W-types in the Effective Topos

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1 Introduction

In this paper we give an explicit description of the W-types in the effective topos **Eff**. The result that the effective topos has W-types is immediate from the observation of Moerdijk and Palmgren (see [5]) that any topos with a natural number object has W-types. Nevertheless, we feel that an explicit construction is of interest.

We limit ourselves to the construction and will not provide an exhaustive justification for the claim that it is the W-type. The missing details can be found in the article [1] (this paper will contain references to the relevant points in that article).

We warn the reader who is concerned about foundations, or is working over an arbitrary topos, that the validity of the description that we give depends on the validity of the internal axiom of choice in **Set**. We refer to the remark at the end of this paper.

A final word about the outline of this paper. Before giving an explicit description of the W-type for a map f in the effective topos, we first isolate a full subcategory of it, the category **Ass** of assemblies, and give an explicit description of the W-types in this category. Then we show that these W-types are also the W-types in the whole category. In the last section we demonstrate how this can be used to give an explicit description of all W-types in the effective topos.

2 The construction of W-types in Ass

In this section we will introduce a full subcategory of the effective topos, the category of assemblies, and show that it has W-types.

An *assembly* is a set X together with a function $E_X : X \to \mathcal{P}_i\mathbb{N}$. Here $\mathcal{P}_i\mathbb{N}$ means the set of inhabited subsets of $\mathbb{N}$. (Such an assembly can be regarded as an object in the effective topos in an obvious way.) If $n \in E_X(x)$, n will be called a *realizer* or a *witness* for x. The set X is the *underlying set* of the assembly (X, E_X) and we will use X both as a name for the assembly and its underlying set.

If Y is some other assembly, a morphism of assemblies from Y to X is a function $f : Y \to X$ (the *underlying function*) with the property that f is *tracked* (or *realized*) by some partial recursive function r. This means that for every element y and every realizer n of y, $r \cdot n$ is defined and equal to a realizer of $f(y)$. (Nota bene: $\cdot$ is the symbol for Kleene application, and partial recursive functions will always be assumed to be coded as natural numbers.)

It is well known that the category of assemblies is a locally cartesian closed regular category with natural number object and finite disjoint sums. We will show that it also has W-types, but first we will give an explicit description of the P_f-functor is the category of assemblies.

So let $f : B \to A$ be a fixed morphism of assemblies and let X be an arbitrary assembly. Now the underlying set of $P_f(X)$ is the set of pairs (a,t) where a is an element of A and t is a function from B_a to X. A natural number n is a realizer for (a,t), if $n =< n_0, n_1 >$ is such that n_0 realizes a and n_1 tracks t (the latter meaning, of course, that for every $b \in B_a$ and every realizer m of b, $n_1 \cdot m$ is defined and equal to a realizer of tb).

The W-type for f is now constructed as follows. First construct the W-type W for the underlying function f in the category **Set**. We now define a function $E : W \to \mathcal{P}\mathbb{N}$ by transfinite induction: $E(\sup_a t)$ consists of those natural numbers $n =< n_0, n_1 >$ such that (i) n_0 realizes a; and (ii) n_1 tracks t, that is, for every $b \in B_a$ and every realizer m of b, $n_1 \cdot m$ is defined and a member of $E(tb)$. We call a member n of $E(w)$ a *decoration* or a *realizer* of the tree $w \in W$. The trees w that have a decoration are called *decorable* and V will be the name of the set of all decorable trees.

The set V is the underlying set of an assembly that has the restriction of E to V as second component. This assembly, that will also be called V, is, we claim, the W-type for f in the category of assemblies. It is not hard to see that it has the structure of a P_f-algebra. Let a be an element of A and t be a function $B_a \to V$. The element $\sup_a(t) \in W$ is actually an element of V, because if n is a realizer of (a,t) in $P_f(V)$, then n is a decoration of $\sup_a(t)$ (this is immediate from the definition of E). So one has a map of assemblies $s : P_f(V) \to V$ that is tracked by the identity.

Now we want to invoke the following theorem (theorem 25 in [1]) to show that V is the W-type.

Theorem 2.1 *Let* **C** *be a locally cartesian closed regular category with a natural number object and finite disjoint sums and let $f : B \to A$ be a morphism in* **C**. *Assume that $(V, s : P_f(V) \to V)$ is a P_f-algebra having the following two properties: (i) its structure map s is mono; (ii) it has no proper P_f-subalgebras. Then V is the W-type for f.*

One immediately sees that s is monic, simply because sup is (in fact, it is not hard to see that s is iso). Therefore we only need to show that V has the second property.

Let $(X, m : P_f(X) \to X)$ be a subalgebra of V. We may assume that the underlying set X is actually a subset of V and that m is the restriction of s to $P_f(X)$ on the level of underlying functions. First of all, we show that $X = V$ on the level of sets. Let P be set of trees $w \in W$ for which we have that:

$$w \in V \Rightarrow w \in X$$

We show that $P = W$ by transfinite induction, which immediately shows that $X = V$ as sets. So suppose $\sup_a t \in W$ and $tb \in P$ for all $b \in B_a$ (here $a \in A, t : B_a \to W$, of course). We want to show that $\sup_a t \in P$, so assume that $\sup_a t \in V$. Because s is iso, we have that $tb \in V$ and hence, by induction hypothesis, $tb \in X$. Since on the level of sets, m is the restriction of s which is a restriction of sup, we have that $\sup_a t$ is in X. This completes the proof.

To show, finally, that the X and V are isomorphic as assemblies, we have to show that the identity map $i : V \to X$ is tracked by some partial recursive function r. For this, let p be a partial recursive function tracking m and let H be the primitive recursive function computing the composition of two partial recursive functions (that is, $H(x,y) \cdot n = x \cdot (y \cdot n)$). Now solve r from the following equation using the

First Recursion Theorem:

$$r \cdot < n_0, n_1 > = p \cdot < n_0, H(r, n_1) >$$

It is easy to see that r tracks i, by proving by transfinite induction that for any tree $w \in W$ and any decoration n of w, $r \cdot n$ is defined and a realizer of $i(w)$.

This completes the proof of the fact that V is the W-type of f in the category of assemblies.

Remark 2.2 The fact that **Ass** has W-types is probably well-known, although the only written proof that we could find is contained in [2]. The argument that we give here is based on some unpublished notes by Ieke Moerdijk.

3 The inclusion i : Ass $\rightarrow$ Eff preserves W-types

In this section we intend to show that the inclusion i : **Ass** $\rightarrow$ **Eff** preserves W-types. What we mean is that the W-type for a map f in **Ass** will be sent to the W-type for the map if in the effective topos. The proof will be very easy once we make the following two observations. But let's fix notation first: let $f : B \rightarrow A$ be a morphism of assemblies and let W be the W-type of f in the category of assemblies.

First of all, observe that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Ass} \arrow[r, "i"] \arrow[d, "P_f"'] & \mathbf{Eff} \arrow[d, "P_{if}"] \\ \mathbf{Ass} \arrow[r, "i"'] & \mathbf{Eff} \end{tikzcd}$$

In words, this means that if one takes an assembly X for which one wishes to compute $P_{if}(iX)$ in the effective topos, one might do this by computing $P_f(X)$ in the category of assemblies and then take the corresponding object in the effective topos. (The reason is that an exponential of two assemblies in the effective topos yields the same object as in assemblies, also when one is working in a slice over an assembly.) This means that iW is a P_{if}-algebra in the effective topos and that the structure map is an isomorphism.

The second observation is that a subobject of an assembly in the effective topos is, up to isomorphism, again an assembly. This is immediate, when one notes that the assemblies are, up to isomorphism, the separated objects of the effective topos. (See [3], pp. 182 and 185.)

This means that the result follows immediately from theorem 2.1. (One could also invoke proposition 1 in [4].)

4 W-types in Eff

In this section we assume the internal axiom of choice (IAC) in **Set** and we construct W-types for all maps f in the effective topos. But first we need to introduce a full subcategory of **Ass**, the category **Pass** of partitioned assemblies.

A *partitioned assembly* is a set X together with a function $\pi_X : X \rightarrow \mathbb{N}$ and a morphism of partitioned assemblies is a function $f : Y \rightarrow X$ which is tracked by some partial recursive function r. This means that for every element y, $r \cdot \pi_Y(y)$ is defined and equal to $\pi_X(f(y))$. Any partitioned assembly is clearly an assembly.

Recall that **Eff** has enough partitioned assemblies: if $(X, =)$ is an object in **Eff** then $(\mathbb{N} \times X', \pi_1)$ where $X' = \{x \in X \mid E[x] \neq \emptyset\}$ and π_1 is the first projection, covers $(X, =)$, and we call such a cover a *pass cover* for $(X, =)$.

By theorem 28 of [1], given a diagram as follows:

$$\begin{tikzcd} B' \arrow[r, "\{\ \}"'] \arrow[d, "\phi"'] & B \arrow[d, "f"] \\ A' \arrow[r, "\{\ \}"'] & A \end{tikzcd}$$

where ϕ a choice map and the square a quasi-pullback, $W := W(f)$ can be constructed as a subquotient of $W' := W(\phi)$. More precisely, consider the following relation on W', defined inductively in the internal logic by: $\mathrm{sup}_\alpha \tau \sim \mathrm{sup}_{\alpha'} \tau'$ iff

$$\{\alpha\} = \{\alpha'\} \wedge \forall \beta \in \phi^{-1}(\alpha), \forall \beta' \in \phi^{-1}(\alpha') : \{\beta\} = \{\beta'\} \rightarrow \tau\beta \sim \tau'\beta'$$

$\sim$ is symmetric and transitive. We construct W by considering the reflexive elements and dividing out by the equivalence relation $\sim$.

Besides, the structure map $\mathrm{sup} : P_f(W) \rightarrow W$ is the unique arrow making the following diagram commute:

$$\begin{tikzcd} P_f(W) \arrow[d, dotted, "\mathrm{sup}"] & R^* \arrow[l, "q^*"] \arrow[r] \arrow[d, "\mathrm{sup}"] \arrow[dr, phantom, "\lrcorner", very near start] & P_\phi(W') \arrow[d, "\mathrm{sup}"] \\ W & R \arrow[l, "q"'] \arrow[r, tail] & W' \end{tikzcd}$$

where R is the object of reflexive elements, q the quotient map and q^* is defined on a pair $(\alpha, \tau : B'_\alpha \rightarrow W')$ such that $\mathrm{sup}_\alpha \tau \in R$ as the pair (a, t), with $a = \{\alpha\}$ and $t : B_a \rightarrow W$ defined by $t(\{\beta\}) = [\tau(\beta)]$ (which is well-defined, because $\mathrm{sup}_\alpha \tau \in R$). (It is in showing that q^* is epi that one uses the fact that ϕ is a choice map.)

Now, assuming the internal axiom of choice (IAC) in **Set**, this is all what we need in order to give an explicit description of the W-types in **Eff**, because of the following lemmas.

Lemma 4.1 *(with IAC) If P is a partitioned assembly, then $(-)^P$ preserves any epi whose domain is also a partitioned assembly.*

Proof: Let $e : X \rightarrow Y$ be an epi and X a partitioned assembly. Let also E be a representative for e (i.e. $e = [E]$) and $\beta : P \times Y \rightarrow \mathcal{P}(\mathbb{N})$ be such that "β is a functional relation" is realized by some $m \in \mathbb{N}$.

There are two recursive functions ϕ_m and ϕ'_m, both recursive in m, such that, for any p, y and for any $n \in \beta(p, y)$: $\phi_m(n)$ is defined and $\phi_m(n) \in E(x, y)$ for some $x \in X$ and for any p, y and for any $n \in \beta(p, y)$: $\phi'_m(n)$ is defined and $\phi'_m(n) = \pi(x)$ for some $x \in X$ (using the facts that i) β is strict, ii) e is epi and iii) E is strict). There is also a recursive function ψ_m, recursive in m, such that, for any p: $\psi_m(\pi(p)) = \pi(x)$ for some $x \in X$ (using the totality of β and the above).

By (IAC), there is a function $f : P \rightarrow X$ such that, for any p: $\psi_m(\pi(p)) = \pi(f(p))$ and f is tracked by any code for ψ_m. In other words, $\alpha(p, x) := \{\pi(p)\}$ iff $f(p) = x$ ($\emptyset$ otherwise) is a functional relation, and a realizer for E_α can be obtained recursively from any code for ψ_m, hence, from any realizer for E_β. Thus, a realizer can be found (using the above and the fact that E is single-valued) for:

$$\bigcap_\beta (E_\beta \rightarrow \bigcup_\alpha (E_\alpha \wedge E_\beta \wedge \bigcap_{p,y} \beta(p, y) \leftrightarrow \bigcup_x (\alpha(p, x) \wedge E(x, y))))$$

But this means exactly that $X^P \rightarrow Y^P$ is epi. $\square$

Lemma 4.2 *(with IAC) Every partitioned assembly is internally projective.*

Proof: Take an epi $e : Y \to X$ and a partitioned assembly P. Consider the following quasi-pullback:

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "e'"'] & Y \arrow[d, "e"] \\ X' \arrow[r] & X \end{tikzcd}$$

where X' and Y' are pass covers of X and $X' \times_X Y$ respectively. Note that e' is epi because quasi-pullbacks preserve epis. By the preceeding lemma, applying $(-)^P$ to this diagram gives us a commutative diagram where the arrow e^P is epi because the other three are.

$$\begin{tikzcd} Y'^P \arrow[r] \arrow[d] & Y^P \arrow[d, "e^P"] \\ X'^P \arrow[r] & X^P \end{tikzcd}$$

□

Lemma 4.3 *(with IAC) Every map between partitioned assemblies is a choice map.*

Sketch of proof: Recall that if (IAC) holds in a topos then it holds in any slice. Therefore, we can use the same kind of arguments as in lemmas 4.1 and 4.2. □

The next lemma now follows easily from the ones above.

Lemma 4.4 *(with IAC) For every arrow $f : B \to A$ in* **Eff** *there exist partitioned assemblies A' and B', covering A and B respectively, and a choice map ϕ such that the following diagram is a quasi-pullback:*

$$\begin{tikzcd} B' \arrow[r, "\{\ \}"] \arrow[d, "\phi"'] & B \arrow[d, "f"] \\ A' \arrow[r, "\{\ \}"'] & A \end{tikzcd}$$

Proof: Choose A' and B' to be pass covers of A and $A' \times_A B$ respectively. □

Using the internal logic of **Eff**, we can now describe W-types as follows:

Theorem 4.5 *(with IAC) If $f : B \to A$ is an arrow in* **Eff** *then $W := W(f) \cong (W_{\mathbf{Set}}(\phi), \sim)$, where ϕ is as in the lemma above, and, for $w = sup_\alpha \tau$, $w' = sup_{\alpha'} \tau'$: $r \Vdash w \sim w'$ if and only if $r =< r_0, r_1, r_2 >$ such that the following hold:*

(1) $r_0 \Vdash Ew \wedge Ew'$

(2) $r_1 \Vdash a = a'$

(3) *for all β, β', m such that $m \Vdash \beta \in \phi^{-1}(\alpha) \wedge \beta' \in \phi^{-1}(\alpha') \wedge b = b'$, $r_2 \cdot m$ is defined and $r_2 \cdot m \Vdash \tau\beta \sim \tau'\beta'$*

where $\alpha = (a, n) \in A', \alpha' = (a', n') \in A', \beta = (b, p) \in B', \beta' = (b', p') \in B'$ and $Ew = \{$decorations of $w\}$.

Moreover, for $a \in |A|$, $T : |B| \times |W| \to \mathcal{P}(\mathbb{N})$, and $w \in W_{Set}(\phi)$, we write:

$$\begin{aligned} E(a,T) \quad &= \quad E(a) \wedge [T \textit{ is a strict and single-valued relation}] \wedge \\ &\quad \bigcap_{b \in |B|} (F(b,a) \leftrightarrow \bigcup_{w \in |W|} T(b,w)) \end{aligned}$$

Then the structure map $sup : P_f(W) \to W$ can be represented by:

$$\begin{aligned} sup((a,T),w) &= E(a,T) \wedge \bigcup_{\alpha,\tau \in |R^*|} [(\{\alpha\} = a) \wedge (sup_\alpha \tau \sim w) \wedge \\ & \bigcap_{\beta \in |B'_\alpha|} (\cup_{w' \in |W|} T(\{\beta\}, w') \wedge (\tau\beta \sim w'))] \end{aligned}$$

Proof: The proof consists in rewriting in terms of realizers the description we gave using the internal logic of **Eff**. □

Remark 4.6 In the absence of the internal axiom of choice in **Set**, the description in theorem 4.5 is no longer valid. In fact, it can be shown that the validity of the description implies the internal axiom of choice in **Set**. We will outline a proof of this statement.

Let B be any set and A a set together with an equivalence relation $\sim$. Let $q : A \to A/\sim$ be the quotient map. We will prove the validity of the statement

$$\forall g \in (A/\sim)^B \, \exists h \in A^B : q \circ h = g$$

in the internal logic of **Set**. This implies the internal axiom of choice.

It is well known that there exists a functor ∇ that embeds **Set** in the effective topos. We change the definition of ∇ a bit in order to get a functor ∇_0:

$$\nabla_0(A) = (A, =)$$

where $n \in [a = a']$ iff $n = 0$ and $a = a'$. ∇_0 is naturally isomorphic to ∇, but the difference is that objects in the image of ∇_0 are partitioned assemblies.

Consider the following two objects of the effective topos: $\nabla_0 B$ and $\nabla_0(A/\sim)$. Observe that any function $g : B \to A/\sim$ gives rise to an element G of $|\nabla_0(A/\sim)^{\nabla_0 B}|$.

Now consider the following morphism in the effective topos: $f : \nabla_0 B \to 1 + \nabla_0(A/\sim)$, the composition of the unique map $\nabla_0 B \to 1$ and the sum inclusion $1 \to 1 + \nabla_0(A/\sim)$. If ϕ is the composition of $\nabla_0 B \to 1$ and $1 \to 1 + \nabla_0 A$, then f and ϕ fit into a (quasi-)pullback of the form considered in lemma 4.4 (minus the requirement that ϕ is a choice map).

The next observation is that the equivalence classes $[a]$ belong to $W(f)$. To be more precise, the elements a belong to the underlying set $W_{\mathbf{Set}}(\phi)$ and $[a =_{W(f)} a']$ is inhabited iff $a \sim a'$. So the element G of $|\nabla_0(A/\sim)^{\nabla_0 B}|$ gives rise to an element $(*, G)$ in $|P_f(W(f))|$ (where $*$ is the unique element of 1).

Now the totality of the morphism sup implies more or less immediately that there exists a morphism $h : B \to A$ such that $q \circ h = g$. But this is precisely what we wanted to prove.

References

[1] Benno van den Berg. Inductive types and exact completion. Available from http://www.mathpreprints.com/math/Preprint/Benno/20031031/1, 2003.

[2] Yuxi Fu. Understanding inductive types in constructions. Technical Report UMCS-93-6-5, Computer Science, University of Manchester, 1993.

[3] J.M.E. Hyland. The effective topos. In A.S. Troelstra and D. van Dalen, editors, *The L.E.J. Brouwer Centenary Symposium*, pages 165–216. North-Holland, Amsterdam, 1982.

[4] M. Jibladze. A presentation of the initial lift-algebra. *Journal of Pure and Applied Algebra*, 116:185–198, 1997.

[5] Ieke Moerdijk and Erik Palmgren. Wellfounded trees in categories. *Annals of Pure and Applied Logic*, 104:189–218, 2000.