39], Theorem 1 on page 60. This shows that $\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is identified with the $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-invariant endomorphisms of $\mathrm{End}\left(A[p^\infty]\times_{\mathrm{Spec}(\mathbb{F}_q)}\mathrm{Spec}(\mathbb{F})\right)$. Suppose that

$$\mathrm{End}(A) \xrightarrow{\sim} \mathrm{End}\left(A\times_{\mathrm{Spec}(\mathbb{F}_q)}\mathrm{Spec}(\mathbb{F})\right) .$$

Then A is hypersymmetric if and only if every element of the ring of endomorphisms of the p-divisible group $A[p^\infty]\times_{\mathrm{Spec}(\mathbb{F}_q)}\mathrm{Spec}(\mathbb{F})$ is invariant under every element of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$. From this we see that there are "many" abelian varieties over a finite field which are not hypersymmetric. Also we see that for a hypersymmetric abelian variety this Galois action is "in diagonal form" for every isoclinic part of $A[p^\infty]$. This can be made precise as follows.

Let K be a finite field and let B be an abelian variety over K. Then B is hypersymmetric if and only if there exists a positive integer n such that the n-th power of the Frobenius π_B of B lies in the center of $\mathrm{End}^0\left(B[p^\infty]\times_{\mathrm{Spec}(K)}\mathrm{Spec}(\mathbb{F})\right)$. *In other words, any two eigenvalues of the action of π_B on the Dieudonné module of* $B[p^\infty]\times_{\mathrm{Spec}(K)}\mathrm{Spec}(\mathbb{F})$ *which have the same p-adic absolute value, differ by a root of unity.*

Remark. In [7] we showed that for any symmetric Newton polygon ξ there exists a hypersymmetric point in the Newton stratum W^0_ξ. This statement does *not* hold for general modular varieties of PEL-type, where a point is said to be hypersymmetric if the underlying abelian variety with prescribed endomorphism ring is hypersymmetric in the sense of [7, Def. 6.4]. Therefore the methods in this article do not apply to all leaves in a modular variety of PEL-type. Explicit conditions for the existence of hypersymmetric points on a Newton stratum can be found in [42].

(2.8) Hecke orbits. Suppose given a field K, and a polarized abelian variety (A,λ) over K. We define the *Hecke orbit* of the moduli point $x := [(A,\lambda)]$ to be the set of points $y = [(B,\mu)]$ over some field L such that there exist a field Ω containing K and L, and an isogeny $\varphi : A \to B$ such that

$$\varphi^*(\mu) = \lambda.$$

Notation. We write $y \in \mathcal{H}(x)$. The set $\mathcal{H}(x)$ is called the *Hecke orbit* of x

Hecke-prime-to-N-orbits. Let $N \in \mathbb{Z}_{>0}$. If in the previous definition moreover the degree of φ is not divisible by any prime dividing N, we say $[(B,\mu)] = y$ is in the Hecke-prime-to-N-orbit of x.
Notation: $y \in \mathcal{H}^{(N)}(x)$.

Hecke-ℓ-orbits. Fix a prime number ℓ different from p. We say $[(B,\mu)] = y$ is in the Hecke-ℓ-orbit of x if in the previous definition moreover the degree of φ and m both are a power of ℓ.
Notation: $y \in \mathcal{H}_\ell(x)$.

Remark. We have given the definition of the so-called $\mathrm{Sp}_{2g}(\mathbb{A}_f)$-Hecke orbit, because for every positive integer multiple N of pd, the prime-to-N Hecke orbit $\mathcal{H}^{(N)}(x)$ of a geometric point $x \in \mathcal{A}_{g,d,n}(k)$ can be described as follows. Consider the tower of moduli spaces $\left(\mathcal{A}_{g,d,m}\right)$, where m runs through all multiples of n which are prime to pd. The group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(N)})$ operates on the tower, where $\mathbb{A}_f^{(N)} = \prod'_{\gcd(\ell,dp)=1} \mathbb{Q}_\ell$. Let $\tilde{x}$ be a k-point of the projective limit $\varprojlim \mathcal{A}_{g,d,m}$ lying above x. Then $\mathcal{H}^{(N)}(x)$ is the image in $\mathcal{A}_{g,d,n}$ of $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(N)}) \cdot \tilde{x}$, the orbit of $\tilde{x}$ under the action of $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(N)})$. See [8], Section 1 for a further discussion.

Hecke stable. We say a set $T \subset \mathcal{A}_g$ is $\mathcal{H}_\ell$-stable if for every $x := [(A,\lambda)] \in T$ we have $\mathcal{H}_\ell(x) \subset T$, and analogously for $\mathcal{H}^{(p)}$-stable. Note that Newton polygon strata are Hecke-stable, and EO-strata and central leaves are $\mathcal{H}^{(p)}$-stable.

The $\mathcal{H}_\alpha$-orbit. By considering Hecke correspondences where all isogenies involved are geometrically successive extensions of α_p, we define the Hecke-α-orbit of a moduli point. If in the notation just used the kernels of φ_1 and of φ_2 are successive extensions of α_p over an algebraically closed field, we write $[(B,\mu)] = y \in \mathcal{H}_\alpha(x)$.

We define $I(y) \subset \mathcal{A}_g$, a maximal $\mathcal{H}_\alpha$-set, by taking the union of all irreducible components of the $\mathcal{H}_\alpha$-orbit of y containing y; this is called the *isogeny leaf* passing through $y = [(B,\mu)]$; we give this the induced reduced scheme structure.

The completion of $\mathcal{A}_{g,d,n}$ over k along an isogeny leaf, in the sense of [29], is etale locally isomorphic to the reduction modulo p of a suitable Rapoport-Zink space, see [36], Th. 6.23 for a more precise statement. This implies that the formal completion $I(y)^{/y}$ of an isogeny leaf at a k-point $y \in \mathcal{A}_{g,d,n}(k)$ is isomorphic to the formal completion at a suitable k-point of the reduction modulo p of a Rapoport-Zink space with reduced structure.

(2.9) Theorem (Oort) ("central leaves and isogeny leaves almost give a product structure on an irreducible component of a Newton polygon stratum"), see [29], Th. 5.3.
Work over an algebraically closed field k. Choose a symmetric Newton polygon ξ, an irreducible component W of $\mathcal{W}_\xi^0(\mathcal{A}_g)$, an irreducible component C of a central leaf, and an irreducible component I of an isogeny leaf. There exist finite surjective morphisms $f\colon T \twoheadrightarrow C$, $g\colon J \twoheadrightarrow I$, a finite surjective morphism

$$\Phi : T \times J \quad \twoheadrightarrow \quad W$$

and a polarization-preserving quasi-isogeny

$$\Theta\colon f^*(A,\lambda) \longrightarrow \Phi^*(B,\mu)\,.$$

such that for every $u \in J$,

$$\Phi(T \times \{u\}) \text{ is a component of a central leaf,}$$

and for every $t \in T$,

$$\Phi(\{t\} \times J) \text{ is a component of an isogeny leaf.}$$

(2.10) Minimal p-divisible groups. Main reference: [31]. Also see [16].
For a pair (m,n) of coprime non-negative integers we define $H_{m,n}$ to be a p-divisible group of dimension m, of height $m+n$, of slope $m/(m+n)$ such that over k the endomorphism ring

$\mathrm{End}(H_{m,n})$ is the *maximal order* in $\mathrm{End}^0(H_{m,n}) := \mathrm{End}(H_{m,n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; this defines this group uniquely up to isomorphism over an algebraically closed field; the group can be constructed over $\mathbb{F}_p$. For $\zeta = \sum_i \ (m_i, n_i)$ we write $H(\zeta) = \prod_i \ H_{m_i,n_i}$. This p-divisible group is defined over $\mathbb{F}_p$; if no confusion can arise we write $H(\zeta)$ over any field $K \supset \mathbb{F}_p$, meaning $H(\zeta) \otimes_{\mathbb{F}_p} K$.

Definition. We say a p-divisible group X is *minimal* if there exists an isomorphism $X_\Omega \cong H(\zeta)$.

(2.11) Theorem (Oort), see [31], Th. 1.2. *Work over an algebraically closed field k. Let X be a p-divisible group, and let ζ be some Newton polygon. Suppose there exists an isomorphism $X[p] \cong H(\zeta)[p]$. There exists a Newton polygon ζ and an isomorphism $X \cong H(\zeta)$.*

Suppose $[(A, \lambda)] = x$ where $A[p^\infty]$ is *minimal* and λ is a *principal* polarization. In this case the central leaf $\mathcal{C}(x)$ is denoted by $\mathcal{Z}_\xi = \mathcal{C}(x)$, where $\xi = \mathcal{N}(A[p^\infty])$, and it is called the *central stream* inside W^0_ξ. Note that a principal quasi-polarization on the minimal p-divisible $A[p^\infty]$ is unique, see [29], 3.7.

Results on ℓ and prime-to-p monodromy and irreducibility. Main reference: [4].

(2.12) Theorem (Chai). *Work over an algebraically closed field k. Suppose $n \in \mathbb{Z}_{\geq 3}$ is relatively prime with pd. Let $Z \subset \mathcal{A}_{g,d,n}$ be a locally closed non-singular subset not contained in the supersingular locus. Suppose Z is $\mathcal{H}^{(pd)}_{\mathrm{Sp}}$-stable. Assume that $\mathcal{H}^{(pd)}_{\mathrm{Sp}}$ acts transitively on $\Pi_0(Z)$. Then Z is* irreducible *and the prime-to-pdn monodromy representation on Z equals*

$$\mathrm{Sp}_{2g}(\hat{\mathbb{Z}}^{(pd)})(n) := \{\gamma \in \mathrm{Sp}_{2g}(\hat{\mathbb{Z}}^{(pd)}) \mid \gamma \equiv 1 \pmod{n}\}.$$

See [4], Prop. 4.5.4 for a proof of 2.12. More precisely, in [4] the degree of the polarization is assumed to be 1, but the proof there gives a proof of 2.12.

Note that 2.12 implies that the image of the ℓ-adic monodromy homomorphism for any leaf not contained in the supersingular stratum $\mathcal{A}_{g,d,n}$ is maximal for ℓ not dividing pdn.

We will need a slightly more general version of the irreducibility part in 2.12.

(2.13) Theorem (Chai). *Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let n, d be positive integers with $\gcd(n, pd) = 1$ and $n \geq 3$. Let $Z \subset \mathcal{A}_{g,d,n}$ be a locally closed subscheme over k smooth over k, not contained in the supersingular locus and stable under all prime-to-pd Hecke correspondences. Let $W \to Z$ be a finite etale scheme over Z, and let*

$$\widetilde{W} := \big(Z \times_{\mathcal{A}_{g,d,n}} \mathcal{A}_{g,d,m} : \ n|m, \ \gcd(m, pd) = 1 \big)$$

be the fiber product of Z with the prime-to-pd tower

$$\widetilde{A}_{g,d} = (\, \mathcal{A}_{g,d,m} : \ n|m, \ \gcd(m, pd) = 1 \,)$$

above $\mathcal{A}_{g,d,n}$. Assume that the prime-to-pd Hecke correspondences lift to W in the sense that the action of the group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(pd)})$ on the restriction to Z of the prime-to-pd tower $\widetilde{A}_{g,d}$ lifts to an action on the tower $\widetilde{W}$. Assume moreover that the prime-to-pd Hecke correspondences on W induced by the Hecke action on the tower $\widetilde{W}$ operate transitively on the set $\Pi_0(W)$ of irreducible components of W. Then W is irreducible. Moreover the prime-to-pdn monodromy of Z is maximal.

The situation of 2.13 is different from that of [4, Prop. 4.4]: in [4] the variety in question is a subvariety of the moduli space $\mathcal{A}_{g,1,n}$, while in 2.13 the variety in question is a cover of subvariety of $\mathcal{A}_{g,d,n}$. However the proof of [4, Prop. 4.4] gives a proof of Theorem 2.13; see also the proof of Prop. 5.5 in [6]. The key point of this argument is the fact that the group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(pd)}), \langle\, ,\, \rangle)$ has no proper subgroup of finite index; see [4, Lemma 3.1]. The assumption in 2.13 implies that $\Pi_0(Z)$ is a surjective image of a quotient of the group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(pd)}), \langle\, ,\, \rangle)$ by a subgroup of finite index. Since every such quotient of $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(pd)}), \langle\, ,\, \rangle)$ is a one-point set, $\Pi_0(Z)$ has only one element, i.e. Z is irreducible. $\square$

(2.14) Unwinding a finite level over a leaf. Main reference: [29], Th. 1.3.
We say that a p-divisible group $X \to S$ is *geometrically fiberwise constant*, abbreviated gfc, if fibers are mutually geometrically isomorphic, see [29], 1.1 for more details. In [29], Th. 1.3 we show that *if S satisfies reasonable finiteness conditions and $X \to S$ is* gfc, *and $i \in \mathbb{Z}_{>0}$ then there exists a finite surjective morphism $T_i = T \twoheadrightarrow S$ such that $X[p^i] \times_S T$ is constant over T.* We need a part of the proof of this theorem which is contained in [34]. And actually we need the part of the proof which goes back to Hasse and Witt, to Dieudonné, and which was described by Zink in [41], §2; also see [34], 1.6.

(2.15) Completely slope divisible p-divisible groups. Let $t \geq r_1 > r_2 > \cdots > r_m \geq 0$ be natural numbers. Let $q = p^t$, and let S be a scheme over $\mathbb{F}_q$. A p-divisible group $Y \to S$ is said to be *completely slope divisible* with respect to the numbers $t \geq r_1 > r_2 > \cdots > r_m \geq 0$ if there exist a filtration

$$0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$$

of Y by p-divisible subgroups over S such that

- each Y_i/Y_{i-1} is a p-divisible group for $i = 1, \ldots, m$,
- the quasi-isogenies $p^{-r_i}\mathrm{Fr}^t : Y_i \to Y_i^{(p^t)}$ are isogenies for $i = 1, \ldots, m$,
- the induced homomorphisms $p^{-r_i}\mathrm{Fr}^t : (Y_i/Y_{i-1}) \to (Y_i/Y_{i-1})^{(p^t)}$ are isomorphisms for $i = 1, \ldots, m$.

See [34, 1.1].

Let $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$ be the slope filtration of a completely slope divisible p-divisible group $Y \to S$ as above. Let $G_i = Y_i/Y_{i-1}$ for $i = 1, \ldots, m$. Let $q = p^t$. By the theory of Φ-etale part of a Frobenius module in [41, §1], for each $b \geq 1$ there exist fppf sheaves of commutative $\mathbb{F}_q$-algebras $\Gamma_{i,b} = \Gamma_{i,b,q}$ on S which are representable by finite etale schemes over S, and natural isomorphisms $\mathcal{O}_S \otimes_{\mathbb{F}_q} \Gamma_{i,b} \xrightarrow{\sim} \mathcal{O}_{G_i[p^b]}$. The sheaf $\Gamma_{i,b}$ is subsheaf of the structure sheaf of $G_i[p^b]$ for the fppf topology fixed under the Frob_S^t-linear map $\Phi_i : \mathcal{O}_{G_i[p^b]} \to \mathcal{O}_{G_i[p^b]}$ induced by the composition of the isomorphism $p^{-r_i}\mathrm{Fr}^t : G_i[p^b] \to G_i[p^b]^{(p^t)}$ and the map $W_s : G_i[p^b]^{(p^t)} \to G_i[p^b]$ coming from the base change by the absolute Frobenius $\mathrm{Frob}_S^t : S \to S$. The group law on $G_i[p^b]$ induces a comultiplication on $\Gamma_{i,b}$, making $\Gamma_{i,b}$ a sheaf of commutative cocommutative bialgebras over $\mathbb{F}_q$. The inclusion maps $G_i[p^b] \hookrightarrow G_i[p^{b+1}]$ induces surjective homomorphism $\Gamma_{i,b+1} \to \Gamma_{i,b}$ between sheaves of bialgebras over $\mathbb{F}_q$.

(2.16) Remarks. (1) A p-divisible group Y over a field $K \supset \mathbb{F}_p$ is completely slope divisible if and only if $Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^{\mathrm{a}})$ is.

(2) Let Y be a p-divisible group over a perfect $K \supset \mathbb{F}_p$. Then Y is completely slope divisible if and only if there exists a finite field $\mathbb{F}_q \subset k$ and isoclinic p-divisible groups $Z_1, \ldots, Z_m$ over $\mathbb{F}_q$ with distinct slopes such that Y is isomorphic to

$$Z_1 \times_{\mathrm{Spec}(\mathbb{F}_q)} \cdots \times_{\mathrm{Spec}(\mathbb{F}_q)} Z_m \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(k)\,.$$

(2.17) Remark. Let Y be a p-divisible group over a field $K \supset \mathbb{F}_p$. By [41, Cor. 13], Y has a slope filtration, i.e. there exist p-divisible subgroups $0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_m = Y$ over K such that each quotient $Z_i := Y_i/Y_{i-1}$ is an isoclinic p-divisible group of slope μ_i over K and $\mu_1 > \mu_2 > \cdots > \mu_m$. We have a natural inclusion

$$\mathrm{End}(Y) \hookrightarrow \mathrm{End}(Z_1) \times \cdots \times \mathrm{End}(Z_m) = \mathrm{End}(Z_1 \times \cdots \times Z_m)\,.$$

That this inclusion is an equality if and only if Y is isomorphic to $Z_1 \times \cdots \times Z_m$. The "if" direction is obvious. For the "only if" part, it suffices to note that the $\mathbb{Z}_p$-span of

$$(\mathbb{Z}_p^\times \cdot \mathrm{Id}_{Z_1}) \times \cdots \times (\mathbb{Z}_p^\times \cdot \mathrm{Id}_{Z_m})$$

in the algebra $\mathrm{End}(Z_1) \times \cdots \times \mathrm{End}(Z_m)$ is equal to $(\mathbb{Z}_p \cdot \mathrm{Id}_{Z_1}) \times \cdots \times (\mathbb{Z}_p \cdot \mathrm{Id}_{Z_m})$. In particular this $\mathbb{Z}_p$-span contains the m idempotents corresponding to the m factors in the product $Z_1 \times \cdots \times Z_m$.

Several notations and prerequisites, perhaps not sufficiently well explained here, are to be found in [8] and [33].

§3. Irreducibility of Newton polygon strata

In order to start the proof of irreducibility of leaves, we first show the irreducibility of Newton polygon strata in the principally polarized case. Note however that *supersingular* Newton polygon strata are reducible (for p large). For the case of elliptic curves in characteristic p this is classical. By Hasse, Deuring, Igusa we know:

$$\sum_{j(E)\ \text{is ss}} \frac{1}{\#(\mathrm{Aut}(E))} \quad = \quad \frac{p-1}{24};$$

this implies that the number of supersingular j-invariants is asymptotically $p/12$ for $p \to \infty$.

For fixed g and large p the supersingular locus has "many components", see [18], 4.9:

$$\#(\Pi_0(W_\sigma)) = H_g(p,1) \quad \text{if} \quad g \quad \text{is odd,}$$

$$\#(\Pi_0(W_\sigma)) = H_g(1,p) \quad \text{if} \quad g \quad \text{is even.}$$

Note that for g fixed, and $p \to \infty$, indeed $\#(\Pi_0(W_\sigma)) \to \infty$.

(3.1) Theorem. *Let ξ be a symmetric Newton polygon which is* not supersingular*. The stratum $W_\xi := \mathcal{W}_\xi(\mathcal{A}_{g,1})$ is* geometrically irreducible.

A *sketch* of the proof below can be found in [30].

Step 1. *The moduli scheme $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is geometrically irreducible.*
This was proved by Faltings, see [11], Korollar on page 364, and this was proved by Chai (in case $p > 2$) in [1]; see [12], Chap. IV Cor. 6.8; for a pure characteristic p proof see [26], Cor. 1.4. $\square$

Step 2. Deformation to $a \leq 1$. *Let (A, λ) be a* principally polarized *abelian variety. There exists a deformation to a principally polarized abelian variety (B, μ) with $\mathcal{N}(A) = \mathcal{N}(B)$ and $a(B) \leq 1$.*
This is a difficult theorem. For the proof we use deformation to $a \leq 1$ for simple p-divisible groups and purity, see [16]. Coroll. 5.12. Once this is established the general result follows by considering deformation of filtered p-divisible groups, see [28], Section 3. $\square$

Corollary. *$W^0_\xi(a \leq 1)$ is dense in W^0_ξ.* $\square$

Step 3. *The Cayley-Hamilton theorem.* In [27] we study deformations of (polarized) p-divisible groups with a-number equal to 1. As a corollary, using the corollary in Step 2, we show that the strata

$$\{W^0_\xi(a \leq 1) \mid \xi\}$$

are ordered by inclusion-in-the-boundary exactly as prescribed by the NP-graph; moreover we compute the dimension of these strata; see [27], Th. 3.4

Step 4. Corollary. *For symmetric Newton polygons $\xi_1 \prec \xi_2$ and for an irreducible component P_1 of $W^0_{\xi_1}$ there exists an irreducible component P_2 of $W^0_{\xi_2}$ such that $P_1 \subset (P_2)^{\mathrm{Zar}}$ and P_2 is unique.*

Corollary. *$W^0_\xi(a \leq 1)$ is dense in W_ξ.*
Corollary. *$\cup_{\xi' \precneqq \xi} W^0_{\xi'}$ is contained in $(W^0_\xi)^{\mathrm{Zar}}$.* $\square$

Step 5. Using the facts

- every non-supersingular Hecke-ℓ orbit is non-finite, see [2], Prop. 1 on page 448,
- EO-strata are Hecke-ℓ stable,
- EO-strata are quasi-affine, see [26], Th. 1.2,

we show:
for every symmetric non-supersingular Newton polygon ξ_2 and every irreducible component P_2 of $W^0_{\xi_2}$ there exists $\xi_1 \precneqq \xi_2$ such that $W^0_{\xi_1} \cap (P_2)^{\mathrm{Zar}} \neq \emptyset$.
Proof. Fix $P_2 \subset W^0_{\xi_2}$. Let $\mathcal{P}$ be a union of irreducible components of $W^0_{\xi_2}$, the union taken over the Hecke-ℓ-orbit of $[P_2]$ in $\Pi_0(W^0_{\xi_2})$.

Consider all elementary sequences, in the sense of [29], which appear on P_2. This is a finite set $\Gamma(P_2)$. On the set of elementary sequences we have a partial ordering, denoted by

$\subset$ in [29], 14.3; i.e. we write $\varphi' \subset \varphi$ if and only if $S_{\varphi'}$ is contained in the closure of S_φ; indeed, this defines a partial ordering by [29], 12.5. Let φ be a *minimal element* in $\Gamma(P_2)$. Let $x \in W^0_{\xi_2} \cap S_\varphi$.
Claim. *$S_\varphi \cap P_2$ is not a finite set.*
Consider $\mathcal{H}_\ell$. Note that $\mathcal{H}_\ell(x) \subset \mathcal{P}$; this follows because any Hecke-ℓ correspondence is etale and finite-to-finite. As x is not contained in the supersingular locus it follows from [2], Prop. 1 on page 448 that this Hecke orbit $\mathcal{H}_\ell(x)$ is non-finite. Note that S_φ and $\mathcal{P}$ are Hecke-ℓ-stable. Hence $\mathcal{H}_\ell(x) \subset (S_\varphi \cap \mathcal{P})$. The Zariski closure $\mathcal{H}$ of $\mathcal{H}_\ell(x)$ is not finite. It contains an irreducible component of positive dimension. As Hecke-ℓ correspondences are etale and finite-to-finte this shows there is an irreducible component $H \subset (S_\varphi \cap P_2)$ of positive dimension containing x. $\square$

Consider the stratum $T_\varphi \subset \mathcal{A}^*_g$ as defined [26], 6.1; this consists of S_φ and all boundary points in $\mathcal{A}^*_g$ corresponding with the elementary sequence φ. The stratum T_φ, in the minimal compactification $\mathcal{A}^*_g$, is quasi-affine, see [26], 6.5 and φ is also minimal on T_φ, see [26], 6.3. Hence there is a point y' in the closure of T_φ not contained in T_φ itself. If $y' \in \mathcal{A}_g$ we write $y = y'$. If $y' \notin \mathcal{A}_g$, using [26], 6.3 we conclude there exists a point $y \in \mathcal{A}_{g,1}$ with $y \notin T_\varphi$. This shows that the Zariski closure of $H \subset \mathcal{A}_{g,1}$ considered above contains a point $y \in \mathcal{A}_{g,1}$ with $y \notin T_\varphi$; hence $y \notin S_\varphi$. Let φ' be the elementary sequence defined by y. Let ξ_1 be the Newton polygon defined by y. As $y \notin S_\varphi$ we have $\varphi' \subsetneqq \varphi$. Because the minimality assumption on φ we see that $y \notin W^0_{\xi_1}$ and $\xi_1 \precneqq \xi_2$. This proves Step 5:

$$W^0_{\xi_1} \cap (P_2)^{\mathrm{Zar}} \quad \neq \quad \emptyset.$$

$\square$

Remark. Instead of the argument above a variant of a method indicated in [2] can be used. If ξ is ordinary, the conclusion of Step 5 follows from the fact that the ordinary locus is dense: either use that $\mathcal{A}_{g,1}$ is irreducible, or use [20]. Suppose ξ is not ordinary (and not supersingular). Choose a point x with minimal EO-type on $\mathcal{P}$ as above, which moreover is defined over a finite field. Consider a Hilbert modular variety $\mathcal{M}$ such that its image in $\mathcal{A}_g$ contains x. On a Hilbert modular variety the Hecke orbit of a non-supersingular is non-finite. Using [26] we see that EO-strata on $\mathcal{M}$ are quasi-affine. Moreover any non-ordinary Newton polygon stratum does not contain a boundary point in the minimal compactification of $\mathcal{M}$. Hence the Zariski closure of $H \subset \mathcal{A}_{g,1}$ considered above contains a point $y \in \mathcal{A}_{g,1}$ with $y \notin S_\varphi$, and we conclude as above.

Step 6. For $\xi_1 \prec \xi_2$ define $i^{\xi_1}_{\xi_2} : \Pi_0(W_{\xi_1}) \to \Pi_0(W_{\xi_2})$ by:

$$i^{\xi_1}_{\xi_2}(P_1) = P_2 \quad \Longleftrightarrow \quad P_1 \subset (P_2)^{\mathrm{Zar}}.$$

Using steps 3, 4 and 5 we conclude:
Corollary. *This map is well-defined.*
Corollary. *The map*

$$i^\sigma_\xi : \Pi_0(W_\sigma) \twoheadrightarrow \Pi_0(W_\xi)$$

is surjective for every ξ. Note that this map is $\mathcal{H}^{\mathrm{Sp}}_\ell$-equivariant.
Corollary. *The map $i^{\xi_1}_{\xi_2}$ is surjective.* $\square$

Step 7. Notation. For $g \in \mathbb{Z}_{>1}$ and $j \in \mathbb{Z}_{\geq 0}$ we write $\Lambda_{g,j}$ for the set of isomorphism classes of polarizations μ on the superspecial abelian variety $A = E^g \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(k)$ such that $\mathrm{Ker}(\mu) = A[F^j]$; here E is a supersingular elliptic curve defined over $\mathbb{F}_p$. Note that $\Lambda_{g,j} \xrightarrow{\sim} \Lambda_{g,j+2}$ under $\mu \mapsto F^t{\cdot}\mu{\cdot}F$.

Fact. Characterization of components of W_σ. *There is a canonical bijective map*

$$\Pi_0(W_\sigma) \quad \xrightarrow{\sim} \quad \Lambda_{g,g-1}.$$

See [18], 3.6 and 4.2; this uses [22], 2.2 and 3.1. ☐

Step 8. Transitivity. *The action of* $\mathcal{H}_\ell^{\mathrm{Sp}}$ *on* $\Lambda_{g,g-1}$ *is transitive.*
Proof. Consider $[(A_1, \mu_1)], [(A_2, \mu_2)] \in \Lambda_{g,g-1}$. Let G_i be the unitary group over defined by the semisimple algebra with involution $(\,\mathrm{End}^0(A_i), *_i\,)$, where $*_i$ is the Rosati involution attached to μ_i. In other words,

$$G_i(R) = \left\{\, y \in (\mathrm{End}^0(A_i) \otimes_{\mathbb{Q}} R)^\times \mid y \cdot *_i(y) = 1 = *_i(y) \cdot y \,\right\}$$

for every commutative $\mathbb{Q}$-algebra R. Denote by $\mathcal{T}$ the affine scheme of finite type over $\mathbb{Q}$ such that

$$\mathcal{T}(R) = \left\{\phi \in \mathrm{Hom}(A_2, A_1) \otimes_{\mathbb{Z}} R \mid \ \phi^*(\lambda_1) = \lambda_2 \ \text{ in } \ \mathrm{Hom}(A_1, A_1^t) \otimes_{\mathbb{Z}} R \,\right\}$$

for all commutative $\mathbb{Q}$-algebra R. It is clear that $\mathcal{T}$ has a natural structure as a left G_1-torsor and also a right G_2-torsor; moreover the left G_1-action commutes with the right G_2-action. We claim that this torsor $\mathcal{T}$ is trivial, i.e. it has a $\mathbb{Q}$-rational point.

We use the left G_1-torsor structure of $\mathcal{T}$. It is clear that $\mathcal{T}(\mathbb{Q}_\ell) \neq \emptyset$ for all prime numbers $\ell \neq p$. We see that $\mathcal{T}(\mathbb{R}) \neq \emptyset$ because the Rosati involutions $*_1$ and $*_2$ are positive definite. The set $\mathcal{T}(\mathbb{Q}_p)$ of $\mathbb{Q}_p$-points of $\mathcal{T}$ is not empty because $(A_1, \lambda_1)[p^\infty] \cong (A_2, \lambda_2)[p^\infty]$ by assumption. Since the group G_1 is connected and simply connected, the Hasse principle applies; see [35], Theorem 6.6. We conclude that $\mathcal{T}(\mathbb{Q}) \neq \emptyset$.

As the torsor $\mathcal{T}$ over $\mathbb{Q}$ is trivial, the weak approximation theorem applies; see [35], 7.3, Theorem 7.7 on page 415. In particular there exists a quasi-isogeny ϕ from A_2 to A_1 which induces an isomorphism from $(A_2, \lambda_2)[p^\infty]$ to $(A_1, \lambda_1)[p^\infty]$. The is prime-to-$p$ quasi-isogeny ϕ defines a prime-to-p Hecke correspondence connecting the two elements $[(A_1, \mu_1)]$, $[(A_2, \mu_2)]$ of $\Lambda_{g,g-1}$. ☐

Corollary. *The action of* $\mathcal{H}_\ell^{\mathrm{Sp}}$ *on* $\Pi_0(W_\sigma)$ *is transitive.*
This follows using Step 7. ☐

Step 9. End of the proof of 3.1. We have seen that the map $i_\xi^\sigma : \Pi_0(W_\sigma) \to \Pi_0(W_\xi)$ is surjective and $\mathcal{H}_\ell^{\mathrm{Sp}}$ equivariant for every ξ; by the previous step this implies that the action of $\mathcal{H}_\ell^{\mathrm{Sp}}$ on $\Pi_0(W_\xi)$ is transitive. By 2.12 this implies that W_ξ is geometrically irreducible for $\xi \neq \sigma$. This finishes a proof of Theorem 3.1. ☐

Remark. The same proof shows that $\mathcal{W}_\xi(\mathcal{A}_{g,1,n})$ is geometrically irreducible for every $n \neq p$.

(3.2) Remark. We have seen that the strata $W^0_\xi := \mathcal{W}^0_\xi(\mathcal{A}_{g,1})$, in the (*principally polarized case*), form a stratification of $\mathcal{A}_{g,1}$; in particular for $\xi_1 \precneqq \xi_2$ the stratum $W^0_{\xi_1}$ is contained in the closure of $W^0_{\xi_2}$.

However there are many cases where $\xi_1 \precneqq \xi_2$ such that a component of $\mathcal{W}^0_{\xi_1}(\mathcal{A}_{g,d})$ is not contained in the closure of $\mathcal{W}^0_{\xi_2}(\mathcal{A}_{g,d})$ for an appropriate choice of d; this has been worked out in [32], §3.

Note the definition of $\mathcal{W}_\xi(\mathcal{A}_{g,d})$ is not given as the closure of $\mathcal{W}^0_\xi(\mathcal{A}_{g,d})$.

We know that the dimension of a central leaf is determined by the related Newton polygon: all leaves in one Newton polygon stratum have the same dimension, independent of the degrees of the polarizations considered. However, different irreducible components of $\mathcal{W}^0_\xi(\mathcal{A}_{g,d})$ for varying d can have different dimension; all this has been computed and discussed in [32].

§4. Irreducibility of leaves

(4.1) Theorem. *Let g, n and d be positive integers with $n \geq 3$ and $\gcd(pd, n) = 1$. Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let $x = [(A, \lambda, \iota)] \in \mathcal{A}_{g,d,n}(k)$ be a geometric point of $\mathcal{A}_{g,d,n}$ not contained in the supersingular stratum. The central leaf $\mathcal{C}(x) \subset \mathcal{A}_{g,d,n}$ passing through x is irreducible.*

Write $d = d'{\cdot}m$, where p does not divide m and d' is a power of p. Suppose that this theorem has been proved with polarizations of degree $(d')^2$ with $\gcd(pd', n) = 1$. The theorem follows using Prop. 2.12 and the remark following that proposition. *From now on in this section we suppose that d is a power of p.*

The starting point of our proof of 4.1 is that the irreducibility non-supersingular Newton polygon strata in $\mathcal{A}_{g,1,n}$ implies the existence of points in any irreducible component of a central leaf C in $\mathcal{A}_{g,1,n}$ which belongs to a specified isogeny class in the Newton polygon stratum containing C.

(4.2) Proposition. *Let g, n, d be positive integers as in 4.1. Let $k \supset \mathbb{F}_p$ be an algebraically closed field. Let C be a central leaf in $\mathcal{A}_{g,d,n}$ over k, and let ξ be the symmetric Newton polygon attached to C. Let W^0_ξ be the open Newton polygon stratum in $\mathcal{A}_{g,1,n}$ with symmetric Newton polygon ξ. Let C' be an irreducible component of C. For any point $y = [(B, \mu_y, \iota_y)] \in W^0_\xi(k)$, there exists a point $x = [(A, \lambda_x, \iota_x)] \in C'(k)$ and a quasi-isogeny $\theta \in \mathrm{Hom}(A_x, B_y) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\theta^*(\mu_y) = \lambda_x$.*

Proof. If C is contained in the supersingular locus, this statement is clear. For the rest of the proof we assume C is not contained in the supersingular locus. By the almost product structure explained in 2.9, we obtain finite surjective morphisms $f\colon T \twoheadrightarrow C$ and $g : J \twoheadrightarrow I$ over the central leaf C and the isogeny leaf I respectively, a finite surjective morphism $\Phi : T \times J \twoheadrightarrow W^0_\xi$, and a quasi -isogeny

$$\Theta \in \mathrm{Hom}\left(A_{T\times J}, B_{W^0_\xi} \times_{W^0_\xi,\Phi} (T \times J)\right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

such that $\Theta^*(\lambda_{A_{T\times J}}) = \mu_{B_{T\times J}}$. Here $\left(A_{T\times J}, \lambda_{T\times J}\right)$ is the pull-back to $T \times J$ of the universal polarized abelian scheme over the central leaf C in $\mathcal{A}_{g,d,n}$, $\left(B_{W^0_\xi} \times_{W^0_\xi,\Phi} (T \times J), \mu_{B_{T\times J}}\right)$ is the pull-back by Φ of the universal principally polarized abelian scheme over the open Newton polygon stratum W^0_ξ in $\mathcal{A}_{g,1,n}$.

Let $T' = f^{-1}(C')$, an open subscheme of T. The image $\Phi(T' \times J)$ of $T' \times J$ under Φ is a union of irreducible components of W^0_ξ, hence equal to W^0_ξ because W^0_ξ is irreducible by 3.1. Let $(t,u) \in (T' \times J)(k)$ be a k-point of the pre-image $f^{-1}(y)$ of y under Φ, and let $x = f(t) = (A_x, \lambda_x, \iota_x)$ be the image of t in C'. Let $\theta = \Theta_t$, the fiber of the quasi-isogeny Θ at t, which is a quasi-isogeny from A_x to B_y such that $\theta^*(\mu_y) = \lambda_x$. $\square$

(4.3) Proposition. *Let g, d, n be positive integers as in 4.1. Let C be a leaf in $\mathcal{A}_{g,d,n}$ over an algebraically closed field $k \supset \mathbb{F}_p$, and let $x_1 = [(B_1, \mu_1, \iota_1)]$, $x_2 = [(B_2, \mu_2, \iota_2)] \in C(k)$ be two hypersymmetric k-points C such that there exists a quasi-isogeny θ from B_2 to B_1 with $\theta^*(\mu_2) = \mu_1$. Then x_2 belongs to the prime-to-p Hecke orbit of x_1, i.e. there exists a quasi-isogeny $\psi \in \mathrm{Hom}(B_2, B_1) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^*(\mu_1) = \mu_2$ and the homomorphism $\psi[p^\infty] : B_2[p^\infty] \to B_1[p^\infty]$ induced by ψ is an isomorphism.*

Reduction of 4.1 to 4.3. Let ξ be the symmetric Newton polygon attached to C. By [7, Prop. 4.1], there exists a hypersymmetric point $y \in W_\xi(k)$. Apply Prop. 4.2 to a hypersymmetric point $y \in W_\xi(k)$, one sees that every irreducible component of C contains a k-point x such that there exists a quasi-isogeny $\theta : B_x \to A_y$ with $\theta^*(\lambda_y) = \mu_x$. Prop. 4.3 tells us that the prime-to-p Hecke correspondences operate transitively on the set of irreducible components of C. By 2.13 this implies Thm. 4.1 .

(4.4) We set up notation for a proof of Prop. 4.3 by group theory. Suppose we have a polarized abelian variety (A, λ) over a field K. Let $*_\lambda : \mathrm{End}^0(A) \to \mathrm{End}^0(A)$ be the Rosati involution attached to the polarization λ. We define a linear algebraic group $G = G_{(A)} = G_{(A,\lambda)}$ over $\mathbb{Q}$ by

$$G(R) = \mathrm{U}\left(\mathrm{End}(A) \otimes R, *\right) := \{x \mid x \cdot *(x) = 1 = *(x) \cdot x\}$$

for every commutative $\mathbb{Q}$-algebra R. In other words, $G_{(A,\lambda)}$ is the unitary group attached to the semisimple $\mathbb{Q}$-algebra with involution $(\mathrm{End}^0(A), *_\lambda)$. Clearly a quasi-isogeny $\beta : B \to A$ induces an isomorphism between $G_{(A,\lambda_A)}$ and $G_{(B,\beta^*\lambda_A)}$.

(4.5) Lemma. *Let (A, λ) be a polarized abelian variety over an algebraically closed field $k \supset \mathbb{F}$. Assume that there is an abelian variety B over $\mathbb{F}$ such that $B \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(k)$ is isogenous to A. The group $G = G^{(A,\lambda)}$ is a connected reductive linear algebraic group.* **Proof.** Let α be a quasi-isogeny from $B \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(k)$ to A, and let μ be a polarization on B which is a multiple of $\alpha^*(\alpha)$. Then α induces an isomorphism between the two semisimple algebras with involution, $(\mathrm{End}^0(A), *_\lambda)$ and $(\mathrm{End}^0(B), *_\mu)$. So we may and do assume that $k = \mathbb{F}$. Changing A by an isogeny, we may assume that $A \cong \prod_{1\le j\le r} C_j^{m_j}$, where each C_j is a simple abelian variety over $\mathbb{F}$, each m_j is a positive integer, and $\mathrm{Hom}(C_j, C_{j'}) = (0)$ if $j \neq j'$. There exist polarizations λ_1, λ_r on $C_1, \ldots, C_r$ such that λ is equal to the product polarization $\lambda_1 \times \cdots \times \lambda_r$ on $A = C_1^{m_1} \times \cdots \times A_r^{m_r}$. Let $*_j$ be the Rosati involution on $\mathrm{End}^0(C_j)$ attached to λ_j. We have a product decomposition $(\mathrm{End}^0(A), *_A) = \prod_{1\le j\le r}(\mathrm{End}^0(C_j^{m_j}), *_j)$ of the semisimple algebra with involution $\mathrm{End}^0(A), *_A)$, which induces a product decomposition $G_{(A,\lambda)} = \prod_{1\le j\le r} G_{(C_j^{m_j},\lambda_j)}$. So we may and do assume that $A \cong C^m$ for a simple abelian variety C over $\mathbb{F}$. The group G is isomorphic the unitary group attached to the semisimple algebra with positive involution $(\mathrm{M}_m(D), *)$, where $D = \mathrm{End}^0(C)$ is a division algebra which has a positive definite involution. A $\mathbb{Q}-$algebra of finite rank with a positive definite involution is called an *Albert algebra*. For the classification, by Albert, for more information and notation and for references, see [24], [8], 10.12 – 10.14.

According to Albert's classification, there are four possible types for the division algebra $D = \mathrm{End}^0(A)$, the endomorphism algebra of an abelian variety A, simple over a field K. In every characteristic all four types do occur for some abelian variety over some field (Albert, Shimura, Gerritzen). However, as Tate proved, over a *finite field* only two types can appear:

(III) a totally definite quaternion division algebra over a totally real number field F,

(IV) a division algebra whose center is a totally imaginary quadratic extension L of a totally real number field F.

See [38]. For further results and references see [33], §5.

Write $G_{\overline{\mathbb{Q}}} := G \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$. We list some of the basic properties of G below. The proofs are straight-forward.

- In the case (III) above, $G_{\overline{\mathbb{Q}}}$ is isomorphic to $\mathrm{Sp}_{2m}^{[F:\mathbb{Q}]}$. So G is connected and simply connected.
- In the case (IV) above, the group $G_{\overline{\mathbb{Q}}}$ is isomorphic to $\mathrm{GL}_N^{[F:\mathbb{Q}]}$, where $N = \sqrt{\dim_L(D)}$. The algebraic group G over $\mathbb{Q}$ is connected; its center is the kernel of the norm homomorphism $\mathrm{Nm}_{L/F} : \mathrm{Res}_{L/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. The derived group G^{der} of G is simply connected. The quotient torus G/G^{der} is isomorphic to the torus $\mathrm{Ker}\left(\mathrm{Nm}_{L/F}\right)$ above.

We have shown that the group G in 4.5 is a product of a finite number of algebraic groups over $\mathbb{Q}$ of type (III) or (IV) above. Clearly G is connected. $\square$

(4.6) We explain the definition of a quasi-bitorsor to be used in the proof of 4.3. Let (A, λ) and (B, μ) be polarized abelian varieties over a field K (eventually we will assume all endomorphisms are defined over K, the polarizations are principal, and the polarized abelian varieties are in the same Hecke orbit).

For a commutative $\mathbb{Q}$-algebra R, a symplectic R-isogeny from (A, λ) to (B, μ) is, by definition, an element of $\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} R$ which has an inverse in $\mathrm{Hom}(B, A) \otimes_{\mathbb{Z}} R$ and which respects the polarizations. Define a functor $\mathcal{T} = \mathcal{T}_{(A,B)} = \mathcal{T}_{((A,\lambda),(B,\mu)))}$ on the category of all commutative $\mathbb{Q}$-algebras by requiring that $\mathcal{T}(R)$ is the set of symplectic R-isogenies from (A, λ) to (B, μ), for every commutative $\mathbb{Q}$-algebra R. (We should not confuse the notions of an R-isogeny and of an isogeny defined over R.) The linear group $G_{(A,\lambda)}$ operates naturally on $\mathcal{T}$ on the right by composing arrows in the following pattern $A \to A \to B$. Similarly, the linear group $G_{(B,\mu)}$ operates on $\mathcal{T}$ on the left by composing arrows in the pattern $A \to B \to B$, compatible with the previous right action by $G_{(A,\lambda)}$. This functor $\mathcal{T}$ is representable, either by the empty scheme, or by a $(G_{(B,\mu)}, G_{(A,\lambda)})$-bitorsor over $\mathbb{Q}$.

Proof of 4.3. Let $G_1 = G_{(B_1,\mu_1)}$, $G_2 = G_{(B_2,\mu_2)}$, $\mathcal{T} = \mathcal{T}_{((B_2,\mu_2),(B_1,\mu_1))}$. We know from 4.5 that G_1 and G_2 are connected reductive linear algebraic groups. We also know that $\mathcal{T}(\mathbb{Q}) \neq \emptyset$ by assumption, so $\mathcal{T}$ is a trivial (G_1, G_2)-bi-torsor over $\mathbb{Q}$. As G_1 is connected, G_1 satisfies weak approximation, see [35], 7.3, Theorem 7.7 on page 415, so the G_1-torsor also satisfies weak approximation. In other words, the image of $\mathcal{T}(\mathbb{Q})$ in $\mathcal{T}(\mathbb{Q}_p)$ is dense for the p-adic topology. Since x_1 and x_2 are on the same leaf C, there exists an isomorphism $\psi_p : (B_2, \mu_2)[p^\infty] \xrightarrow{\sim} (B_1, \mu_1)[p^\infty])$. Because B_1 and B_2 are hypersymmetric, $\mathrm{Hom}(B_2, B_1) \otimes_{\mathbb{Z}} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Hom}(B_2[p^\infty], B_1[p^\infty])$, and ψ_p corresponds to a $\mathbb{Q}_p$-point t_p of $\mathcal{T}$ under the above isomorphism. Let U_p be the open subset of $\mathcal{T}(\mathbb{Q}_p)$ consisting of all elements $t'_p \in \mathcal{T}(\mathbb{Q}_p)$ such that t'_p induces an isomorphism from $(B_2, \mu_2)[p^\infty]$ to $(B_1, \mu_1)[p^\infty]$; it is not empty because $t_p \in U_p$. By the weak approximation property of $\mathcal{T}$, see [35], 7.3, Theorem 7.7 on page 415, there exist a an element $t \in \mathcal{T}(\mathbb{Q}) \cap U$ under the natural injection $\mathcal{T}(\mathbb{Q}) \hookrightarrow \mathcal{T}(\mathbb{Q}_p)$. The quasi-isogeny ψ from B_2 to B_1 corresponding to t satisfies the required properties. We have proved 4.3 and 4.1 $\square$

Remark. The assumption that x_1, x_2 are hypersymmetric is used in the proof above to guarantee that U_p is not empty.

(4.7) Theorem. *Let* (A, λ) *be a* principally polarized *abelian variety which is* not supersingular. *Suppose* $A[p^\infty]$ *is a* minimal *p-divisible group. The central stream* $\mathcal{Z}_\xi$ *passing through* $[(A, \lambda)] =: x \in \mathcal{A}_{g,1}$ *is geometrically irreducible.*

Note that 4.7 is a special case of 4.1. We state this special case explicitly because the proof below is completely different from the proof of 4.1 given above.

The proof of 4.7 is divided into three steps.
Step 1. *For every Newton polygon* ξ *the central stream* $\mathcal{Z}_\xi \subset \mathcal{A}_{g,1}$ *is an* EO-stratum:

$$\mathcal{Z}_\xi := \mathcal{C}(x) = S_\varphi, \quad \text{where} \quad \varphi := ((A, \lambda)[p] \mod \cong).$$

Proof of Step 1. The central stream is defined by the fact that it is the central leaf where the p-divisible groups are minimal. For a minimal p-divisible group X over k, we know that $X[p]$ determines the isomorphism class of X, see [31], 1.2; see 2.11. Moreover, a principal quasi-polarization on a minimal p-divisible group is unique, see [29], 3.7. This proves that the usual inclusion $\mathcal{C}(x) \subset S_\varphi$ in this case is an equality. $\square$

Step 2. Use notation as in [26]. *Let* $\varphi = \{\varphi(1), \cdots, \varphi(g)\}$ *be an elementary sequence, and let* $S_\varphi \subset \mathcal{A}_{g,1}$ *be the corresponding* EO-*stratum. Write* $g = 2r$, *respectively* $g = 2r - 1$, *i.e.* $r = \lceil g/2 \rceil$.
Then

$$\varphi(r) = 0 \quad \Longleftrightarrow \quad S_\varphi \subset W_\sigma .$$

Proof of Step 2. Let $N_1 \subset \cdots \subset N_{2g} = N = X[p]$ be a final filtration. Suppose $\varphi(r) = 0$; we see that N_{g+r}/N_r is annihilated by F and by V. Hence X/N_r is superspecial. Hence X is supersingular. We conclude $S_\varphi \subset W_\sigma$.

Define $u(\xi)$ to be the elementary sequence of the minimal p-divisible group with Newton polygon equal to ξ. Suppose ξ is "almost supersingular", i.e. either $\xi = (r, r-1) + (1, 1) +$

$(r-1,1)$ or $\xi = (r, r-1) + (r-1, r)$. Direct computation shows that

$$u(\xi) := \mathrm{ES}(H(\xi)[p]) = \{0, \cdots, \varphi(r-1) = 0, \varphi(r) = 1, \cdots, 1\}.$$

In this case $S_{u(\xi)} \not\subset W_\sigma$. For every φ with $\varphi(r) \neq 0$ we have $\varphi \succ \mathrm{ES}(H(\xi)[p])$. Using [26], Th. 1.3 we conclude

$$\varphi(r) \neq 0 \quad \Longrightarrow \quad S_{u(\xi)} \subset S_\varphi; \quad \text{hence} \quad S_\varphi \not\subset W_\sigma.$$

$\square$

Step 3. End of the proof of 4.7. In [10], Th. 11.5, we see that certain EO-strata are geometrically irreducible. By Step 2 we see that these are exactly the EO-strata not contained in the supersingular locus. Using moreover Step 1 we conclude that $\mathcal{Z}_\xi = S_{u(\xi)}$ is geometrically irreducible for every $\xi \neq \sigma$ and $u(\xi) := \mathrm{ES}(H(\xi)[p])$. This proves Theorem 4.7. $\square$

§5. Monodromy

In this section we show that the p-adic monodromy of a leaf is maximal using the method in [6, § 5] as indicated in [6, 6.3].

Maximality of monodromy is an irreducibility statement relative to the base. We have already seen that every non-supersingular leaf C is irreducible, so the maximality of p-adic monodromy means that certain profinite etale cover $\mathcal{S}$ of C is irreducible, or equivalently every finite quotient cover $\mathcal{S}_b$ of C is irreducible. The covers $\mathcal{S}_b$ are described in 5.13. The point of the proof is to show that the fiber in $\mathcal{S}_b$ above a hypersymmetric point of the leaf C belongs to the same prime-to-pd Hecke orbit, so $\mathcal{S}_b$ is irreducible by 2.13.

This phenomenon that the p-adic monodromy is related to the prime-to-p Hecke correspondences is reminiscent of the product formula as envisioned by H. Hida.

(5.1) Definition. Let K be a field of characteristic $p > 0$, let $\overline{K}$ be an algebraic closure of K, and let K^{perf} be the perfection of K in $\overline{K}$. Let $\mathrm{Gal}_K := \mathrm{Gal}(\overline{K}/K^{\mathrm{perf}})$, naturally isomorphic to $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. Let X be a p-divisible group over K. Let X_0 be a p-divisible group over a finite subfield $\mathbb{F}_q \subset K$ such that

(i) there exists a quasi-isogeny

$$\psi_0 \colon X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K}) \longrightarrow X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$$

(ii) the natural map can_q in the diagram

$$\mathrm{End}^0(X_0) \xrightarrow{\mathrm{can}_q} \mathrm{End}^0\left(X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\mathbb{F})\right) \xrightarrow{\sim} \mathrm{End}^0\left(X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K})\right)$$

is an isomorphism.

Consider the set

$$\mathrm{QIsog}[X_0, X] := \mathrm{QIsog}\left(X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K}), X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right)$$

of quasi-isogenies, which has the following properties:

- $\mathrm{QIsog}[X_0, X]$ has a natural structure as a right $\mathrm{End}^0(X_0)^\times$-torsor and a natural continuous left action by Gal_K.
- The right action of $\mathrm{End}^0(X_0)^\times$ on $\mathrm{QIsog}[X_0, X]$ is compatible with the left Galois action, i.e. ${}^\sigma(\psi \cdot u) = ({}^\sigma\psi) \cdot u$ for all $\sigma \in \mathrm{Gal}_K$, all $\psi \in \mathrm{QIsog}[X_0, X]$ and all $u \in \mathrm{End}^0(X_0)^\times$.

Define a continuous homomorphism $\rho_{X,\psi_0}\colon \mathrm{Gal}_K \longrightarrow \mathrm{End}^0(X_0)^\times$ by

$$ {}^\sigma\psi_0 = \psi_0 \cdot \rho_{X,\psi_0}(\sigma) \qquad \forall\, \sigma \in \mathrm{Gal}_K \ . $$

We call ρ_{X,ψ_0} the *p-adic monodromy homomorphism* of X with respect to ψ_0.

(5.2) Remarks. (1) Suppose that t is a positive integer which is a common multiple of the denominators of the slopes of X. Then for $q = p^t$, there exists a *decent* p-divisible group X_0 over the finite field $\mathbb{F}_q$ which satisfies the requirements (i), (ii); see 5.3.

(2) It is not difficult to see that if X_0, X_1 are two p-divisible groups over $\mathbb{F}_q$ satisfying (i) and (ii), then X_1 is up to isogeny a Galois twist of X_0 via a continuous homomorphism

$$ \chi\colon\ \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \longrightarrow Z\big(\mathrm{End}^0(X_0)\big)^\times $$

where $Z\big(\mathrm{End}^0(X_0)\big)$ is the center of $\mathrm{End}^0(X_0)$. In other words the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on $\mathrm{QIsog}[X_0, X_1]$ via a character χ as above. If in addition X_0 and X_1 are decent, then the character χ has finite image; see 5.3.

(3) Suppose that μ is a polarization on X, and μ_0 is a polarization on X_0, and $\psi_0^*(\mu) = \mu_0$ as polarizations on $X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K})$. Denote by U the subgroup of $\mathrm{End}^0(X_0)^\times$ consisting of all quasi-isogenies from X_0 to itself which preserve the polarization μ_0. Let

$$ \mathrm{UQIsog}[X_0, X] := \mathrm{UQIsog}\left(X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K}), X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right) $$

be the set of all elements ψ in $\mathrm{QIsog}\left(X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K}), X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right)$ such that $\psi^*(\mu) = \mu_0$. Then $\mathrm{UQIsog}[X_0, X]$ is stable under both the left action by Gal_K and the right action by the subgroup U of $\mathrm{End}^0(X_0)^\times$. So the p-adic monodromy homomorphism $\rho_{X,\psi_0}\colon \mathrm{Gal}_K \to \mathrm{End}^0(X_0)^\times$ factors through the subgroup $U \subset \mathrm{End}^0(X_0)^\times$.

(4) If ψ_0 is an isomorphism from $X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K})$ to $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$, then

$$ \rho_{X,\psi_0}(\mathrm{Gal}_K) \subseteq \mathrm{Aut}(X_0)\,. $$

Similarly in the situation of (2) above, if ψ_0 is an isomorphism from $(X_0, \mu_0) \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\overline{K})$ to $(X, \mu) \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$, then

$$ \rho_{X,\psi_0}(\mathrm{Gal}_K) \subseteq \mathrm{Aut}(X_0, \mu_0)\,. $$

(5) Suppose that X_2 is a p-divisible group over $\mathbb{F}_q$, ψ_2 is an element of $\mathrm{QIsog}[X_2, X]$, and δ is the quasi-isogeny from $X_2 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\mathbb{F})$ to $X_0 \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\mathbb{F})$ such that. Then

$$ \rho_{X,\psi_2}(\sigma) = \delta^{-1} \circ \rho_{X,\psi_0}(\sigma) \circ {}^\sigma\delta \qquad \forall \sigma \in \mathrm{Gal}_K \tag{†} $$

where $\psi_2 = \psi_0 \circ \delta$. In particular if $\psi_1 = \psi \circ \gamma$, for an element $\gamma \in \mathrm{End}^0(X_0)^\times$, then

$$\rho_{X,\psi_1}(\sigma) = \gamma^{-1} \cdot \rho_{X,\psi_0}(\sigma) \cdot \gamma \qquad \forall\, \sigma \in \mathrm{Gal}_K \ .$$

The formula (†) also shows that if $K \supset \mathbb{F}$, then the p-adic monodromy homomorphism for a p-divisible group X is unique up to conjugation.

(6) Suppose that $\beta : X \to X'$ is a quasi-isogeny of p-divisible groups over K, and $\psi' = \beta \circ \psi$. Then $\rho_{X',\psi'} = \rho_{X,\psi}$.

(7) From (1) and (4) above, we see that the p-adic monodromy homomorphism ρ_X attached to a p-divisible group is unique up to
(a) conjugation, and
(b)multiplication by a continuous homomorphism $\chi : \mathrm{Gal}_K \longrightarrow Z(\mathrm{End}^0(X))^\times$.

(5.3) Remark. (1) Recall that a p-divisible group Y over a perfect field $L \supset \mathbb{F}_p$ is said to be *decent* for if its associated isocrystal is generated by elements v_i satisfying an equation of the form $F^t v = p^{r_i} v$ with $t, r_i \in \mathbb{Z}_{>0}$; see [36, 2.13]. The group $G_{m,n}$ in 2.4 is a decent p-divisible group over $\mathbb{F}_p$; its base change to $\mathbb{F}_{p^{m+n}}$ satisfies requirement (i) in 5.1.

(2) If Y_1, Y_2 are decent p-divisible groups over a perfect field $K \subset \mathbb{F}_p$, then there exists a finite extension field L of K such that the canonical map

$$\mathrm{Hom}(Y_1 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L), Y_2 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \mathrm{Hom}(Y_1 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K}), Y_2 \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K}))$$

is an isomorphism.

(3) From (2) above and 5.2 (1) and (4), we see that if we require the "base quasi-isogeny" ψ_0 in definition 5.1 to be decent, then the p-adic monodromy homomorphism attached to a p-divisible group X is unique up to
(a) conjugation, and
(b) multiplication by a homomorphism $\chi : \mathrm{Gal}_K \longrightarrow Z(\mathrm{End}^0(X))^\times$ with finite image.

(5.4) We discuss a variant of Definition 5.1 on p-adic monodromy in the case of a completely slope divisible p-divisible group over an integral base $\mathbb{F}_p$-scheme, and also in the case of a p-divisible group with constant Newton polygon over a noetherian normal integral $\mathbb{F}_p$-scheme.

Let $0 = Y_0 \subset Y_1 \subset \cdots \subset Y_m = Y$ be the slope filtration of a completely slope divisible p-divisible group $Y \to S$ with respect to natural numbers $t \geq r_1 > r_2 > \cdots > r_m \geq 0$ as in 2.15, where S is an integral $\mathbb{F}_q$ scheme, $q = p^t$. The p-adic monodromy homomorphism for $Y \to S$ factors through the fundamental group $\pi_1(S)$ of S, and can be computed via a projective system of finite etale coverings of S. We follow the notation in 2.15.

For every geometric point $\bar{s}$ of S, we get homomorphisms

$$\rho_{G_i,b,\bar{s}} \colon \pi_1(S, \bar{s}) \longrightarrow \mathrm{Aut}\left((\Gamma_{i,b})_{\bar{s}}\right)$$

from the fundamental group of S to the group of automorphisms of the commutative bialgebra $(\Gamma_{i,b})_{\bar{s}}$ over $\mathbb{F}_q$. The homomorphisms $\rho_{G_i,b,\bar{s}}$ are compatible if we fix i and let b vary. Taking the limit, we get homomorphisms

$$\rho_{G_i,\bar{s}}\colon \pi_1(S,\bar{s}) \longrightarrow \varprojlim_b \operatorname{Aut}\left((\Gamma_{i,b})_{\bar{s}}\right) .$$

Notice that the target $\varprojlim_b \operatorname{Aut}\left((\Gamma_{i,b})_{\bar{s}}\right)$ of the homomorphism $\rho_{G_i,\bar{s}}$ is naturally isomorphic to the opposite of the group of automorphisms of $G_i \times_S \bar{s}$. Let

$$\rho_{Y,\bar{s}}\colon \pi_1(S,\bar{s}) \longrightarrow \prod_{i=1}^{m} \varprojlim_b \operatorname{Aut}\left((\Gamma_{i,b})_{\bar{s}}\right)$$

be the product of the homomorphisms $\rho_{G_i,\bar{s}}$. The composition of $\rho_{Y,\bar{s}}$ with the inverse map on $\pi_1(S,\bar{s})$ can be regarded as a homomorphism from $\pi_1(S,\bar{s})$ to the group of automorphisms $\operatorname{Aut}(Y_{\bar{s}})$ of the geometric fiber $Y_{\bar{s}}$ of Y; see 2.16 and 2.17. We denote the composition also by $\rho_{Y,\bar{s}}$ if no confusion is possible.

Suppose that $X \to S$ is a p-divisible group with constant Newton polygon over a noetherian normal integral $\mathbb{F}_p$-scheme S. By [34, Thm. 2.1], there exists a completely slope divisible p-divisible group $Y \to S$ and a quasi-isogeny $\phi\colon Y \to X$ over S. Let $\bar{s}$ be a geometric point of S. The quasi-isogeny $\phi_{\bar{s}}$ induces an injection from $j_{\phi,\bar{s}}\colon \operatorname{Aut}(Y_{\bar{s}})$ to $\operatorname{End}^0(X_{\bar{s}})^\times$. Let $\rho_{X,\phi,\bar{s}}$ be the composition of $\rho_{Y,\bar{s}}$ with $j_{\phi,\bar{s}}$; we call it the *p-adic monodromy* attached to $Y \xrightarrow{\phi} X \to S$ with base point $\bar{s}$. It is not difficult to check that the homomorphism

$$\rho_{X,\phi,s}\colon \pi_1(X,\bar{s}) \to \operatorname{End}^0(X_{\bar{s}})^\times$$

is independent of the choice of quasi-isogeny ϕ, so the subscript "ϕ" can be dropped from the notation $\rho_{X,\phi,s}$.

(5.5) Remarks. (1) The construction/definition of $\rho_{Y,\bar{s}}$ is independent of the choice of $q = p^t$ in the following sense. Suppose that $\mathcal{O}_S$ contains $\mathbb{F}_{q^a}$ for some positive integer a. Then $C_{i,b,q^a} = C_{i,b,q} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^a}$ for all $i = 1, \ldots, m$ and all $b \geq 1$. Consequently the homomorphisms $\rho_{Y,q,\bar{s}}$ and $\rho_{Y,q^a,\bar{s}}$ defined using q and q^a respectively are equal, as homomorphisms from $\pi_1(S,\bar{s})$ to $\operatorname{Aut}(Y_{\bar{s}})$.

(2) The construction/definition of the p-adic monodromy homomorphism in 5.4 coincides with Definition 5.1 up to conjugation. Let S be an integral scheme with function field K. Let η be the generic point of S, and let $\bar{\eta}$ be the geometric generic point of S. Let $Y \to S$ be a completely p-divisible group as in 5.4. For $i = 1, \ldots, m$, let X_i be the p-divisible group over $\mathbb{F}_q$ with $X_i[p^b] = \operatorname{Spec}\left((\Gamma_{i,b})_{\bar{\eta}}\right)$ over $\mathbb{F}_q$; each X_i is decent by construction. Let $X = X_1 \times_{\operatorname{Spec}(\mathbb{F}_q)} \cdots \times_{\operatorname{Spec}(\mathbb{F}_q)} X_m$. The theory of Φ-etale part in [41] defines an isomorphism

$$\psi\colon X \times_{\operatorname{Spec}\mathbb{F}_q} \bar{\eta} \xrightarrow{\sim} Y_{\bar{\eta}}\,.$$

The p-adic monodromy homomorphism

$$\rho_{Y_\eta,\psi_0}\colon\ \operatorname{Gal}(K^{\mathrm{a}}/K^{\mathrm{perf}}) \longrightarrow \operatorname{End}^0(X)^\times = \operatorname{End}^0(X_1)^\times \times \cdots \times \operatorname{End}^0(X_m)^\times$$

defined in 5.1 is compatible with the homomorphism

$$\rho_{Y_\eta,\psi}\colon \pi_1(S,\bar\eta) \to \mathrm{Aut}(X) = \mathrm{Aut}(X_1)\times\cdots\times\mathrm{Aut}(X_m)$$

via the natural surjection $\mathrm{Gal}(K^{\mathrm{a}}/K^{\mathrm{perf}}) \twoheadrightarrow \pi_1(S,\bar\eta)$ and the inclusion $\mathrm{Aut}(X)\hookrightarrow \mathrm{End}^0(X)^\times$. To see this, it suffices to check for the case when $m=1$, because $Y_{\bar\eta}$ splits canonically into a product $(Y_1)_{\bar\eta}\times(Y_2/Y_1)_{\bar\eta}\cdots\times(Y_m/Y_{m-1})_{\bar\eta}$, which induces a splitting of ψ as a product $\psi_1\times\cdots\psi_m$, and the $\rho_{Y_\eta,\psi}$ is the product of the p-adic monodromy homomorphisms $\rho_{(Y_i/Y_{i-1})_{\eta,\psi_i}}$. Assume that Y is isoclinic and completely p-divisible, let $C(b)$ be the etale sheaf commutative $\mathbb{F}_q$-bialgebras over S attached to $Y[p^b]$ as before. The theory of Φ-etale part shows that the projective system $\varprojlim_b \mathcal{I}som\,(C(b), C(b)_{\bar\eta}\times S)$ of sheaves on S_{fppf} is isomorphic to the projective system $\varprojlim_b \mathcal{I}som\,\left(X[p^b]\times_{\mathrm{Spec}(\mathbb{F}_q)} S, Y[p^b]\right)$ of sheaves. The claim on compatibility follows.

(5.6) Theorem. *Let $k\supset\mathbb{F}_p$ be an algebraically closed field. Let n be a positive integers with $n\ge 3$ with $\gcd(n,pd)=1$. Let C be a leaf over k in $\mathcal{A}_{g,d,n}$, and let $x=[(A_x,\lambda_x,\eta_x)]$ be a k-point on C. Then the p-adic monodromy homomorphism*

$$\rho_{(A,\lambda)[p^\infty]/C,x}\colon \pi_1(C,\,x) \longrightarrow \mathrm{Aut}\,((A_x,\lambda_x)[p^\infty])$$

for the polarized p-divisible group attached to the universal polarized abelian scheme (A,λ) over C is surjective.

Write $d=d'{\cdot}m$, where p does not divide m and d' is a power of p. Suppose that this theorem has been proved with polarizations of degree $(d')^2$. Then the theorem follows using Proposition 2.13 and the remarks following that theorem. *From now on we suppose that d is a power of p.*

Remark. It suffices to prove 5.6 for one k-point $x\in C(k)$. In the following we will take a hypersymmetric point $x_0\in C(k)$ and prove 5.6 in the case $x=x_0$.

We will first prove Theorem 5.6 in the special case 5.9 when $A_0[p^\infty]$ is completely slope divisible. The basic argument is more transparent and not burdened with extra technical complications such as 5.11.

(5.7) Let k be an algebraically closed field as before. Let $C\subset\mathcal{A}_{g,d,n}$ be a leaf over k such that the restriction to C of the p-divisible group attached to the universal abelian scheme is completely slope divisible, or equivalently there exists a point $x=[(A_x,\lambda_x,\iota_x)]$ on C such that $A_x[p^\infty]$ is completely slope divisible. Assume in addition that C is not contained in the supersingular locus. By Thm. 4.1, the leaf C contains a k-point $x_0=[(A_0,\lambda_0,\iota_0)]$ which is hypersymmetric. Moreover C is smooth and irreducible. It is convenient to use the definition of p-adic monodromy explained in 5.4, because the p-divisible group $A[p^\infty]\to C$ is completely slope divisible. Let

$$0=Y_0\subset Y_1\subset\cdots\subset Y_m=A[p^\infty]$$

be the slope filtration of $A[p^\infty]$ over C. Let

$$\rho_{A[p^\infty]/C,x_0}\colon \pi_1(\,C,x_0) \longrightarrow \mathrm{Aut}((A_0,\lambda_0)[p^\infty])$$

be the p-adic monodromy homomorphism explained in 5.4. Here the polarization λ of the abelian scheme $A \to C$ induces a polarization of the p-divisible group $A[p^\infty] \to C$, again denoted by λ, which is compatible with the slope filtrations on $A[p^\infty] \to C$ and on the dual of $A[p^\infty] \to C$. Hence the p-adic monodromy homomorphism factors through the subgroup $\mathrm{Aut}((A_0, \lambda_0)[p^\infty])$ of $\mathrm{Aut}(A_0[p^\infty])$.

Let $L_{\mathbb{Z}_p} := \mathrm{Aut}((A_0, \lambda_0)[p^\infty])$, and denote by $L_{\mathbb{Z}/p^b\mathbb{Z}}$ the image of $L_{\mathbb{Z}_p}$ in the finite quotient group $\mathrm{Aut}((A_0, \lambda_0)[p^b])$ for positive integers b. Clearly we have

$$L_{\mathbb{Z}_p} = \varprojlim_b L_{\mathbb{Z}/p^b\mathbb{Z}}\,,$$

the projective limit of the finite groups $L_{\mathbb{Z}/b\mathbb{Z}}$, where the transition maps in the projective system are all surjective. The p-adic monodromy homomorphism

$$\rho_{A[p^\infty]/C,x_0} \colon \pi_1(C, x_0) \longrightarrow \mathrm{Aut}((A_0, \lambda_0)[p^\infty])$$

comes from a right etale $L_{\mathbb{Z}_p}$-torsor $\mathcal{T}$ over C, constructed from the projective system

$$\left(\prod_{i=1}^{m} \Gamma_{i,b}\right)_b$$

of etale covers of C, analogous to the "reduction of the structural group" procedure. Put it differently, we have a projective system of finite etale $L_{\mathbb{Z}/p^b\mathbb{Z}}$-torsors $\mathcal{T}_b$ over C such that $\mathcal{T} = \varprojlim_b \mathcal{T}_b$ and $\prod_{i=1}^m \Gamma_{i,b}$ is the contraction product $\mathcal{T}_b \times^{L_{\mathbb{Z}/p^b\mathbb{Z}}} (\Gamma_{i,b})_{x_0}$, i.e. the finite etale cover of C attached to the $L_{\mathbb{Z}/p^b\mathbb{Z}}$-torsor $\mathcal{T}_b$ and the natural action of $\mathcal{T}_b$ on the fiber $(\Gamma_{i,b})_{x_0}$ of $\Gamma_{i,b}$ at x_0. By construction, for each $b \geq 1$ the composition of the homomorphism $\pi_1(C, x_0) \to L_{\mathbb{Z}/pbn\mathbb{Z}}$ associated to the $L_{\mathbb{Z}/p^b\mathbb{Z}}$ and the inclusions

$$L_{\mathbb{Z}/p^b\mathbb{Z}} \hookrightarrow \mathrm{Aut}(A_0[p^b]) = \prod_{i=1}^{m} \mathrm{Aut}\left((\Gamma_{i,b})_{x_0}\right)$$

is equal to the homomorphism attached to the family of etale covers $(\Gamma_{i,b})_{i=1,\ldots,m}$. The inverse limit of the image in $\mathrm{Aut}(A_0[p^b])$ of this homomorphism $\pi_1(C, x_0) \to \mathrm{Aut}(A_0[p^b])$, as b goes to ∞, is equal to the image of the p-adic monodromy homomorphism $\rho_{(A,\lambda)[p^\infty]/C,x_0}$. The natural projections $\mathcal{T}_{b+1} \to \mathcal{T}_b$ are surjective, and $\mathcal{T}$ is the projective limit of the $\mathcal{T}_b$'s.

The set of k-points of the $L_{\mathbb{Z}_p}$-torsor $\mathcal{T}$ can be canonically identified with the set of isomorphism classes of quadruples $(B, \lambda_B, \iota, \psi)$, where $[(B, \lambda_B, \psi)]$ is an k-point of C, in particular ι is a symplectic level-n structure of the principally polarized abelian variety (B, λ), and

$$\psi \colon (A_0, \lambda_0)[p^\infty] \xrightarrow{\sim} (B, \lambda_B)[p^\infty]$$

is an isomorphism. The set of k-points of the $L_{\mathbb{Z}/p^b\mathbb{Z}}$-torsor $\mathcal{T}_b$ is the set of isomorphism classes of quadruples $[(B, \lambda_B, \iota, \psi_b)]$, where $[(B, \lambda_B, \iota)]$ is an k-point of C, and $\psi_b \colon (A_0, \lambda_0)[p^b] \xrightarrow{\sim} (B, \lambda_B)[p^b]$ is an isomorphism which can be lifted to an isomorphism $\psi \colon (A_0, \lambda_0)[p^\infty] \xrightarrow{\sim} (B, \lambda_B)[p^\infty]$ of polarized p-divisible groups.

More generally, for a k-scheme S, the set of S-points of the $L_{\mathbb{Z}/p^b\mathbb{Z}}$-torsor $\mathcal{T}_b$ can be described similarly. It is the set of isomorphism classes of quadruples $[(B, \lambda_B, \iota, \psi_b)]$, where

$[(B, \lambda_B, \iota)]$ is an S-point of C as before, so that $B \to S$ is a g-dimensional abelian scheme, λ_B is a polarization on $B \to S$ of degree d, ι is a level-n structure for $B \to S$, and all geometric fibers of $(B, \lambda_B)[p^\infty] \to S$ are isomorphic to $(A_0, \lambda_0)[p^\infty]$ after suitable extension of base fields. The fourth element ψ_b in the above quadruple is a an isomorphism

$$\psi_b\colon \prod_{i=1}^{m} (X_i/X_{i-1})[p^b] \times_{\mathrm{Spec}(k)} S \xrightarrow{\sim} \prod_{i=1}^{m} (Y_i/Y_{i-1})[p^b] \times_C S$$

which respects the polarizations induced by λ_0 and λ_B respectively, such that for each $N \geq b$ the isomorphism ψ_b over S can be lifted etale locally to a polarization-preserving isomorphism

$$\psi_N\colon \prod_{i=1}^{m} (X_i/X_{i-1})[p^N] \times_{\mathrm{Spec}(k)} S \xrightarrow{\sim} \prod_{i=1}^{m} (Y_i/Y_{i-1})[p^N] \times_C S\,.$$

In the above $0 \subset X_1 \subset \cdots \subset X_m = A_0[p^\infty]$ denotes the slope filtration on $A_0[p^\infty]$, and $0 \subset Y_1 \subset \cdots \subset Y_m = A[p^\infty]$ is the slope filtration on the completely slope divisible p-divisible group $A[p^\infty] \to C$. Notice that the base change to S of the slope filtration for $A[p^\infty] \to C$ is the slope filtration for $B[p^\infty] \to S$.

Our goal is to prove a strong version of the maximality of the p-adic monodromy homomorphism $\rho_{A[p^\infty]/C,x_0}$, that its image is equal to $\mathrm{Aut}\left((A_0, \lambda_0)[p^\infty]\right)$. This is equivalent to the statement that the $L_{\mathbb{Z}/p^b\mathbb{Z}}$-torsor $\mathcal{T}_b$ is geometrically irreducible for every $b \geq 1$.

(5.8) Notation as in 5.7. To prove that $L_{\mathbb{Z}/p^b\mathbb{Z}}$-torsor $\mathcal{T}_b$ is irreducible, we will use the prime-to-p Hecke correspondences on $L_{\mathbb{Z}/p^b}$. These correspondences come from the action of the locally compact group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ on the prime-to-p tower $\left(\mathcal{T}_{b;c}\right)_c$ defined below, where c runs through all positive multiples of n such that $\gcd(c, p) = 1$, For each positive integer multiple c of n not dividing p, let $C_c := C \times_{\mathcal{A}_{g,d,n}} \mathcal{A}_{g,d,c}$, and let $\mathcal{T}_{b;c} := \mathcal{T}_b \times_C C_c$. Thus $\mathcal{T}_{b;c}$ is a torsor over $\mathcal{T}_b$ for the group $\mathrm{Sp}_{2g}(\mathbb{Z}/c\mathbb{Z})(n)$, the group of all elements of $\mathrm{Sp}_{2g}(\mathbb{Z}/c\mathbb{Z})$ which are congruent to Id modulo n. Let $\mathcal{T}_b^{\sim}$ be the projective limit of the tower $\left(\mathcal{T}_{b;c}\right)_c$.

The action of the group $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$ on the prime-to-p tower $C^{\sim} := \left(C \times_{\mathcal{A}_{g,d,n}} \mathcal{A}_{g,d,c}\right)_c$ lifts to an action on $\mathcal{T}_b^{\sim}$, described below.

First we describe the set of S-valued points of $\mathcal{T}_b^{\sim}$ for any k-scheme S. It consists of isomorphism classes of all quadruples $(B, \lambda_B, \tilde{\iota}, \psi_b)$, where

- $(B, \lambda_B) \to S$ is a g-dimensional polarized abelian scheme over S up to prime-to-p quasi-isogenies, $\deg(\lambda_B) = d^2$,
- $\tilde{\iota}\colon \mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}) \times S \xrightarrow{\sim} \underline{\mathrm{H}}_1(B/S, \mathbb{A}_f^{(p)})$ is a symplectic isomorphism such that the image of $[(B, \lambda_B, \tilde{\iota})]$ in $\mathcal{A}_{g,d,n}$ factors through $C \hookrightarrow \mathcal{A}_{g,d,n}$, and
- $\psi_b = \left(\psi_{b,i}\right)_{1\leq i\leq m}$ is a family of isomorphisms $\psi_{b,i}$, $i = 1, \ldots, m$ as in 5.7, so that each $\psi_{b,i}$ is an isomorphism $(X_i/X_{i-1})[p^b] \times_{\mathrm{Spec}(k)} S \xrightarrow{\sim} (Y_i/Y_{i-1})[p^b] \times_C S$ such that for each $N \geq b$ $\psi_{b,i}$ can be extend etale locally to an isomorphism $(X_i/X_{i-1})[p^N] \times_{\mathrm{Spec}(k)} S \xrightarrow{\sim} (Y_i/Y_{i-1})[p^N] \times_C S$.

An isomorphism between two quadruples $(B, \lambda_B, \tilde{\iota}, \psi_b)$ and $(B', \lambda_{B'}, \tilde{\iota}', \psi'_b)$ is a prime-to-p isogeny β from B to B' which sends $\tilde{\iota}$ to $\tilde{\iota}'$ and ψ_b to ψ'_b

Under the above description of points of the projective limit $\mathcal{T}_b^{\sim}$, the symplectic group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\,,\,\rangle)$ operates on the right of on $\mathcal{T}_b^{\sim}$. Suppose that γ is an element of the symplectic group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\,,\,\rangle)$ and $[(B, \lambda_B, \iota^{\sim}, \psi_b)]$, is a point of $\mathcal{T}_b^{\sim}$, then

$$[(B, \lambda_B, \tilde{\iota}, \psi_b)] \cdot \gamma = [B, \lambda_B, \tilde{\iota} \circ \gamma, \psi_b)]\,.$$

As usual, the action of the group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\,,\,\rangle)$ on $\mathcal{T}_b^{\sim}$ induces a family of algebraic correspondences on $\mathcal{T}_b$. Recall that A_0 is a hypersymmetric abelian variety; hence the natural map $\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}(A_0[p^\infty])$ is an isomorphism. Let $*_0$ be the Rosati involution attached to the polarization λ_0 of A_0. Let H be the unitary group over $\mathbb{Q}$ attached to the semisimple algebra $\mathrm{End}^0(A_0)$ with involution $*_0$; it is the linear algebra group over $\mathbb{Q}$ such that for every commutative $\mathbb{Q}$-algebra R the set $H(R)$ of R-valued points of H consists of all elements $x \in (\mathrm{End}^0(A_0) \otimes_{\mathbb{Q}} R)^\times$ such that $*_0(x) \cdot x = x \cdot *_0(x) = 1$ in $(\mathrm{End}^0(A_0) \otimes_{\mathbb{Q}} R)^\times$. In other words $H(R)$ is the group of symplectic R-isogenies from A_0 to A_0. We abuse the notation and denote by $H(\mathbb{Z}_p)$ the subgroup of $H(\mathbb{Q}_p)$ consists of symplectic $\mathbb{Q}_p$-isogenies which are automorphisms of $A_0[p^\infty]$. Since A_0 is hypersymmetric, the group $H(\mathbb{Z}_p)$ is canonically isomorphic to the group $\mathrm{Aut}((A_0, \lambda)[p^\infty])$ of automorphisms of the polarized p-divisible group $(A_0, \lambda_0)[p^\infty]$.

(5.9) Theorem. *Notation as in* 5.7 *and* 5.8. *In particular* $k \supset \mathbb{F}_p$ *is an algebraically closed field,* $\gcd(n, p) = 1$, $n \geq 3$, *and* C *is a leaf in* $\mathcal{A}_{g,d,n}$ *over* k *such that the* p*-divisible of the universal abelian scheme* $A \to C$ *is completely slope divisible.*

(i) *For any two* k*-points* y_1, y_2 *of* $\mathcal{T}_b$ *above* x_0, *there exists an element* δ *in the group* $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\,,\,\rangle)$ *such that* y_1 *belongs to the image of* y_2 *under the algebraic correspondence induced by* δ. *In particular the Hecke correspondences induced by elements of* $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\,,\,\rangle)$ *operates transitively on the set* $\Pi_0(\mathcal{T}_b)$ *of irreducible components of* $\mathcal{T}_b$ *for every positive integer* b.

(ii) *The* $L_{\mathbb{Z}/p^n\mathbb{Z}}$*-torsor* $\mathcal{T}_b$ *over* k *is irreducible for every* $b \geq 1$. *Therefore the* p*-adic monodromy homomorphism*

$$\rho_{(A,\lambda)[p^\infty]/C, x_0} \colon \pi_1(C, x_0) \longrightarrow \mathrm{Aut}((A_0, \lambda_0)[p^\infty]$$

attached to the leaf C *in* $\mathcal{A}_{g,d,n}$ *over* k *is surjective.*

Proof. The first assertion in statement (ii) follows from (i) by 2.13. by the argument of [4, Prop. 4.4]; see also the proof of Prop. 5.5 in [6]. Of key importance to this argument is the fact that the group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\,,\,\rangle)$ has no proper subgroup of finite index; see [4, Lemma 3.1]. The second part of (ii) follows from the first part.

As indicated in [6, 6.3.5], the methods in [6, §5] can be used to give a proof of statement (i). Write $y_1 = [(A_0, \lambda_0, \iota, \psi_b)]$ $y_2 = [(A_0, \lambda_0, \iota, \psi'_b)]$ in the notation of 5.7, where the ι and ι' are symplectic level-n structures on (A_0, λ_0), and ψ_b, ψ'_b are automorphisms of $(A_0, \lambda_0)[p^b]$ induced by automorphisms of $(A_0, \lambda_0)[p^\infty]$. Pick a point $z = [(A_0, \lambda_0, \tilde{\iota}, \psi_b)]$ of $\mathcal{T}^{\sim}$ above y_1. Then

$z' = [(A_0, \lambda_0, \tilde{\iota}, \psi_b')]$ is a point above y_2. Denote by $H(\mathbb{Z}_{(p)})$ the intersection $H(\mathbb{Q}) \cap H(\mathbb{Z}_p)$ in $H(\mathbb{Q}_p)$. Each element $\gamma \in H(\mathbb{Z}_{(p)})$ can be regarded as a prime-to-p symplectic isogeny from A_0 to itself. Moreover γ induces an isomorphism from $(A_0, \lambda_0, \tilde{\iota}, \psi_b)$ to $(A_0, \lambda_0, \gamma^{(p)} \circ \tilde{\iota}, \gamma_{p,b} \circ \psi_b)$, where $\gamma^{(p)}$ is the automorphism of $\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)})$ induced by γ, and $\gamma_{p,b}$ is the automorphism of $(A_0, \lambda_0)[p^b]$ induced by γ. By the weak approximation theorem applied to the connected linear algebraic group H over $\mathbb{Q}$, there exists an element $\xi \in H(\mathbb{Z}_{(p)})$ such that $\xi_{p,b} \circ \psi_b = \psi_b'$. Let δ be the element of $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\, ,\, \rangle)$ such that $\xi^{(p)} \circ \tilde{\iota} = \tilde{\iota} \circ \delta$. Then we have

$$z = [(A_0, \lambda_0, \xi^{(p)} \circ \tilde{\iota}, \psi_b')] = z' \cdot \delta\,,$$

so y_1 is an image of y_2 under the prime-to-p Hecke correspondence induced by δ. We have proved that the image of the p-adic monodromy for C is equal to $L_{\mathbb{Z}}$. □

We formulate 5.11 and the related Lemma 5.10, convenient for the exposition of the proof of 5.6.

(5.10) Lemma (Zink). *Let Y be a p-divisible group over a field $K \supset \mathbb{F}_p$ and let L be a field containing K. Then there exist isogenies $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ with X, Z completely slope divisible, with the following properties*

(1) *Every isogeny $\alpha'\colon X' \to Y$ of p-divisible groups with X' completely slope divisible factors through α, i.e. there exist an isogeny $\delta\colon X' \to X$ such that $\alpha' = \alpha \circ \delta$. Similarly every isogeny $\beta' : Y \to Z'$ with Z' completely slope divisible factors through β, i.e. there exists an isogeny $\epsilon\colon Z \to Z'$ such that $\beta' = \epsilon \circ \beta$. These properties characterize α and β respectively. Moreover these properties hold after extension of base fields $K \to L$.*

(2) *The isogeny α induces an inclusion*

$$\mathrm{End}(X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)) \supset \mathrm{End}(Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))\,;$$

the isogeny β induces an inclusion

$$\mathrm{End}(Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)) \subset \mathrm{End}(Z \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L))\,.$$

Proof. The construction of $\beta\colon Y \to X$ is implicit in the proof of [41, Thm. 7]; it is given by an inductive procedure. We recall briefly the construction. In Lemma 9 of [41], a "formula" for β is given in terms of Dieudonné modules when the base field K is perfect and Y is isoclinic. This construction in terms of Dieudonné modules can be reformulated in terms of the "small image" of a suitable homomorphism between p-divisible groups, which settles the case when Y is isoclinic. An induction on the length of the slope filtration finishes the proof. See [41] for details. □

(5.11) Proposition (Zink). *Notation as in 5.6. There exists an abelian scheme B over the leaf C and a p-primary isogeny $\beta\colon A \to B$ over C with the following properties.*

(i) *The p-divisible group $B[p^\infty] \to C$ is completely slope divisible.*

(ii) *For every geometric point $x \in C(\Omega)$, the isogeny $\beta_x\colon A_x \to B_x$ induces an inclusion*

$$\mathrm{End}(A_x[p^\infty]) \subseteq \mathrm{End}(B_x[p^\infty])\,.$$

In particular β_x induces an inclusion $\mathrm{End}(A_x)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)} \subseteq \mathrm{End}(B_x)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$.

Proof. Clearly it suffices to establish the existence of an isogeny $\beta[p^\infty]\colon A[p^\infty] \to B[p^\infty]$ between p-divisible groups satisfying (i) and (ii). Apply 5.10 to the generic fiber of $A[p^\infty] \to C$, one obtains the desired p-primary isogeny $\beta_{K(C)} : A[p^\infty]_{K(C)} \to B[p^\infty]_{K(C)}$ over the spectrum of the function field $K(C)$ of C, which satisfies the properties in 5.10. Prop. 14 in [41] and the proof of Thm. 7 in [41] show that the p-divisible group $B[p^\infty]$ over $K(C)$ extends to a completely slope divisible p-divisible group $B[p^\infty]$ over C, and the isogeny over $K(C)$ extends to C.

We know that the p-divisible groups $A[p^\infty]$ over C is geometrically fiberwise constant by assumption, while the p-divisible group $B[p^\infty]$ over C is geometrically fiberwise constant because it is completely slope divisible. Because C is irreducible, we conclude that any isogeny between $A[p^\infty]$ and $B[p^\infty]$ is also geometrically fiberwise constant. This implies that the same procedure, which produced the isogeny $\beta_{K(C)}$ over the function field $K(C)$ from the p-divisible group $A[p^\infty]$ over $K(C)$, also produces the fiber β_y of β at y when applied to the fiber $A[p^\infty]_y$ of $A[p^\infty]$, for every geometric point $y \in C(k)$. ∎

(5.12) Choose notation as in 5.7 and 5.8. Let $\beta\colon A \to B$ be as in Prop. 5.11, and let $\beta_0\colon A_0 \to B_0$ be the fiber of β over x_0. Let μ be a polarization of B over C such that $\beta^*(\mu)$ is a multiple of λ. Let μ_0 be the fiber of μ at x_0, a polarization of B_0, and let ι_0' be the level-n structure on B_0 induced by β_0. Write $x_0 = (A_0, \lambda_0, \iota_0)$, $y_0 = (B_0, \lambda_0', \iota_0')$. Let

$$\rho_{(B,\lambda')[p^\infty]}\colon \pi_1(C, x_0) \to \mathrm{Aut}((B_0, \lambda_0')[p^\infty])$$

be the p-adic monodromy homomorphism attached to $(B, \lambda') \to C$.

We use the definition of p-adic monodromy in 5.4. By the property of β stated in Prop. 5.11, the isogeny β_0 identifies $\mathrm{Aut}((A_0,\lambda_0)[p^\infty])$ with a subgroup of $\mathrm{Aut}(B_0,\mu_0)$. We know that the image of $\rho_{(B,\mu)}$ is contained in the subgroup $\mathrm{Aut}((A_0,\lambda_0)[p^\infty])$ of $\mathrm{Aut}(B_0,\mu_0)$. We must show that image of $\rho_{(B,\mu)}$ is equal to $\mathrm{Aut}((A_0,\lambda_0)[p^\infty])$.

Let H be the unitary group over $\mathbb{Q}$ attached to $(\mathrm{End}^0(A_0), *_{\lambda_0}) = (\mathrm{End}^0(B_0), *_{\mu_0})$. Let $L_{\mathbb{Z}_p} = L(A_0,\lambda_0)_{\mathbb{Z}_p} := \mathrm{Aut}((A_0,\lambda_0)[p^\infty]$, $L'_{\mathbb{Z}_p} := L(B_0,\mu_0)_{\mathbb{Z}_p} := \mathrm{Aut}((B_0,\mu_0)[p^\infty]$. Let $L'_{\mathbb{Z}/b\mathbb{Z}}$ be the image of $L'_{\mathbb{Z}_p}$ in $\mathrm{Aut}((B_0,\mu)[p^b])$, and let $L_{\mathbb{Z}/b\mathbb{Z}}$ be the image of $L_{\mathbb{Z}_p}$ in $\mathrm{Aut}((B_0,\mu)[p^b])$. So $L_{\mathbb{Z}_p}$ and $L'_{\mathbb{Z}_p}$ are both compact open subgroups of $H(\mathbb{Q}_p)$.

(5.13) The p-adic monodromy representation attached to the polarized p-divisible group $(B[p^\infty],\mu) \to C$ comes from a right $L'_{\mathbb{Z}_p}$-torsor $\mathcal{T}$ over C defined the same way as before. As before, the right $L'_{\mathbb{Z}_p}$-torsor $\mathcal{T} \to C$ is the projective of a tower of right $L'_{\mathbb{Z}/b\mathbb{Z}}$-torsors $\mathcal{T}_b$, and the transition morphisms $\mathcal{T}_{b_1} \to \mathcal{T}_b$ are finite etale surjective if $b|b_1$.

The set of k-points $\mathcal{T}$ can be described as follows. It consists of all isomorphism classes of quadruples $(A_x, \lambda_x, \iota_x, \psi)$, where $[(A_x, \lambda_x, \iota_x)] =: x$ is a k-point of C, and

$$\psi\colon (B_0, \mu_0)[p^\infty] \xrightarrow{\sim} (B_x, \mu_x)[p^\infty]$$

is an isomorphism. In the above B_x (resp. μ_x) denotes the fiber of B (resp. μ) above the point $x \in C$. The set of k-points of $\mathcal{T}_b$ and the set of S-points of $\mathcal{T}_b$ of a k-scheme S can be described explicitly in a similar way as in 5.7 and 5.8.

We know that the image of the p-adic monodromy representation is contained in $L_{\mathbb{Z}_p}$, so the right $L'_{\mathbb{Z}_p}$-torsor $\mathcal{T}$ over C is of the form $\mathcal{S} \times^{L_{\mathbb{Z}_p}} L'_{\mathbb{Z}_p}$, the push-forward via $L_{\mathbb{Z}_p} \to L'_{\mathbb{Z}_p}$ of a right $L_{\mathbb{Z}_p}$-torsor $\mathcal{S}$ over C. As before, the $L_{\mathbb{Z}_p}$-torsor $\mathcal{S} \to C$ is a projective limit of $L_{\mathbb{Z}/b\mathbb{Z}}$-torsors $\mathcal{S}_b \to C$ Below are explicit descriptions $\mathcal{S}$.

(I) k-points of $\mathcal{S}$.

The set of k-points of the $L_{\mathbb{Z}_p}$-torsor $\mathcal{S} \to C$ is the set of isomorphism classes of quadruples $(A_x, \lambda_x, \iota_x, \psi)$, where

- $[(A_x, \lambda_x, \iota_x)] =: x$ is a k-point of C, and
- $\psi\colon (B_0, \mu_0)[p^\infty] \xrightarrow{\sim} (B_x, \mu_x)[p^\infty]$ is an isomorphism of polarized p-divisible groups which induces an isomorphism $(A_0, \lambda_0)[p^\infty] \xrightarrow{\sim} (A_x, \lambda_x)[p^\infty]$.

(II) D-points of $\mathcal{S}$ for a k-scheme D.

It is more convenient to describe the set $\mathcal{S}_b(D)$ of D-points for members $\mathcal{S}_b$ in the tower $(\mathcal{S}_b)_{b\geq 1}$ whose limit is $\mathcal{S}$. The set $\mathcal{S}_b(D)$ consists of all isomorphism classes of quadruples

$$(A_D \to D, \lambda_D, \iota_D, \psi_b)$$

where $(A \to D, \lambda_D, \iota_D)$ defines a D-point of the leaf C, and

$$\psi_b : \prod_{i=1}^{m} (X_i/X_{i-1})[p^b]) \times_{\mathrm{Spec}(\mathbb{F}_q)} D \xrightarrow{\sim} \prod_{i=1}^{m} (Z_i/Z_{i-1})[p^b] \times_C D$$

is an isomorphism over D satisfying the following properties

(1) The isomorphism ψ_b respects the polarizations induced by μ_0 and μ_D respectively.

(2) For every geometric point $y \in D$, there exist isomorphisms

$$\psi_y\colon (B_0, \mu_0)[p^\infty] \xrightarrow{\sim} (B_y, \mu_y)[p^\infty] \quad \text{and} \quad \phi_y\colon (A_0, \lambda_0)[p^\infty] \xrightarrow{\sim} (A_y, \mu_y)[p^\infty]$$

such that $\psi_y \circ \beta_0 = \beta_y \circ \phi_y$ and the fiber of ψ_b at y is induced by ψ_y.

(3) For every integer $N > b$, there exists an isomorphism

$$\psi_N : \prod_{i=1}^{m} (X_i/X_{i-1})[p^N]) \times_{\mathrm{Spec}(\mathbb{F}_q)} D \xrightarrow{\sim} \prod_{i=1}^{m} (Z_i/Z_{i-1})[p^N] \times_C D$$

such that properties (1), (2) hold with b replaced by N.

The following notations are used in the above.

- $\beta : A \to B$ is the isogeny over the leaf C in 5.11,
- (B_D, μ_D) is the pull-back to D of the polarized abelian scheme (B, μ) as in 5.11,
- $0 = Z_0 \subset Z_1 \subset \cdots Z_m = B[p^\infty]$ is the slope filtration of the completely slope divisible p-divisible group $B[p^\infty]$ over C,
- $0 = X_0 \subset X_1 \subset \cdots \subset X_m = B_0[p^\infty]$ is the slope filtration of the completely slope divisible p-divisible group $B_0[p^\infty]$.

Write $L'_{\mathbb{Z}_p} = \coprod_{i=1}^m (L_{\mathbb{Z}_p} \cdot \zeta_i)$, where $\zeta_1 = 1, \ldots, \zeta_m$ is a system of representatives of $L_{\mathbb{Z}_p} \backslash L'_{\mathbb{Z}_p}$, with $\zeta_1 = 1$. Then

$$L'_{\mathbb{Z}_p} = \coprod_{i=1}^m \mathcal{S} \times^{L_{\mathbb{Z}_p}} (L_{\mathbb{Z}_p} \cdot \zeta_i),$$

a disjoint union of contraction products, and $\mathcal{S}$ is equal to $\mathcal{S} \times^{L_{\mathbb{Z}_p}} (L_{\mathbb{Z}_p} \cdot \zeta_1)$.

Proof of 5.6. We need to show that the $L_{\mathbb{Z}_p}$-torsor $\mathcal{S}$ is irreducible, or equivalently each of the $L_{\mathbb{Z}/b\mathbb{Z}}$-torsors $\mathcal{S}_b \to C$ is irreducible.

The group $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\, ,\, \rangle) \cong \mathrm{Sp}(\mathrm{H}_1(B_0, \mathbb{A}_f^{(p)}), \langle\, ,\, \rangle)$ operates on the torsor $\mathcal{S} = \varprojlim \mathcal{S}_b$, inducing prime-to-$p$ Hecke correspondences on each $\mathcal{S}_b$.

The argument of Thm. 5.9 applies to the present situation and shows that any two points y_1, y_2 of $\mathcal{S}_b$ lying above the hypersymmetric point x_0 belong to the same prime-to-p Hecke orbit: Suppose that $y_i = [(A_0, \lambda_0, \iota_0, \psi_{i,b})]$, $i = 1, 2$, where $\psi_{i,b} \colon (B_0, \mu_0)[p^b] \xrightarrow{\sim} (B_0, \mu_0)[p^b]$ is the image of an element $\psi_i \in L_{\mathbb{Z}_p}$, and $\psi_2 \in L_{\mathbb{Z}_p} \cdot \psi_1$. Let $\tilde{\iota}$ be a prime-to-p symplectic level structure of (A_0, λ_0) above ι_0. Let U_b be the open subset of $L_{\mathbb{Z}_p}$ consisting of all elements of $\gamma_p \in L_{\mathbb{Z}_p}$ such that the image of $\gamma_p \circ \psi_1$ in $L'_{\mathbb{Z}/b\mathbb{Z}}$ is $\psi_{2,b}$. By the weak approximation theorem applied to the connected $\mathbb{Q}$-group H, there exists an element $\gamma \in H(\mathbb{Q})$ such that the image of $\gamma \in H(\mathbb{Q}_p)$ belongs to U_b. Let δ be the element of $\mathrm{Sp}(\mathrm{H}_1(A_0, \mathbb{A}_f^{(p)}), \langle\, ,\, \rangle)$ such that $\gamma^{(p)} \circ \tilde{\iota} = \tilde{\iota} \circ \delta$. Then y_1 is an image of y_2 under the prime-to-p Hecke correspondence induced by δ. We conclude by 2.13 that $\mathcal{S}_b$ is irreducible for every $b \geq 1$ by 2.13. Therefore $\mathcal{S}$ is irreducible and the image of the p-adic monodromy homomorphism $\rho_{(A,\lambda)[p^\infty]/C, x_0}$ is equal to $\mathrm{Aut}((A_0, \lambda_0)[p^\infty])$. This finishes the proof of 5.6. $\square$

(5.14) Remark. As an alternative proof of 5.6, we indicate how 5.6 can be deduced from its special case 5.9. Let $\alpha \colon B \to A$ be a p-primary isogeny which is a multiple of the quasi-isogeny β^{-1}, and let $\mu' := \alpha^*(\lambda)$. Let $d' > 0$ be the square root of the degree of μ'. Let μ'_0 be the fiber of μ' at $x_0 \in C$, and let ι'_0 be the symplectic level-n structure on B_0 corresponding to the level-n structure on A_0 under the p-primary isogeny α. Let y_0 be the k-point $[(B_0, \mu'_0, \iota'_0)]$ in $\mathcal{A}_{g,d',n}$, and let $C' = \mathcal{C}(y_0)$ be the leaf in $\mathcal{A}_{g,d',n}$ passing through y_0. Let $(B', \nu) \to C'$ be the universal polarized abelian scheme over the leaf C' in $\mathcal{A}_{g,d',n}$, and let $(B', \nu)[p^\infty] \to C'$ be the associated polarized p-divisible group. Let C'_{perf} be the perfection of C'.

We know from 4.1 that the leaf C' is (geometrically) irreducible. By [34, Prop. 1.3], the completely slope divisible group $B'[p^\infty] \times_{C'} C'_{\mathrm{perf}} \to C'_{\mathrm{perf}}$ over C'_{perf} splits into a direct sum of isoclinic completely slope divisible groups with distinct slopes. From 5.9 we know that the

p-adic monodromy for $(B', \mu')[p^\infty] \to C'$ is equal to $L'_{\mathbb{Z}_p} = \mathrm{Aut}((B'_0, \mu'_0)[p^\infty])$. According to the theory of Φ-etale part, there exists an *irreducible* profinite etale Galois cover $\mathcal{Z}$ of C'_{perf} with group $L'_{\mathbb{Z}_p}$ such that $(B', \mu')[p^\infty] \times_{C'_{\mathrm{perf}}} \mathcal{Z}$ is a constant polarized p-divisible group over $\mathcal{Z}$.

By 5.11, the kernel of the isogeny $\alpha_0 \colon B_0 \to A_0$ is stable under the subgroup $L_{\mathbb{Z}_p} = \mathrm{Aut}((A_0, \lambda_0)[p^\infty])$ of $L_{\mathbb{Z}_p}$. Let $D = \mathcal{Z}/L_{\mathbb{Z}_p}$ be the finite etale cover of C'_{perf} corresponding to the subgroup $L_{\mathbb{Z}_p}$. On D we have a p-primary isogeny $\alpha' \colon B' \times_{C'} D \to A'$ such that the fiber α'_{y_0} of α' at y_0 is equal to α_{x_0}. Moreover the polarization ν on $B' \times_{C'} D \to D$ descends to a polarization λ' on $A' \to D$ such that $(\alpha')^*(\lambda') = \nu$ and the fiber of λ' at y_0 is equal to λ'_0.

The polarized abelian scheme $(A', \lambda') \to D$ defines a morphism $D \to \mathcal{A}_{g,d,n}$ which factorizes as $D \to C \hookrightarrow \mathcal{A}_{g,d,n}$ by construction, such that (A', λ') is the pull-back to D of the universal polarized abelian scheme over C. By functoriality of the fundamental group, the p-adic monodromy for the leaf C contains the p-adic monodromy attached to the polarized p-divisible group $(A', \lambda')[p^\infty]$ over D. Since the latter is equal to $L_{\mathbb{Z}_p}$ by construction, the image of the p-adic monodromy representation for C contains $L_{\mathbb{Z}_p}$, hence is equal $L_{\mathbb{Z}_p}$.

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