Relative Completions

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Abstract

We introduce a relativised version of the regular and exact completion. This is motivated by the fact that the standard constructions are often not applicable in a constructive context. We show that our construction gives more general results, for instance in the study of realizability categories over an arbitrary base topos.

1 Introduction

Since the discovery that realizability toposes enjoy a certain universal property, a lot of work has been done on the study of regular and exact completions, especially their applications to categories that play a prominent role in realizability. The most important (and best-known) results in this area are, that, starting from a partial combinatory algebra $\mathbb{A}$, the category of Assemblies $\mathbf{Ass}(\mathbb{A})$ is the regular completion of the category of Partitioned Assemblies $\mathbf{PAss}(\mathbb{A})$, and that the realizability topos $\mathbf{RT}(\mathbb{A})$ is the exact completion of $\mathbf{PAss}(\mathbb{A})$. These results are useful, because they give a simple presentation of a realizability topos and also display some of its structure. An important restriction is, however, that they rely on an essential use of the axiom of choice in the base topos. For example, if one is to show that the Effective Topos arises as an exact completion, then one has to show that **Eff** has enough projectives. But in order to do so, one cannot avoid an appeal to choice in **Set**.

This paper is intended as a first attempt at analysing what happens if we wish to refrain from using choice. Put differently, what happens when we do not work over the base topos **Set**, but over an arbitrary topos $\mathcal{E}$, in which the axiom of choice fails? Can the construction of a realizability topos then still be seen as a solution to a universal problem? Is it still some kind of completion of the category of Partitioned Assemblies?

The paper is structured in the following manner: section 2 will contain some basic definitions and notation. We assume that the reader has knowledge about the standard completions; for those who do not, there are various references, such as [9], [2] or [5]. We will only rehearse some notation concerning these completions. There will also be definitions of the categories that we will be mostly interested in, namely categories of (Partitioned) Assemblies over an arbitrary base topos $\mathcal{E}$, and categories associated to internal locales in a topos. Over **Set**, these definitions are to be found in various places, such as [7] or [9], but since we do not work over **Set** any longer, there are some subtleties that need attention, whence the inclusion of precise definitions.

Section 3 contains the central definitions of the *relative regular completion* and the *relative exact completion*. The idea is, that we do not simply form the completion of a category $\mathcal{C}$, but take into account that there is a functor $F : \mathcal{E} \to \mathcal{C}$, which bears information about how $\mathcal{C}$ is related to the base topos. We get the following picture:

$$\mathcal{E} \xrightarrow{F} \mathcal{C} \xrightarrow{y} \mathcal{C}_{reg} \xrightarrow{P_\Sigma} \mathcal{C}_{\mathcal{E}/reg}$$

where $\mathcal{C}_{\mathcal{E}/reg}$ denotes the relative regular completion. In fact, it will be constructed from $\mathcal{C}_{reg}$ as a category of fractions. Thus we get a quotient functor P_Σ as in the picture above. After explaining the construction, we give some simple examples, and we also show, that the construction may be viewed as a Kock-Zöberlein-doctrine.

The focus of sections 4, 5 and 6 is an analysis of the functor P_Σ: this is mainly motivated by the fact that the definition of section 3 is not very elegant, and far from convenient to work with. Therefore we give two different presentations of the relative completion: the first one (section 4) makes use of pushouts in the category of regular categories, and the second one (section 5) is based on topologies. This enables us to identify some situations in which the relative completion of a category is somewhat better behaved than in general. In particular, we find a simple condition under which $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$. Section 6 is devoted to a more detailed analysis of the situation where the relative completion is a reflective subcategory of the ordinary completion, that is, when the functor P_Σ has a full and faithful right adjoint.

With the theory from sections 5 and 6, we have the major ingredients for our characterization of assemblies, which, together with locales, will be carried out in section 7. This will also answer the initial question that we posed, namely that the realizability topos can still be seen as a completion of the category of partitioned assemblies, namely the relative exact completion.

Finally, we present a number of open questions related to our constructions, to which we think it would be nice to have an answer.

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2 Preliminaries

Completions. As said in the introduction, we assume familiarity with the basic theory of completions, mostly the regular completion. For convenience, we only repeat the usual notational conventions for the regular completion $\mathcal{C}_{reg}$ of a category $\mathcal{C}$: objects will be written $\begin{pmatrix} X \\ f\downarrow \\ Y \end{pmatrix}$, and maps will be pictured as

$$\begin{pmatrix} X \\ f\downarrow \\ Y \end{pmatrix} \xrightarrow{[k]} \begin{pmatrix} P \\ g\downarrow \\ Q \end{pmatrix}$$

where $[k]$ is an equivalence class of maps $k : X \to Y$ in $\mathcal{C}$, and where k, k' are equivalent iff their composites with g are equal. We also recall that the projectives in $\mathcal{C}_{reg}$ are precisely the objects in the image of the inclusion $y : \mathcal{C} \to \mathcal{C}_{reg}$. Finally, if e is a regular epi in $\mathcal{C}$, then $y(e)$ is not regular, unless e is split. This observation will be crucial for the comparison of the standard completion and our relativised version.

Assemblies. Let $\mathcal{E}$ be an arbitrary topos, and consider an internal partial combinatory algebra (pca) $\mathbb{A}$ in $\mathcal{E}$. We will assume that the reader is familiar with the (tripos-theoretic) construction of the realizability topos $\mathbf{RT}_{\mathcal{E}}(\mathbb{A})$, and some of its basic properties. For a standard reference, see [10], in which it is also explained that, in absence of choice, there is an internal version and an external version of the realizability tripos associated with a pca, the difference being that for the internal version one takes a definable object of designated truth-values. For our application, it turns out that an assumption on the tripos is needed: every inhabited subobject of $\mathbb{A}$ must have a global element. The external tripos always satisfies this property, but we run into trouble once we consider the internal variant. In section 7 it will be pointed out why this assumption is needed for our approach.

We start by defining the categories of *Partitioned Assemblies*, $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$, and *Assemblies*, $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$. The objects of $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ are pairs (X, E_X), where X is an object of $\mathcal{E}$, and $E_X : X \to \mathbb{A}$ is a map in $\mathcal{E}$ to the internal pca $\mathbb{A}$. An arrow from (X, E_X) to (Y, E_Y) is a map $f : X \to Y$ in $\mathcal{E}$ such that $\mathcal{E} \models \exists a : \mathbb{A} \forall x : X.\ a \bullet E_X(x)\downarrow\ \wedge\ a \bullet E_X(x) = E_Y(fx)$. An assembly is also a pair (X, E_X), but now $E_X : X \to \mathcal{P}_i(\mathbb{A})$, where $\mathcal{P}_i(\mathbb{A})$ stands for the object of inhabited subsets of $\mathbb{A}$. Similarly, a map $f : X \to Y$ is a map of assemblies if we have $\mathcal{E} \models \exists a : \mathbb{A} \forall x : X \forall b \in E_X(x).\ a \bullet b\downarrow\ \wedge\ a \bullet b \in E_Y(fx)$.

In the definition of an assembly, we might just as well take functions into the object of nonempty subsets of $\mathbb{A}$ rather than the inhabited subsets, since this gives equivalent categories.

As usual, we have an embedding $\nabla : \mathcal{E} \to \mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$, that has a faithful left adjoint, denoted Γ. ∇ preserves regular epis, although $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ is not a regular category. We use the same notation ∇, Γ to denote the localization of $\mathcal{E}$ in $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$. Again, ∇ is a regular functor.

Lemma 2.1 *The category $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ of assemblies is equivalent to the full subcategory of $\mathbf{RT}_{\mathcal{E}}[\mathbb{A}]$ on the subobjects of objects of the form $\nabla(X)$.*

Proof. This is straightforward.

□

In [6], it is explained that there is a monad on the category of ordered partial combinatory algebras, based on the fact that the collection of non-empty downsets in a pca inherits the combinatorial structure. This generalizes to pcas in an arbitrary topos. Thus we get that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is equivalent to $\mathbf{PAss}_{\mathcal{E}}(I\mathbb{A})$, where I is the nonempty (or, equivalently, the inhabited) downset-monad.

In the classical case, one has a convenient characterization of regular epis in Assemblies; this goes through in the general setting:

Lemma 2.2 *In $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$, a map $e' : (Y', E_{Y'}) \to (X, E_X)$ is regular epi if and only if it is isomorphic (over (X, E_X)) to a map $e : (Y, E_Y) \to (X, E_X)$ that satisfies $E_X(x) = \bigcup_{e(y)=x} E_Y(y)$.*

Proof. As usual.

□

Lemma 2.3 *The functor $\Gamma : \mathbf{Ass}_{\mathcal{E}}(\mathbb{A}) \to \mathcal{E}$ preserves regular projectives.*

Proof. Its right adjoint preserves regular epis.

□

Lemma 2.4 *An object (X, E_X) in $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is projective if and only if it is a partitioned assembly and X is projective in $\mathcal{E}$.*

Proof. Observe first that any assembly can be covered by a partitioned assembly, namely cover (X, E_X) by (Q, E_Q), where $Q = \{(x, a) | a \in E_X(x)\}$. Moreover, the partitioned assemblies are closed under finite limits. Now if (X, E_X) is projective, then this cover has a section, presenting (X, E_X) as a regular subobject of a partitioned assembly, hence as a partitioned assembly. Also, X is projective in $\mathcal{E}$ by the previous lemma.

Conversely, any partitioned assembly (X, E_X) with X projective in $\mathcal{E}$ is projective. For let $e : (Y, E_Y) \to (X, E_X)$ be regular epi. Then $e(y) = x$ implies $E_Y(y) = E_X(x)$. So take any section in $\mathcal{E}$, and it will be tracked by the identity.

□

We refer to the covering Q as in the lemma as the *canonical covering* of (X, E_X).

From this lemma it follows that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is in general not equivalent to the regular completion of $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$, since in this completion, every partitioned assembly is projective.

Finally, we recall a folklore theorem [3]:

Theorem 2.5 *Let $\mathcal{P}$ be a tripos on a category $\mathcal{C}$, let $\mathcal{C}[\mathcal{P}]$ denote the resulting topos and write $\nabla : \mathcal{C} \to \mathcal{C}[\mathcal{E}]$ for the constant objects functor. Then $\mathcal{C}[\mathcal{P}]$ is the ex/reg-completion of its full subcategory on the subobjects of objects in the image of ∇.*

For us, the main implication of this theorem is, that the realizability topos $\mathbf{RT}[\mathbb{A}]$ is the *ex/reg*-completion of $\mathbf{Ass}(\mathbb{A})$.

Locales. Let H be a locale in $\mathcal{E}$. The category of elements for H, denoted $\int_{\mathcal{E}} H$ has pairs (X, α) as objects, where X is an object of $\mathcal{E}$, and $\alpha : X \to H$ a map into the locale H; maps are arrows $f : X \to X'$ for which $\alpha(x) \leq \alpha'(f(x))$ for all $x \in X$. In case that $\mathcal{E} = \mathbf{Set}$, this is the usual category $Fam(H)$, the coproduct completion of H, viewed as a small category. $\int_{\mathcal{E}} H$ is a regular category.

Given H, form a new locale by taking non-empty downsets in H, denoted I^*H, ordered by inclusion. There is an embedding $H \to I^*H$ (which is given by $a \mapsto \downarrow(a)$), that induces an embedding $\int_{\mathcal{E}} H \to \int_{\mathcal{E}}(I^*H)$.

3 A Universal Construction

We fix a category $\mathcal{E}$ with finite limits (this is the minimum amount of structure required for the construction; in most applications however, $\mathcal{E}$ will be a topos). Consider the category $\mathcal{E}/\mathbf{LEX}$. Objects are left exact functors $F : \mathcal{E} \to \mathcal{C}$ with $\mathcal{C}$ a lex category, and morphisms are commutative triangles of lex functors. Similarly, we have a category $\mathcal{E}/\mathbf{REG}$ where all categories and functors involved are regular, and $\mathcal{E}/\mathbf{EX}$, where all categories and functors are exact. The theorem that we aim for is the following:

Theorem 3.1 *The forgetful functor $\mathcal{E}/\mathbf{REG} \to \mathcal{E}/\mathbf{LEX}$ has a left biadjoint.*

Proof. Send $F : \mathcal{E} \to \mathcal{C}$ to the composite

$$\mathcal{E} \xrightarrow{F} \mathcal{C} \xrightarrow{y} \mathcal{C}_{reg} \xrightarrow{P_\Sigma} \mathcal{C}_{reg}[\Sigma^{-1}].$$

Here, $\mathcal{C}_{reg}[\Sigma^{-1}]$ refers to the category obtained from $\mathcal{C}_{reg}$ by formally inverting all arrows in a class Σ. This class of arrows Σ is defined as follows: consider a regular epi $f : X \to Y$ in $\mathcal{E}$. The functor F sends f to Ff, and the embedding y takes this to yFf. In $\mathcal{C}_{reg}$, the arrow yFf has a regular epi-mono factorization, as in the diagram:

$$yFX \xrightarrow{[1]} \!\!\!\!\rightarrow \begin{pmatrix} FX \\ Ff\downarrow \\ FY \end{pmatrix} \rightarrowtail^{[Ff]} yFY.$$

The reflection of $F : \mathcal{E} \to \mathcal{C}$ in $\mathcal{E}/\mathbf{REG}$ must be a regular functor, which means that the arrow $[Ff]$ has to be inverted. So define Σ_0 to be the class of all the arrows $[Fe]$ that arise as in diagrams such as the one above. Then define Σ to be the least class of maps containing Σ_0, with the properties that

- All isomorphisms are in Σ,
- If two out of three sides of a commutative triangle are in Σ, then so is the third,

- Σ is pullback-stable,
- If $e^*\sigma \in \Sigma$ for some regular epi e, then $\sigma \in \Sigma$.

Following Bénabou, we call a collection of arrows Σ satisfying these closure properties a *local pullback congruence*. Now it follows from the theory of categories of fractions that $\mathcal{C}_{reg}[\Sigma^{-1}]$ is a regular category, and that P_Σ is a regular functor (see [1], Theorem 2.2.2).

For the universal property, consider any left exact functor $G : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a regular category, and where the composite GF is regular. Then in the diagram below:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] & \mathcal{C} \arrow[r, "y"] \arrow[drr, "G"'] & \mathcal{C}_{reg} \arrow[r, "P_\Sigma"] \arrow[dr, dashed, "\hat{G}"] & \mathcal{C}_{reg}[\Sigma^{-1}] \arrow[d, dashed, "\tilde{G}"] \\ & & & \mathcal{D} \end{tikzcd}$$

the regular functor $\hat{G}$ arises because of the universal property of $\mathcal{C}_{reg}$. $\hat{G}$ inverts all arrows in Σ_0 and therefore also all arrows in Σ. Hence the universal property of the category of fractions gives us the required regular $\tilde{G}$.

$\square$

We introduce the following terminology: given $F : \mathcal{E} \to \mathcal{C}$ left exact, we shall write $\mathcal{C}_{\mathcal{E}/reg}$ for the value (at F) of the biadjoint of theorem 3.1, and we call it the *relative regular completion* of $\mathcal{C}$ (relative to $\mathcal{E}$).

One can summarize the idea behind the construction as follows: the ordinary regular completion $y : \mathcal{C} \to \mathcal{C}_{reg}$ sends regular epis to epis which are not regular (except for those that have a splitting), so it destroys the regular structure that exists in $\mathcal{C}$. The fraction construction tries to restore as much of this structure as possible.

Although we concentrate on the relative regular completion in this paper, we mention that there is also a natural notion of a relative exact completion:

Theorem 3.2 *The forgetful functor from $\mathcal{E}/\mathbf{EX} \to \mathcal{E}/\mathbf{REG}$ has a left biadjoint.*

Proof. Send $F : \mathcal{E} \to \mathcal{C}$ to the composite

$$\mathcal{E} \xrightarrow{F} \mathcal{C} \xrightarrow{y} \mathcal{C}_{ex/reg}.$$

It is easily seen that this gives the required propery.

$\square$

Let us denote these biadjoints by $(-)_{\mathcal{E}/reg}$,$(-)_{\mathcal{E}/ex/reg}$ and their composite by $(-)_{\mathcal{E}/ex}$.

Our motivating examples, namely Partitioned Assemblies and locales, will appear in section 7. At this point, we will give some simpler examples, to give the reader a feel for the construction.

Examples 3.3 1. First of all, let's see what happens when applying this construction to the identity on $\mathcal{C}$, when $\mathcal{C}$ is an arbitrary finite limit category. If there exists a regular epi in $\mathcal{C}$ which has no section, then the relative completion $\mathcal{C}_{\mathcal{C}/reg}$ will preserve such a map, and hence $\mathcal{C}_{\mathcal{C}/reg}$ will differ from $\mathcal{C}_{reg}$. Somewhat more generally, let $\mathcal{C}' \to \mathcal{C}$ be the inclusion of a subcategory which is closed under finite limits. Then the relative completion may be seen as the closest approximation to $\mathcal{C}_{reg}$, in which the regular structure of $\mathcal{C}'$ is preserved.

2. Now let $\mathcal{C}$ be regular, and consider again the identity functor on $\mathcal{C}$. Then the relative completion of $\mathcal{C}$ is equivalent to $\mathcal{C}$ itself (and in fact, the quotient functor from $\mathcal{C}_{reg}$ to $\mathcal{C}_{\mathcal{C}/reg} \simeq \mathcal{C}$ is simply the left adjoint to the inclusion $y : \mathcal{C} \to \mathcal{C}_{reg}$).

3. On the other hand, we might take any finite limit category for $\mathcal{E}$, and the functor which has the terminal object 1 of $\mathcal{C}$ as constant value. Then the relative completion coincides with the ordinary regular completion. (For, each map in Σ is an isomorphism.) Somewhat more generally, if F sends every regular epi to an isomorphism, or even to a split epi, then $\mathcal{C}_{reg} \simeq \mathcal{C}_{\mathcal{E}/reg}$.

4. The ordinary regular completion will always send non-equivalent categories to non-equivalent completions. The relative version need not do so; as an example, take $F : \mathcal{E} \to \mathcal{C}$ such that $\mathcal{C}_{\mathcal{E}/reg}$ is not equivalent to $\mathcal{C}_{reg}$. Then consider $(\mathcal{C}_{\mathcal{E}/reg})_{\mathcal{E}/reg}$ and $(\mathcal{C}_{reg})_{\mathcal{E}/reg}$. These are easily seen to be equivalent, since they have the same universal property. For a more general treatment of this phenomenon, see section 6.

As one expects, the relative completion may be viewed as a KZ-doctrine on the category $\mathcal{E}/\mathbf{LEX}$, just as the ordinary completions are KZ-doctrines on **LEX**, the category of all small left exact categories. (For an account of KZ-doctrines, see [8].) Let us exhibit part of the relevant structure:

Lemma 3.4 *Suppose $F : \mathcal{E} \to \mathcal{D}$ is a regular functor, and $\mathcal{D}$ a regular category. Then the embedding $\mathcal{D} \to \mathcal{D}_{\mathcal{E}/reg}$ has a regular left adjoint, and the counit of the adjunction is an isomorphism.*

Proof. Consider

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] & \mathcal{D} \arrow[r, "y"] \arrow[drr, "1"'] & \mathcal{D}_{reg} \arrow[r, "P_\Sigma"] \arrow[dr, "a'"] & \mathcal{D}_{\mathcal{E}/reg} \arrow[d, dashed, "a"] \\
& & & \mathcal{D}.
\end{tikzcd}$$

Because $\mathcal{D}$ is regular, there is an extension a' with $a' \circ y \cong 1$, and since F is regular, so is $a' \circ y \circ F \cong F$, and by the universal property of $\mathcal{D}_{\mathcal{E}/reg}$ this gives the functor a as in the diagram. a sends an object of $\mathcal{D}_{\mathcal{E}/reg}$ to its image in $\mathcal{D}$, and since a' inverts all morphisms in Σ, one easily finds that $a \dashv P_\Sigma \circ y$.

□

Theorem 3.5 *$\mathcal{E}/\mathbf{REG}$ is equivalent to the category of algebras for the monad $(-)_{\mathcal{E}/reg}$.*

Proof. The previous lemma stated that every object of $\mathcal{E}/\mathbf{REG}$ carries an algebra structure (which is then unique up to isomorphism, since it is left adjoint to the unit). Conversely, let $\mathcal{D}$ have an algebra structure $a : \mathcal{D}_{\mathcal{E}/reg} \to \mathcal{D}$. Composed with P_Σ, we get a map $a \circ P_\Sigma : \mathcal{D}_{reg} \to \mathcal{D}$, which is an algebra map. Then apply the fact that the ordinary regular completion is a KZ-doctrine, so that $a \circ P_\Sigma$ is left adjoint to the unit at $\mathcal{D}$. In particular, $\mathcal{D}$ is regular, and so is the functor $a \circ P_\Sigma$. Because of the universal property of $\mathcal{D}_{\mathcal{E}/reg}$, $a \circ P_\Sigma$ has an extension, which must be isomorphic to a. Thus a is also regular. It is also evident that $F : \mathcal{E} \to \mathcal{D}$ is regular. Thus F is indeed an object of $\mathcal{E}/\mathbf{REG}$.

□

4 Algebraic Presentation

In this section we give an alternative characterization of the category $\mathcal{C}_{\mathcal{E}/reg}$, and derive some consequences. First, we show that $\mathcal{C}_{\mathcal{E}/reg}$ can be constructed as a pseudo-pushout in the category of regular categories. It is essential that the base category $\mathcal{E}$ is regular, so that the embedding $\mathcal{E} \to \mathcal{E}_{reg}$ has a regular left adjoint r.

Proposition 4.1 *Let $\mathcal{E}$ be regular, $\mathcal{C}$ have finite limits and let $F : \mathcal{E} \to \mathcal{C}$ preserve finite limits. The following square is a pseudo-pushout in* **REG***:*

$$\begin{tikzcd}
\mathcal{E}_{reg} \arrow[r, "F_{reg}"] \arrow[d, "r"'] & \mathcal{C}_{reg} \arrow[d, "P_\Sigma"] \\
\mathcal{E} \arrow[r, "P_\Sigma \circ y \circ F"'] & \mathcal{C}_{\mathcal{E}/reg}.
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{C} \arrow[d, "y"] \\
\mathcal{E}_{reg} \arrow[r, "F_{reg}"] \arrow[d, "r"'] & \mathcal{C}_{reg} \arrow[d, "P_\Sigma"] \\
\mathcal{E} \arrow[r, "P_\Sigma \circ y \circ F"'] & \mathcal{C}_{\mathcal{E}/reg}.
\end{tikzcd}$$

First, the large square commutes since $r \circ y \cong Id$. Also, the top square commutes, so we have $P_\Sigma \circ F_{reg} \circ y \cong (P_\Sigma \circ y \circ F) \circ r \circ y : \mathcal{E} \to \mathcal{C}_{\mathcal{E}/reg}$. Both $P_\Sigma \circ F_{reg}$ and $(P_\Sigma \circ y \circ F) \circ r$ are regular functors from $\mathcal{E}_{reg}$ to $\mathcal{C}_{\mathcal{E}/reg}$, and hence determined up to isomorphism by their composites with $y : \mathcal{E} \to \mathcal{E}_{reg}$. These are isomorphic, so it follows that $P_\Sigma \circ F_{reg} \cong (P_\Sigma \circ y \circ F) \circ r$, and the below square commutes.

For the universal property we take regular functors $G : \mathcal{C}_{reg} \to \mathcal{D}$ and $H : \mathcal{E} \to \mathcal{D}$, such that $H \circ r \cong G \circ F_{reg}$. Then $H \cong H \circ r \circ y \cong G \circ F_{reg} \circ y$ is regular, and thus $G \circ y \circ F : \mathcal{E} \to \mathcal{D}$. By the universal property of $\mathcal{C}_{\mathcal{E}/reg}$, we obtain a factorization $G \cong K \circ P_\Sigma$. It only remains to be checked that $H \cong K \circ (P_\Sigma \circ y \circ F)$. But

$$\begin{aligned} H &\cong H \circ r \circ y \\ &\cong G \circ F_{reg} \circ y \\ &\cong (K \circ P_\Sigma) \circ F_{reg} \circ y \\ &\cong K \circ (P_\Sigma \circ y \circ F), \end{aligned}$$

which completes the proof.

□

We can also easily show the analogous statement for the relative exact completion (for this to make sense, assume $\mathcal{E}$ to be exact):

Proposition 4.2 *Let $\mathcal{E}$ be exact, $\mathcal{C}$ have finite limits and let $F : \mathcal{E} \to \mathcal{C}$ preserve finite limits. The following square is a pseudo-pushout in* **EX***:*

$$\begin{tikzcd} \mathcal{E}_{ex} \arrow[r, "F_{ex}"] \arrow[d, "r"'] & \mathcal{C}_{ex} \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{C}_{\mathcal{E}/ex}. \end{tikzcd}$$

Proof. Apply the *ex/reg*-construction to the pushout of proposition 4.1. The *ex/reg*-construction is a left bi-adjoint, and therefore preserves pseudo-pushouts.

□

Before we have a look at some of the consequences of these presentations, we show that the situation is surprisingly similar to some constructions in algebra. For instance, let R be a ring, M a monoid and $f : R \to M$ a map of monoids. If we write $F(R)$ and $F(M)$ for the free rings on R and M, we construct a ring $F_R(M)$ by forming the pushout

$$\begin{tikzcd} F(R) \arrow[r, "F(f)"] \arrow[d] & F(M) \arrow[d] \\ R \arrow[r] & F_R(M). \end{tikzcd}$$

The ring $F_R(M)$ is the free ring on M such that $R \to F_R(M)$ is a ringhomomorphism, i.e. for any ring N and any map of monoids $k : M \to N$ such that $kf : R \to N$ is a ringhomomorphism, there is a unique $\hat{k} : F_R(M) \to N$ through which k factors.
Observe that it now follows that the relative exact completion can also be obtained as a category of fractions; this follows from the fact that for any functor $P : \mathcal{C} \to \mathcal{D}$ in **LEX**, and any class of maps Ξ in $\mathcal{C}$, the the following square is a pushout, where $P\Xi$ denotes the image of Ξ under P:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{P} & \mathcal{D} \\ {\scriptstyle P_\Xi}\downarrow & & \downarrow{\scriptstyle P_{P\Xi}} \\ \mathcal{D}[\Xi^{-1}] & \longrightarrow & \mathcal{D}[P\Xi^{-1}]. \end{array}$$

Combined with the fact that $\mathcal{E}$ is a localization of $\mathcal{E}_{ex}$ (and may therefore be seen as a category of fractions), we see that $\mathcal{C}_{ex} \to \mathcal{C}_{\mathcal{E}/ex}$, being a pushout of $\mathcal{E}_{ex} \to \mathcal{E}$, is itself of this form.
As a simple corollary of proposition 4.1, we get that the ordinary regular completion coincides with the relative completion when the base category $\mathcal{E}$ satisfies the axiom of choice (meaning that every regular epi splits):

Corollary 4.3 *If $\mathcal{E}$ is regular and $\mathcal{E} \models AC$, then $\mathcal{C}_{reg} \simeq \mathcal{C}_{\mathcal{E}/reg}$.*

Proof. If every regular epi splits in $\mathcal{E}$, then $\mathcal{E} \simeq \mathcal{E}_{reg}$ (see, for instance [9]). So, in the pushout square of proposition 4.1 the left-hand map is an equivalence, and therefore the right-hand map is an equivalence, too.

□

A converse to this corollary holds if we assume the functor $F : \mathcal{E} \to \mathcal{C}$ to be full:

Proposition 4.4 *If $F : \mathcal{E} \to \mathcal{C}$ is full, and $\mathcal{C}_{reg} \simeq \mathcal{C}_{\mathcal{E}/reg}$, then $\mathcal{E} \models AC$.*

Proof. Consider a regular epi $e : X \to Y$ in $\mathcal{E}$. This is sent to $yF(e) : yF(X) \to yF(Y)$ in $\mathcal{C}_{reg}$. This map is again regular epi, because the composite yF is now a regular functor. This in turn means that the mono part of the reg-epi/mono factorization of $yF(e)$ is an isomorphism. Thus it has an inverse

$$yF(Y) \xrightarrow{[k]} \begin{pmatrix} FX \\ Fe\downarrow \\ FY \end{pmatrix}$$

and the underlying arrow $k : FY \to FX$ is easily seen to be a splitting for Fe. Now F is full, so k is in the image of F, say $k = Fh$, and h is a splitting for e.
□

Similar statements hold when $\mathcal{E}$ is an exact category.

Finally, we show that relative completions, just like ordinary completions, inherit chaotic stiuations. The notion of a chaotic situation was formulated in [9]: $\mathcal{C}$ has $\mathcal{E}$ as a chaotic situation if $\mathcal{E}$ is a topos, and if there is an embedding $F : \mathcal{E} \to \mathcal{D}$ which has a faithful left adjoint G. So assume that this is the case. Because $\mathcal{E}$ is regular, the universal property of $\mathcal{C}_{reg}$ gives an extension of G to $\hat{G} : \mathcal{C}_{reg} \to \mathcal{E}$, which is left adjoint to the composite $y \circ F : \mathcal{E} \to \mathcal{C}_{reg}$. Thus we get

$$\begin{tikzcd}
\mathcal{E}_{reg} \arrow[r, "F_{reg}"] \arrow[d, "r"'] & \mathcal{C}_{reg} \arrow[d, "P_\Sigma"] \arrow[ddr, bend left, "\hat{G}"] & \\
\mathcal{E} \arrow[r, "P_\Sigma \circ y \circ F"'] \arrow[drr, bend right] & \mathcal{C}_{\mathcal{E}/reg} & \\
& & \mathcal{E}
\end{tikzcd}$$

and there is a factorization through the pushout $\tilde{G} : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{E}$. It is easily verified that this map is again faithful, and left adjoint to the embedding of $\mathcal{E}$ in $\mathcal{C}_{\mathcal{E}/reg}$.

5 Sheaves

Next, we concentrate on a presentation in terms of sheaves. We make use of the notion of a quasi-topology and of a topology on $\mathcal{C}$. These were introduced in [9], but we provide a short recapitulation.

Definition 5.1 Let $\mathcal{C}$ be a finite limit category. A *quasi-topology* on $\mathcal{C}$ is a family $J(X)$ for each object X of $\mathcal{C}$, of maps with codomain X, subject to the following conditions:

- $1_X \in J(X)$
- for $f : Y \to X$, if $g \in J(X)$ then $f^*g \in J(Y)$ (where f^*g denotes the pullback of g along f)
- if $g \circ h \in J(X)$, then $g \in J(X)$
- if $f : Y \to X \in J(X)$ and $g \in J(Y)$ then $f \circ g \in J(X)$.

Definition 5.2 A map $h : Z \to X$ is *closed* for a quasi-topology J if for every $f : Y \to X$, $f^*h \in J(Y)$ implies that f factors through h.

Definition 5.3 A quasi-topology J is a *topology* if for every map $f : Y \to X$ there is a $g : V \to W \in J(W)$ and a closed $h : W \to X$ such that f factors through $h \circ g$ and vice versa.

The point of these definitions is, that topologies on $\mathcal{C}$ correspond to universal closure operators on $\mathcal{C}_{reg}$ (and on $\mathcal{C}_{ex}$). A (quasi-)topology J is called *subcanonical* if every map in J is regular epi.

Construction 5.4 Consider again a functor $F : \mathcal{E} \to \mathcal{C}$. We will construct a quasi-topology on $\mathcal{C}$ by defining:

- $f \in K(C)$ iff there is a diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & C \arrow[d, "\alpha"] \\ F(E') \arrow[r, "Fe"] & F(E) \end{tikzcd}$$

  where e is a regular epi in $\mathcal{E}$, and the square is a pullback.

- J is the closure if K under composition and under right-halves, i.e. if $hk \in K$ then so does h.

The verification that J is a quasi-topology on $\mathcal{C}$ is straightforward. Now there is a technical lemma to be proved:

Lemma 5.5 *Let $f : X \to Y$ be a map in $\mathcal{C}$, inducing a mono $[f] : f \to yY$ in $\mathcal{C}_{reg}$. Then $f \in J(Y)$ implies $[f] \in \Sigma$.*

We give the proof of this in the appendix, since it is purely technical.

Theorem 5.6 *The quasi-topology J is a topology iff $\mathcal{C}_{\mathcal{E}/reg}$ is a reflective subcategory of $\mathcal{C}_{reg}$ (in which case it is of the form sheaves for the induced closure operator on $\mathcal{C}_{reg}$).*

Proof. If J is a topology, then there is an induced universal closure operator on $\mathcal{C}_{reg}$, with the property that for any arrow $f : C' \to C$, $[f] : f \to yC$ is dense iff $f \in J$. Using the previous lemma, we get that $[f]$ dense implies $[f] \in \Sigma$. From this, it follows that all dense monos are in Σ.

On the other hand, all maps in Σ_0 are dense, and hence are all monos in Σ. We conclude that the class of dense maps coincides with the class Σ.

□

This theorem shows that in some cases, the relative completion may be seen as a category of sheaves for a universal closure operator; the next section studies this situation in some more detail, and we will see that this gives a more manageable presentation than one in terms of categories of fractions.

It is clear that, in general, $\mathcal{C}$ is not a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$, and also, that the image of $F : \mathcal{E} \to \mathcal{C}$ need not be so. The following is an obvious criterion:

Lemma 5.7

1. *$\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$ iff every map in J is regular epi;*

2. *$Im(F)$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$ iff objects in the image of F think that all maps in J are regular epi. By this, we mean that for every map $f : X \to Y$ in $J(X)$ with kernel f_0, f_1, and every map $m : X \to F(W)$ for which $mf_0 = mf_1$, there is a unique extension of m along f.*

Proof. For 1), clearly, every map in J is regular epi iff J is subcanonical, see [9]. But then we find that $\mathcal{C}$ is a full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$.

2) is treated similarly.

□

As example 3.3.4. showed, non-equivalent categories may yield the same completion. The following lemma provides some insight:

Lemma 5.8 *Let $F : \mathcal{E} \to \mathcal{C}$ be given and consider $y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$. Define $\mathcal{D}$ to be the full subcategory of $\mathcal{C}_{\mathcal{E}/reg}$ on the objects in the image of y. Then $\mathcal{C}_{\mathcal{E}/reg} \simeq \mathcal{D}_{\mathcal{E}/reg}$.*

Proof. We have a factorization of y as

$$\begin{tikzcd} \mathcal{E} \arrow[r, "F"] & \mathcal{C} \arrow[r, "y"] \arrow[d, "G"'] & \mathcal{C}_{\mathcal{E}/reg} \\ & \mathcal{D} \arrow[ur, "\mu"'] & \end{tikzcd}$$

Consider $y' : \mathcal{D} \to \mathcal{D}_{\mathcal{E}/reg}$. By the universal property of $\mathcal{C}_{\mathcal{E}/reg}$, the map $y \circ G : \mathcal{C} \to \mathcal{D}_{\mathcal{E}/reg}$ can be extended along y to give a map $\hat{G} : \mathcal{C}_{\mathcal{E}/reg} \to \mathcal{D}_{\mathcal{E}/reg}$. On the other hand, the universal property of $\mathcal{D}_{\mathcal{E}/reg}$ gives an extension $\hat{\mu}$ of μ along y'. Then $\hat{\mu}$ and $\hat{G}$ are pseudo-inverses of each other.

□

6 Minimal Coverings and Sheaves

We further analyse the situation of the previous section, in which the relative completion was reflective in the ordinary completion. To this end, we first introduce a technical notion, called a minimal covering.

Let $\mathcal{C}$ be a lex category, $\mathcal{D}$ a regular category and let $F : \mathcal{C} \to \mathcal{D}$ be a left exact full and faithful functor. First we recall that a map $k : FC \to D$ is called $\mathcal{C}$-*projecting* [9] if every other map $FC' \to D$ factors through e. Then we define:

Definition 6.1 $F : \mathcal{C} \to \mathcal{D}$ is called a *minimal covering* iff for every D in $\mathcal{D}$ there is a C in $\mathcal{C}$ and a $\mathcal{C}$-projecting regular epi $e : FC \to D$.

The connection between minimal coverings, topologies and regular completions can be formulated as:

Lemma 6.2 *Let $F : \mathcal{C} \to \mathcal{D}$ be a lex, full and faithful functor, with $\mathcal{D}$ regular. Consider the extension $\hat{F} : \mathcal{C}_{reg} \to \mathcal{D}$. Then the following are equivalent:*

1. *The functor $\hat{F}$ has a right adjoint G with $GF \cong y$ and $\hat{F}G \cong Id$;*

2. *F is a minimal covering, and every object in D embeds into an object in the image of F;*

3. *There is a subcanonical topology on $\mathcal{C}$ such that $\mathcal{D}$ is equivalent to the category of sheaves for the induced universal closure operator on $\mathcal{C}_{reg}$.*

Proof. First assume 2. We define $G : \mathcal{D} \to \mathcal{C}_{reg}$ as follows. An object D gives a composite map

$$FC \xrightarrow{e} \!\!\!\!\rightarrow D \rightarrowtail^{m} FC'$$

with e a regular epi. Since F is full, there is a map $f : C \to C'$ in $\mathcal{C}$ with $Ff = me$. This map f is the value of G on D. This is well-defined, because any other cover e' will factor through e and vice versa. (note in particular that $G(FC) = C$.) For arrows, consider the diagram

$$\begin{tikzcd} FC \arrow[r, dashed, "\bar{f}"] \arrow[d, two heads, "e_1"'] & FB \arrow[d, two heads, "e_2"] \\ D \arrow[r, "f"] \arrow[d, tail, "m_1"'] & E \arrow[d, tail, "m_2"] \\ FC' & FB'. \end{tikzcd}$$

The lifting $\bar{f}$ exists because fe_1 factors through e_2. Since F is a full embedding, $\bar{f}$ is of the form $Fh : FB \to FC$, and h, in turn, represents an arrow in $\mathcal{C}_{reg}$ from GD to GE. The adjointness is easily verified, just as the facts $\hat{F}G \cong Id$ and $GF \cong y$. This proves 1).

For the converse, if a right adjoint G exists with $GF = y$ and counit iso, then cover an object D in $\mathcal{D}$ as follows: G sends D to some map $k : C \to C'$. This gives $Fk : FC \to FC'$. The image of Fk is D, so the factorization of $Fk = me$ gives a cover of D. Also, D embeds into FC'. If $p : FB \to D$ is any arrow, then Gp is a map in $\mathcal{C}_{reg}$ from $GFB = yB$ to k. Thus it has a representative $h : B \to C$. This shows that p factors through e. Therefore e is a cover with the required properties.

For the equivalence between 1) and 3), we start from the correspondence of topologies on $\mathcal{C}$ and universal closure operators on $\mathcal{C}_{reg}$. Thus, any topology gives a category $\mathcal{D}$ of sheaves, and the condition $GF \cong y$ corresponds to this topology being subcanonical, i.e. to the condition that $\mathcal{C}$ is full in $\mathcal{D}$.

$\square$

Let us remark that if the right adjoint is regular, then it is automatically an equivalence, since $\mathcal{D}$ then has the same universal property as $\mathcal{C}_{reg}$.

For the remainder of this section we assume that $F : \mathcal{E} \to \mathcal{C}$ is such, that the induced class J is a subcanonical topology. By the above lemma, this means that $\mathcal{C}_{\mathcal{E}/reg}$ is reflective in $\mathcal{C}_{reg}$. In this case, we make the following easy observations:

Lemma 6.3 *If J is a subcanonical topology, then:*

1. *$\mathcal{C}_{\mathcal{E}/reg}$ is the full subcategory of $\mathcal{C}_{reg}$ on the objects $\begin{pmatrix} X \\ f\downarrow \\ Y \end{pmatrix}$ for which f is closed w.r.t J;*

2. *the functor $F : \mathcal{E} \to \mathcal{C}$ is regular;*

3. *the functor $P_\Sigma \circ y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ is regular and is a minimal covering;*

4. *an object is projective in $\mathcal{C}_{\mathcal{E}/reg}$ if it is isomorphic to an object of the form $P_\Sigma y(X)$ for which X is projective w.r.t. all regular maps in J in $\mathcal{C}$. Thus $P_\Sigma y$ preserves projectives.*

Proof. Item 1) is direct from the correspondence between topologies on $\mathcal{C}$ and closure operators on $\mathcal{C}_{reg}$. 2) follows from the definition of J, 3) follows from Lemma 6.2 and 4) follows from the observation that the regular epis in $\mathcal{C}_{\mathcal{E}/reg}$ are the maps for which the underlying arrow is a map in J.

□

This lemma gives a good description of the properties of $\mathcal{C}_{\mathcal{E}/reg}$ as a subcategory of $\mathcal{C}_{reg}$. Moreover, we show that $\mathcal{C}_{\mathcal{E}/reg}$ is in fact the largest such subcategory:

Theorem 6.4 *Let J be a topology. Then $\mathcal{C}_{\mathcal{E}/reg}$ is characterized as the largest category with the following properties:*

- *$\mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ is a minimal covering;*
- *the composite $\mathcal{E} \to \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$ is regular.*

*This means, that any other category of $\mathcal{E}/$**REG** that satisfies those properties will be a full reflective subcategory of $\mathcal{C}_{\mathcal{E}/reg}$.*

Proof. The previous lemma showed that $\mathcal{C}_{\mathcal{E}/reg}$ indeed has those properties. If some category D also has them, then this implies that there is a topology H on $\mathcal{C}$ such that D is sheaves for the induced closure operator on $\mathcal{C}_{reg}$. Moreover, from the fact that $\mathcal{E} \to \mathcal{C} \to \mathcal{D}$ is regular, we find that this topology H is *larger* than J, because maps in Σ are dense for it. Therefore, any map f in $\mathcal{C}$ that is closed for H is automatically closed for J. Now $\mathcal{D}$ is the full subcategory of $\mathcal{C}_{reg}$ on the H-closed maps, whereas $\mathcal{C}_{\mathcal{E}/reg}$ is the full subcategory on the J-closed maps. Hence $\mathcal{D}$ is contained in $\mathcal{C}_{\mathcal{E}/reg}$. A reflection is obtained via the universal property of $\mathcal{C}_{\mathcal{E}/reg}$.

□

It would be desirable to know what the role of the objects of $\mathcal{C}$ inside the category $\mathcal{C}_{\mathcal{E}/reg}$ is. It is clear that they are not, in general, the projective objects. A closely related question is: given a minimal covering as in Lemma 6.2, how can we, categorically, distinguish the objects of $\mathcal{C}$ inside $\mathcal{D}$? Although we could not provide a full answer to this question, the following is worth noticing: the objects if $\mathcal{C}$ are precisely the objects which are projective w.r.t. a certain class of regular epimorphisms. This class can be described in various ways;

for instance, it is the class of regular epis that are preserved by the inclusion $\mathcal{D} \to \mathcal{C}_{reg}$. Unfortunately, we could not find a description of this class that makes no reference to the category $\mathcal{C}$.

We conclude this section with a remark about the topology J and a question. The smallest possible topology is that of the split epis: this is obtained by taking a functor $F : \mathcal{E} \to \mathcal{C}$ that sends every regular epi of $\mathcal{E}$ to a split epi (cf. example 3.3.3). On the other hand, the largest topology that we can obtain is the topology consisting of all regular epis in $\mathcal{C}$. The tripos-theoretic examples that we will deal with in section 7 will be instances of this. It might be good to know necessary and sufficient conditions on $F : \mathcal{E} \to \mathcal{C}$ under which J is a topology consisting of all regular epis.

7 Assemblies and Locales

We have now enough concepts and facts to give a categorical characterization of the category of assemblies $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$. We stress, that the approach below uses a mild assumption on the partial combinatory algebra: each inhabited subobject $B \subset \mathbb{A}$ should have a global element $1 \to B$.

Our first aim is to show the following theorem:

Theorem 7.1 *The categories $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})_{\mathcal{E}/\mathbf{REG}}$ and $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ are equivalent.*

First, it is easily seen that the inclusion $i : \mathbf{PAss}_{\mathcal{E}}(\mathbb{A}) \to \mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is a minimal covering: we already described how to cover an assembly (X, E_X) with a partitioned assembly (Q, E_Q). If $m : (Y, E_Y) \to (X, E_X)$ is any map with (Y, E_Y) partitioned, then we have $\mathcal{E} \models \exists r : r \bullet E_Y(y) \in E_X(m(y))$. Now we use our assumption and pick a global element $r : 1 \to \mathbb{A}$ Now put $\bar{m}(y) = (m(y), r \bullet E_Y(y))$. Thus we have a lifting $\bar{m} : (Y, E_Y) \to (Q, E_Q)$, which shows that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is a reflective subcategory of $(\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{reg}$. Denote the topology on $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ corresponding to this closure operator with M.

The corresponding universal closure operator may therefore be described in the following manner: given $f : (X, \alpha) \to (Y, \beta)$, we define an equivalence relation on $X : x \sim x' \Leftrightarrow f(x) = f(x') \wedge \alpha(x) = \alpha(x')$. This induces an object $(X/{\sim}, \hat{\alpha})$ and a factorization of f through $(X/{\sim}, \hat{\alpha})$, where $(X, \alpha) \to (X/{\sim}, \hat{\alpha})$ is regular epi.

Now it is easily derived that if a map is in the corresponding topology, then it must be a cartesian map, hence in the topology induced by the regular epis in the image of ∇. Conversely, for such a regular epi $\nabla(e)$, we see that it already a sheaf (considered as an object of $(\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{reg}$. So the two topologies coincide. $\square$

Note, that the topology J on the category $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ consists of all regular epis. This implies that $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ is, up to isomorphism, the unique category for which there is a minimal covering $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}) \to \mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$ which preserves the regular structure; indeed, by the theory of the previous section, a minimal cover corresponds to a topology in which every map is regular epi (because $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$

must be a full subcategory); for any such minimal cover $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}) \to \mathcal{D}$ we know that $\mathcal{D}$ is a reflective subcategory of $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$. But this means that the topology corresponding to $\mathcal{D}$ is larger that J, which is impossible since it must consist of regular epis.

The above theorem has the following corollary:

Corollary 7.2 $(\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{\mathcal{E}/ex} \simeq \mathbf{RT}[\mathbb{A}]$.

Proof. This is a straightforward consequence of Theorem 2.5, because we have $(\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{\mathcal{E}/ex} \simeq ((\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{\mathcal{E}/reg})_{ex/reg} \simeq (\mathbf{Ass}_{\mathcal{E}}(\mathbb{A}))_{ex/reg} \simeq \mathbf{RT}[\mathbb{A}]$. $\square$

Our second application concerns locales; in [9] one finds the following theorem:

Theorem 7.3 (Menni) *Let H be a locale, and let I^*H denote the non-empty downsets in H. Then $Fam(H)_{reg} \simeq Fam(I^*H)$.*

We will generalize this to an arbitrary locale in an arbitrary topos. So let $\mathcal{E}$ be such a topos, and let H be a locale in $\mathcal{E}$. Then, with notation as in section 2, we get:

Theorem 7.4 *The categories $(\int_{\mathcal{E}} H)_{\mathcal{E}/reg}$ and $\int_{\mathcal{E}}(I^*H)$ are equivalent.*

Proof. This is virtually the same construction as for assemblies. There is an embedding of $\int_{\mathcal{E}} H$ into $\int_{\mathcal{E}}(I^*H)$, via $(X, \alpha) \mapsto (X, \alpha')$ with $\alpha'(x) = \downarrow(\alpha(x))$. We cover an object (Y, β) of $\int_{\mathcal{E}}(I^*H)$ with (Q, π), with $Q = \{(y, a) | a \in \beta(y)\}$, and $\pi(y, a) = \downarrow(a)$. Then one shows that maps $f : (Y, \beta) \to (Y', \beta')$ lift to these covers. Also, one embeds (Y, β) in $(Y, \top)$, where $\top(y) = H$. For any functor $G : \int_{\mathcal{E}} H \to \mathcal{D}$ in $\mathcal{E}/REG$, the extension $\hat{G} : \int_{\mathcal{E}}(I^*H) \to \mathcal{D}$ is defined by sending an object (Y, β) to the image of the map $G(Q, \pi) \to G(Y, \top)$. This gives the universal property. $\square$

8 Discussion and Open Questions

There are a lot of interesting open questions, to which we have not provided any answers. The typical type of theorems that are proved about completions are of the form: the regular/exact completion of $\mathcal{C}$ has property X iff $\mathcal{C}$ has property Y, where Y is usually some weakened version of X. For example, $\mathcal{C}_{ex}$ is locally cartesian closed iff $\mathcal{C}$ has weak dependant products [4]. Or: $\mathcal{C}_{ex}$ is a topos iff $\mathcal{C}$ has weak dependant products and a generic proof [9]. For the relativised version, the same questions can be asked, but there are even more basic questions:

1. How can we characterize those objects of $\mathcal{E}/\mathbf{REG}$ that are in the image of $(-)_{\mathcal{E}/reg}$?

2. How can we characterize those objects of $\mathcal{C}_{\mathcal{E}/reg}$ that are in the image of $y : \mathcal{C} \to \mathcal{C}_{\mathcal{E}/reg}$?

3. Same questions for the relative exact completion.

The problem here seems to be to find the right relativation of the notion of projectivity, which is at the heart of the answers to the classical questions.

Another interesting point is, that our main examples were tripos-theoretic in nature. This suggests that a uniform treatment should be possible. Is there an operation on indexed pre-orders that corresponds, on the level of their categories of elements, to the relative regular completion? In fact, we can give an affirmative answer here, but this will be the subject of another paper.

Furthermore, in our treatment of assemblies, we used the assumption that the underlying pca had enough global elements. Although this is not a severe limitation (it is certainly satisfied if the terminal object 1 is projective, for instance), we feel obliged to say a word about what would happen if we omitted it. From the constructions of the canonical covers, it is clear that this approach makes essential use of the assumption, so the theorem $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A}) \simeq (\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{\mathcal{E}/reg}$ would be simply false if we drop it. However, one might try something along the following line: redefine the categories $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ and $\mathbf{Ass}_{\mathcal{E}}(\mathbb{A})$, by taking the same objects, but as morphisms $(X, \alpha) \to (Y, \beta)$ total relations $R \subset X \times Y$ for which $\mathcal{E} \models \exists a : \mathbb{A} \forall x : X \exists y : Y(R(x, y) \wedge a \bullet \alpha(x) \downarrow \wedge \beta(y) = a \bullet \alpha(x))$. This certainly circumvents the need for global elements, but now the relationship of these newly defined categories with the realizability topos is less clear...

Another problem concerning partitioned assemblies is the following: if $\mathcal{E}$, the underlying topos, does not have choice, then $(\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{ex}$ differs from $\mathbf{RT}[\mathbb{A}]$ (in fact, the latter is a reflective subcategory of the former). But can it happen that $(\mathbf{PAss}_{\mathcal{E}}(\mathbb{A}))_{ex}$ is still a topos? We know (see [9]), that this is equivalent to asking whether $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ has a generic proof, which in turn implies that $\mathcal{E}$ has a generic proof. The only examples of toposes with generic proofs that we are aware of, are toposes that satisfy the axiom of choice or that arise as a coproduct completion of a small category. The latter can be characterized as Grothendieck toposes of the form sheaves for the double negation topology on an atomic site. We could not answer the question whether $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ having a generic proof implies that $\mathcal{E} \models AC$, although it can be shown that if the classical construction of a generic proof in $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ still works, that this implies choice for $\mathcal{E}$; in other words, if $\mathbf{PAss}_{\mathcal{E}}(\mathbb{A})$ has a generic proof and $\mathcal{E}$ does not have choice, then this generic proof is not the usual one!

Finally, we would like to see more mathematically interesting examples of relative completions that are not of the above kind.

9 Appendix: Proof of Lemma 5.5

We prove lemma 5.5, stating that for every map $f : X \to Y$ be a map in $\mathcal{C}$, inducing a mono $[f] : f \to yY$ in $\mathcal{C}_{reg}$: $f \in J(Y)$ implies $[f] \in \Sigma$.

Proof. We show this by induction on the structure of J. Observe, for the basic case, that if f is of the form $F(e)$ with e regular epi in $\mathcal{E}$, then $[f] \in \Sigma_0$ (and vice versa). Next, if f arises as the pullback of such a map $F(e)$ as in the left diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "\alpha"] \\
FE' \arrow[r, "Fe"] & FE
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{pmatrix} M \\ p\downarrow \\ N \end{pmatrix} \arrow[dr, dashed, "{[u]}"] \arrow[drr, "{[r]}"] \arrow[ddr, "{[s]}"'] & & \\
& \begin{pmatrix} X \\ f\downarrow \\ Y \end{pmatrix} \arrow[r] \arrow[d, "{[f]}"] & \begin{pmatrix} FE' \\ Fe\downarrow \\ FE \end{pmatrix} \arrow[d, "{[Fe]}"] \\
& y(Y) \arrow[r, "y(\alpha)"] & y(FE)
\end{tikzcd}$$

then we can show that the square in the diagram on the right is a pullback in $\mathcal{C}_{reg}$: for consider another object $(p : M \to N)$ in $\mathcal{C}_{reg}$, and maps $[r] : p \to Fe, [s] : p \to y(Y)$, such that $[Fe] \circ [r] = y(\alpha) \circ [s]$, i.e., $Fe \circ r = \alpha \circ s$. Since the left diagram is a pullback in $\mathcal{C}$, there is a unique $u : M \to X$, such that $\pi_Y \circ u = s, \pi_{FE} \circ u = r$. No in $\mathcal{C}_{reg}$, u induces a map $[u] : p \to f$, because (writing p_0, p_1 for the kernel pair of p), $f \circ u \circ p_0 = s \circ p_0 = s \circ p_1 = f \circ u \circ p_1$. This map $[u]$ is the unique map that makes $[r]$ and $[s]$ factor through the object f. Hence the square is a pullback. Since Σ was closed under pullbacks and the right-hand map was in it, so is the left-hand map.

Then, suppose that h is in $J(Y)$ because $f = hg$ is in $J(Y)$. By induction hypothesis, $[f] = [hg]$ is in Σ. We need to verify that $[h] \in \Sigma$. But $[h]$ is mono, and Σ is a pullback congruence, so if $[hg]$ will be inverted, so will $[h]$, and therefore $[h] \in \Sigma$.

Finally, consider a composite of such arrows (it suffices to look only at a binary composite): suppose $h \in J(Y), g \in J(Y')$, so that, by induction hypothesis, $[h] : h \to y(Y), [g] : g \to y(Y') \in \Sigma$. We must show that $[hg] : hg \to y(Y) \in \Sigma$. First, consider the following pullbacks, where the first one is in $\mathcal{C}$, and the second one in $\mathcal{C}_{reg}$:

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, "1"] \arrow[dddr, "g"'] & & \\
& X \times_Y Y' \arrow[r] \arrow[dd] & X \arrow[d, "g"] \\
& & Y' \arrow[d, "h"] \\
& Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

$$
\begin{tikzcd}
\begin{pmatrix} X \\ g\downarrow \\ Y' \end{pmatrix} \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& \begin{pmatrix} X \times_Y Y' \\ \downarrow \\ Y' \end{pmatrix} \arrow[r, two heads] \arrow[d, tail] & \begin{pmatrix} X \\ g\downarrow \\ Y' \\ h\downarrow \\ Y \end{pmatrix} \arrow[d, tail, "{[g]}"] \\
& yY' \arrow[r, two heads, "{[1]}"] & \begin{pmatrix} Y' \\ h\downarrow \\ Y \end{pmatrix}
\end{tikzcd}
$$

In the second diagram, the object $X \times_Y Y' \to Y'$ is the projection as in the first pullback. It is easily verified that the outer square of the second diagram commutes, so there is a factorization through the pullback. Now, since $[g] : g \to yY'$ in Σ, so is $[\pi_{Y'}] : \pi_{Y'} \to yY'$. Because pullbacks along regular epimorphisms reflect Σ-maps, we obtain that $[g] : hg \to h$ is also in Σ. Using that Σ is closed under composition, we find that $[hg] : hg \to h \to yY \in \Sigma$, and our induction is complete.

$\square$

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