Arithmetic is Categorical

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May 30, 2011; revised September 26, 2014

Abstract

We show that in the Effective Topos, there is exactly one model of intuitionistic $\mathrm{I}\Sigma_1$ (the basic theory of the nonnegative integers with induction for Σ_1-formulas). This generalizes and reinterprets a similar theorem by Charles McCarty. We conclude that in the Effective Topos, first-order arithmetic is essentially finitely axiomatized.

In [3], McCarty showed that in the Friedman-McCarty realizability model of intuitionistic set theory, there is only one model of Heyting Arithmetic. See also [4]. The present note strengthens this result and reinterprets it. For unexplained notions concerning the Effective Topos, consult [6].

Let $\mathrm{I}\Sigma_1^i$ be the theory in the language $\{0, S, +, \cdot, \leq\}$ axiomatized by the axioms of $Q_{\leq}$ (see [1]) and induction for Σ_1-formulas; but based on intuitionistic logic.

Theorem 0.1 *In the effective topos $\mathcal{E}ff$ there exists (up to isomorphism) precisely one model of $\mathrm{I}\Sigma_1^i$, namely the standard model N (the canonical structure on the natural numbers object).*

Proof. We recall that $\mathrm{I}\Sigma_1^i$ proves decidability of all Δ_0-formulas. Hence every model of $\mathrm{I}\Sigma_1^i$ must be a decidable object in $\mathcal{E}ff$, and therefore isomorphic to a modest set (X, E) (see [6], p.153).

Since such a model (X, E) has an element 0 and an injective endofunction S, there is an embedding from N into it: a function $i : \mathbb{N} \to X$ such that for some total recursive function t we have $t(n) \in E(n)$ for all $n \in \mathbb{N}$. The decidability of (X,E) means that there is a partial recursive function d which is defined on the set $(\bigcup_{x\in X} E(x))^2$, and satisfies:

$$d(k,l)=0 \;\Leftrightarrow\; \text{there is } x\in X \text{ with } k,l\in E(x)$$

Now if $x\in X$ is in the image of i then for each $a\in E(x)$ there is a unique $n\in\mathbb{N}$ such that $d(a,t(n))=0$; and this n can be found recursively in a. We conclude:

The map i embeds N as $\neg\neg$-closed subobject in (X,E)

Therefore, if the function i is surjective, it is an isomorphism.

For the sake of a contradiction, suppose i is not an isomorphism. Then there is an element $c\in X$ which is not in the image of i, and by decidability of the linear order and the fact that i embeds N as downwards closed subset (which is all provable in $\mathrm{I}\Sigma_1^i$) we have $\mathcal{E}\!f\!f \models \forall n{:}N.i(n)<c$.

Now, we can copy what is essentially McCarty's argument. Since $\mathrm{I}\Sigma_1^i$ proves the representability and totality of all primitive recursive functions, let $\exists z T'(e,x,y,z)$ and $\exists w U'(x,i,w)$ be Σ_1-formulas (so T' and U' are Δ_0) representing the Kleene T-predicate $T(e,x,y)$ and result extracting function $U(x)=i$, respectively. Define the subobject A of (X,E) internally by

$$A \;=\; \{x<c \,|\, \forall y<c\neg\exists z<c\exists w<c(T'(x,x,y,z)\wedge U'(y,1,w))\}$$

Then since A is given by a Δ_0-formula, A is a decidable subobject of (X,E) and hence $i^{-1}(A)$ is a decidable subobject of N; which means that

$$R \;=\; \{n\in\mathbb{N} \,|\, \mathcal{E}\!f\!f\models n\in i^{-1}(A)\}$$

is a recursive subset of $\mathbb{N}$.

Moreover, for the following subsets of $\mathbb{N}$:

$$\begin{array}{rcl} A_0 & = & \{n\in\mathbb{N} \,|\, \varphi_n(n)=0\} \\ A_1 & = & \{n\in\mathbb{N} \,|\, \varphi_n(n)=1\} \end{array}$$

we have $A_0\subset R$ and $A_1\cap R=\emptyset$.

So, R is a recursive separation of the sets A_0 and A_1, but it is well-known that this is impossible. ∎

Corollary 0.2 *Let* IZF *be intuitionistic set theory (as formulated in, e.g., [2]). Then* IZF *does not prove that there is a model of classical $I\Sigma_1$. Moreover,* IZF *does not prove that there is a model of* $\mathrm{I}\Sigma_1^i$ *which is not a model of full Heyting Arithmetic.*

Proof. In [6],section 3.5, it is shown that the Friedman-McCarty realizability interpretation of IZF can be seen as an interpretation of IZF in $\mathcal{E}ff$. Any model as in the corollary would thus give rise to one such model in $\mathcal{E}ff$, which we have shown not to exist. ∎

We conclude that whoever predicates his notion of truth on the effective topos, must accept the following nonstandard conclusions:

a) Classical $I\Sigma_1$ is 'inconsistent' (it has no models)

b) Heyting Arithmetic is essentially finitely axiomatized (it is equivalent to $\mathrm{I}\Sigma_1^i$).

Remark 0.3 Both in [5] and [7], 'realizability-like' toposes are presented in which nonstandard models of PA do exist.

References

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