Ordered Partial Combinatory Algebras

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1 Introduction

Partial Combinatory Algebras, models for a form of Combinatory Logic with partial application, have been studied for the last thirty years because of their close connection to Intuitionistic Logic (see, for example, [11]).

From the "algebraic" side, Partial Combinatory Algebras gave rise to the construction of *elementary toposes* as shown in [4]: for every partial combinatory algebra $\mathbb{A}$ we have the *realizability topos* $\mathbf{RT}[\mathbb{A}]$. The best known of these toposes is Hyland's *Effective Topos* (see [3]).

This paper is motivated by the question: what would be a good category for partial combinatory algebras (pca's), such that the construction of a realizability topos $\mathbf{RT}[\mathbb{A}]$ out of $\mathbb{A}$ becomes a functor with nice properties? Of course, this depends on one's point of view as to which category these realizability toposes live in. Some functoriality is obtained in John Longley's thesis [8]; he defines a 2-category of pca's, such that morphisms in this category correspond to certain exact functors between realizability toposes.

In this paper, we are mainly interested in *geometric morphisms* between realizability toposes. Our approach is both a refinement and an analysis of Longley's. First, we propose the notion of *ordered partial combinatory algebra* (opca), a generalization of pca. The standard construction of realizability toposes goes through for these ordered pca's. This is reviewed in the first section.

However, the context of opca's allows some constructions which are not avaliable for pca's. This becomes apparent when we introduce a 2-category for ordered pca's, **OPCA+**. On this category, there is a 2-monad, the non-empty downset monad, T. Whereas Longley's morphisms are certain total relations, we are able to work with functions and recover his category as follows: Longley's 2-category of pca's is a full subcategory of the Kleisli category $\mathbf{Kl}(T)$ for our monad T, on objects with are in fact genuine pca's. There is a 2-functor from $\mathbf{Kl}(T)$ to the 2-category of realizability triposes and exact functors between them; this functor is locally an equivalence, so that, up to 2-isomorphism, maps in $\mathbf{Kl}(T)$ between two fixed opca's are the same as exact functors between the associated triposes.

The next step is to impose a restriction on opca-maps, obtaining a subcategory **OPCA**, to which the monad T restricts. The idea is that the maps in **OPCA** are precisely the maps which induce geometric morphisms between triposes. Then we obtain a 2-functor from the Kleisli category for the monad on **OPCA** to the 2-category of triposes and geometric morphisms, and this 2-functor is again a local equivalence.

In the third section we focus on (pseudo-) algebras for our monad, and we consider the category $\mathbf{Pass}(\mathbb{A})$ of Partitioned Assemblies associated to an ordered pca $\mathbb{A}$. We obtain the result that $\mathbf{Pass}(\mathbb{A})$ is regular if and only if $\mathbb{A}$ has a pseudo-algebra structure. Moreover, this category is a regular completion (of a category that is again of the form $\mathbf{Pass}(\mathbb{B})$) if and only if $\mathbb{A}$ is equivalent to a free algebra $T\mathbb{B}$.

Then we discuss some applications of our framework. The first one concerns relative realizability (see [1]); the main result is a characterization of those sub-opca's $\mathbb{A}$ of some $\mathbb{B}$ for which there is a local map from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$. In other words, we give a necessary and sufficient condition so that the relative realizability topos $\mathbf{RT}[\mathbb{B},\mathbb{A}]$ coincides with $\mathbf{RT}[\mathbb{B}]$.

A second application is a presentation of a hierarchy of realizability toposes, induced by the sequence of opca's $\mathbb{A}, T\mathbb{A}, T^2\mathbb{A}, \ldots$. The fact that certain hierarchies can be presented in this tripos-theoretic way was already conjectured by Menni [9].

Finally, we give a slight generalization of a theorem by Johnstone and Robinson, stating that the Effective Topos is not equivalent to any topos obtained from a total combinatory algebra.

2 Definitions and Basic Properties

This section sets out the definitions and reviews basic properties. We define ordered pca's, the standard realizability tripos $I(\mathbb{A})$ for an ordered pca $\mathbb{A}$ and the associated categories of assemblies and partitioned assemblies. Most of the well-known properties of these structures for ordinary pca's carry over easily to the ordered case; proofs are omitted.

2.1 Ordered Pca's

Definition 2.1 An *ordered pca* is a triple $\mathbb{A} = (A, \leq, \bullet)$, where $\leq$ partially orders the set A, and where $\bullet$ is a partial function from $A \times A$ to A. We write $a \bullet b{\downarrow}$ or $ab{\downarrow}$ if (a, b) is in the domain of $\bullet$, in which case $a \bullet b$ or ab denote the value. We require that the following conditions are satisfied:

1. For all $a, b \in A$: if $ab{\downarrow}$, $a' \leq a$ and $b' \leq b$, then $a'b'{\downarrow}$ and $a'b' \leq ab$.
2. There are elements k and s of A that satisfy
   - for all $a, b \in A$: $ka{\downarrow}$ and $kab{\downarrow}$ and $kab \leq a$,

- for all $a, b, c \in A$: $sa{\downarrow}$ and $sab{\downarrow}$ and if $(ac)(bc){\downarrow}$, then $sabc{\downarrow}$ and $sabc \leq (ac)(bc)$.

Of course, every ordinary pca can be seen as an ordered pca, by taking the discrete ordering.

The motivating example for the definition of ordered pca's in [12] (where they are called $\leq$-pca's; however, this terminology is hard to pronounce) is the following: given a pca A, the set of nonempty subsets of A (or the set of nonempty finite subsets of A) forms an ordered pca (but not a pca!) by putting

$$\alpha \bullet \beta = \{xy \mid x \in \alpha, y \in \beta\}$$

(This is defined if for all $x \in \alpha$ and $y \in \beta$, $xy{\downarrow}$)

A fundamental property of pca's is their so-called *combinatorial completeness*. Up to $\leq$, this remains true for ordered pca's:

Proposition 2.2 (Combinatorial completeness) *Let $\mathbb{A}$ be an ordered pca. For every term t composed of elements of A, application and variables x, $x_1, \ldots, x_n$, there is a term $[\Lambda x.t]$, containing at most the variables $x_1, \ldots, x_n$, such that for all elements $a, a_1, \ldots, a_n \in A$: if $t[a/x, a_1/x_1, \ldots, a_n/x_n]{\downarrow}$ then*

$$([\Lambda x.t][a_1/x_1, \ldots, a_n/x_n])a{\downarrow}$$

and

$$([\Lambda x.t][a_1/x_1, \ldots, a_n/x_n])a \leq t[a/x, a_1/x_1, \ldots, a_n/x_n]$$

As was already remarked in [12], the proof is an easy adaptation of the standard case.

From Proposition 2.2 it follows that there are pairing operations, written j, j_0, j_1 that satisfy

$$j_0(j(a, b)) \leq a, \qquad j_1(j(a, b)) \leq b.$$

It is well-known that every pca is either infinite or consists of only one element (One way of understanding this is to observe first that, using k and s one can construct all the *numerals* $\bar{0}, \bar{1}, \ldots$, and then to remark that these all have to be distinct, if $k \neq s$). For ordered pca's there are other possibilities, as becomes apparent after the following definition:

Definition 2.3 An ordered pca is called *trivial* if it has a least element, and it is called *pseudo-trivial* if there is an element that serves both as k and as s.

An example of a pseudo-trivial ordered pca that is not trivial is provided by a meet-semilattice (without a least element, of course; application is given by meet). We have the following characterization:

Lemma 2.4 *For any ordered pca $\mathbb{A}$ the following statements are equivalent:*

1. *$\mathbb{A}$ is pseudo-trivial,*

2. *there is an element u such that $u \leq k = true$ and $u \leq sk = false$,*

3. *any two elements have a lower bound (not necessarily a meet),*

4. *there are natural numbers n, m such that $n \neq m$, but $\overline{n}$ and $\overline{m}$ have a lower bound ($\overline{n}$ denotes the element that corresponds to n for some coding of the natural numbers).*

Proof. (1) $\Rightarrow$ (3): consider the element $u = skkk = kskk$. We have $skkk \leq kk(kk) \leq k$, but also $kskk \leq sk$. Now $kxy \leq x$, so $(skkk)xy \leq x$. And $skxy \leq y$, so $(kskk)xy \leq y$, and we have found that $(skkk)xy = (kskk)xy = uxy$ is a lower bound of any x and y.

(2) $\Rightarrow$ (1): take u with $u \leq k$ and $u \leq sk$. Then uks is a lower bound for k and s, and this lower bound serves both as k and as s.

(3) $\Rightarrow$ (1), (2), (4) are trivial.

(4) $\Rightarrow$ (2): suppose $m > n$ and $x \leq \overline{m}$ and $x \leq \overline{n}$. We have, by combinatorial completeness, terms $zero$ and $pred$, that test for zero and take the predecessor. To be more precise: $zero \bullet \overline{p} \leq k$ if $p = 0$, and $zero \bullet \overline{p} \leq sk$ if $p \neq 0$, $pred \bullet \overline{p} \leq \overline{p-1}$: Now we find that $zero(pred^n \bullet \overline{m}) \leq sk$ and $zero(pred^n \bullet \overline{m}) \leq k$. So for x this implies $zero(pred^n \bullet x) \leq sk$ and $zero(pred^n \bullet x) \leq k$.

□

2.2 Triposes for Ordered Pca's

By now, the construction of a tripos, and hence of a realizability topos out of a partial combinatory algebra is standard. (The reference [4] is just as standard.) We give the straightforward generalization to ordered pca's.

Given a ordered pca $\mathbb{A} = (A, \leq, \bullet)$, define $I(\mathbb{A})$ as the set of all *downsets* in A, that is,

$$I(\mathbb{A}) = \{\alpha \subseteq A \mid \forall a \in \alpha, \forall a' \in A(a' \leq a \rightarrow a' \in \alpha)\}.$$

The *standard realizability tripos* on $\mathbb{A}$, also denoted $I(\mathbb{A})$ assigns to any set X the set of functions $I(\mathbb{A})^X$; reindexing is given by composition. The tripos structure is a straightforward generalisation of the pca case: for $\phi, \psi \in I(\mathbb{A})^X$, we put

$$\phi \vdash \psi \quad \text{iff} \quad \exists a \in A \; \forall x \in X \; \forall b \in \phi(x) : ab{\downarrow} \;\; \& \;\; ab \in \psi(x)$$

We leave the rest of the structure to the reader.

The topos represented by the tripos $I(\mathbb{A})$ is denoted by $\mathbf{RT}[\mathbb{A}]$.

Remark. It is easily seen that $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{Set}$ if $\mathbb{A}$ is trivial. Moreover, if $\mathbb{A}$ is a meet-semilattice, then $\mathbf{RT}[\mathbb{A}]$ is a filter quotient of the presheaf topos $\mathbf{Set}^{\mathbb{A}^{op}}$ (see [12]).

Remark. A possible confusion might arise if one considers pca's like Scott's $P(\omega)$ or some examples from domain theory, which have a partial order such that requirement 1. of Definition 2.1 is satisfied. Considered as opca, $P(\omega)$ is trivial, so $\mathbf{RT}[P(\omega)] \simeq \mathbf{Set}$, in contrast to the realizability topos over $P(\omega)$ as pca!

2.3 Toposes, Assemblies and Partitioned Assemblies

This section contains some straightforward generalizations of well-known facts about the Effective Topos.

The category of *assemblies* over $\mathbb{A}$, $\mathbf{Ass}(\mathbb{A})$, has as objects pairs of form (X, ϵ_X) where X is a set and $\epsilon_X : X \to I(\mathbb{A})$ a function such that $\epsilon_X(x) \neq \emptyset$ for each $x \in X$; a morphism $(X, \epsilon_X) \to (Y, \epsilon_Y)$ is a function $f : X \to Y$ such that there is an $a \in \mathbb{A}$ such that for all $x \in X$ and all $b \in \epsilon_X(x)$, $ab\downarrow$ and $ab \in \epsilon_Y(f(x))$ (one says that f is *tracked* by a).

The category of *partitioned assemblies* over $\mathbb{A}$, $\mathbf{Pass}(\mathbb{A})$, is the full subcategory of $\mathbf{Ass}(\mathbb{A})$ on objects (X, ϵ_X) where for each $x \in X$, $\epsilon_X(x)$ is a principal downset $\downarrow(a) = \{b \in \mathbb{A} \mid b \leq a\}$. When working in $\mathbf{Pass}(\mathbb{A})$ we will simply take ϵ_X to be a function $X \to \mathbb{A}$.

$\mathbf{Ass}(\mathbb{A})$ and $\mathbf{Pass}(\mathbb{A})$ are full subcategories of $\mathbf{RT}[\mathbb{A}]$ and closed under finite limits. We have the usual results, that $\mathbf{Ass}(\mathbb{A})$ is equivalent to the category of $\neg\neg$-separated objects in $\mathbf{RT}[\mathbb{A}]$, and $\mathbf{Pass}(\mathbb{A})$ is equivalent to the category of projective objects of $\mathbf{RT}[\mathbb{A}]$. $\mathbf{RT}[\mathbb{A}]$ has enough projectives, and is therefore the exact completion of $\mathbf{Pass}(\mathbb{A})$; $\mathbf{Ass}(\mathbb{A})$ is the regular completion of $\mathbf{Pass}(\mathbb{A})$.

3 A 2-Category for Ordered PCA's

In Longley's thesis [8], we find a description of a 2-category of pca's. The definition of a morphism between two pca's is chosen in such a way, that there is a correspondence between such morphisms and certain exact functors between the associated realizability toposes.

In Longley's framework, a morphism $\phi : \mathbb{A} \to \mathbb{B}$ of pca's is defined to be a total relation from $\mathbb{A}$ to $\mathbb{B}$, for which there is an element $r \in \mathbb{B}$ such that if $\phi(a, b)$, $\phi(a', b')$ and $aa'\downarrow$ hold, then $rbb'\downarrow$ and $\phi(aa', rbb')$ hold.

In the context of ordered pca's, we can redefine this with functions (instead of relations), and recover Longley's definition with the help of the monad structure on ordered pca's, discussed in 3.2.

Now the succes of Longley's definition is easily seen to depend crucially on the following theorem by Pitts (see [10], section 4.9):

Theorem 3.1 *Let $\mathbb{A}$ and $\mathbb{B}$ be pca's. There is a one-to-one correspondence between*

1. **Set**-*indexed functors from $I(\mathbb{A})$ to $I(\mathbb{B})$ that preserve $T, \wedge$ and $\exists$, and*
2. *functions $f : \mathbb{A} \to P(\mathbb{B})$ such that $f(a) \neq \emptyset$ for all a, and moreover $\bigcap_{a,a' \in Dom(\bullet)} f(a) \to (f(a') \to f(aa')) \neq \emptyset$.*

We will also base our definition on this theorem ourselves, but we are more interested in geometric morphisms than in exact functors, so an important part of our approach will be a characterization of those functions between ordered pca's that induce geometric morphisms between the realizability toposes.

3.1 The category OPCA+

As a first approximation, we present a category for ordered pca's, that is suitable for studying exact functors between realizability triposes, and generalizes Longley's 2-category for pca's. The objects are, of course, ordered pca's. For morphisms, we introduce the following definition:

Definition 3.2 Let $\mathbb{A}$ and $\mathbb{B}$ be ordered pca's, and let $f : \mathbb{A} \to \mathbb{B}$ be a function. We say that f is a *morphism of ordered pca's* (or *opca-map*) if:

- there exists an element $r \in \mathbb{B}$ such that $aa'{\downarrow} \Rightarrow (r \bullet f(a)) \bullet f(a'){\downarrow}$ and $(r \bullet f(a)) \bullet f(a') \leq f(aa')$.
- there exists an element $u \in \mathbb{B}$ such that $a \leq a' \Rightarrow u \bullet f(a) \leq f(a')$

It is easily verified that composition is well-defined. We will write **OPCA+** for this category.

Next, we observe that the Hom-sets of this category are pre-ordered sets if we define, for $f, g : \mathbb{A} \to \mathbb{B} : f \leq g$ iff $\exists b \in \mathbb{B} : b \bullet f(a){\downarrow} \;\; \& \; b \bullet f(a) \leq g(a)$ for all $a \in \mathbb{A}$. Since composition of morphisms preserves this ordering, in the sense that $f \leq g \Rightarrow fh \leq gh$ and $kf \leq kg$, we see that **OPCA+** is a pre-order enriched category. We write $f \sim g$ for $f \leq g \;\&\; g \leq f$, and we say that f and g are *isomorphic* as morphisms.

It is good to observe that a map $f : \mathbb{A} \to \mathbb{B}$ provides us with a description of $\mathbb{A}$ as an *internal* ordered pca in the topos $\mathbf{RT}[\mathbb{B}]$. The underlying set of this (canonically projective) object is the underlying set of $\mathbb{A}$, and the existence predicate is given by $E_f(a) = {\downarrow}(f(a))$. Moreover, if we have $f, g : \mathbb{A} \to \mathbb{B}$, then $f \leq g$ iff, internally in $\mathbf{RT}[\mathbb{B}]$, the identity on $\mathbb{A}$ is a map $(\mathbb{A}, E_f) \to (\mathbb{A}, E_g)$.

Remarks. The structure of the category **OPCA+** is not particularly impressive. We mention the following:

1. (This generalizes an observation by Longley.) The terminal object in **OPCA+** is the one-point ordered pca. For any other trivial $\mathbb{A}$, there is, for any $\mathbb{B}$, always a morphism $f : \mathbb{B} \to \mathbb{A}$. This f is unique up to isomorphism. Trivial ordered pca's are also pseudo-initial, in the sense that for any other ordered pca $\mathbb{B}$, there is always a map into $\mathbb{B}$, and any two such maps are isomorphic.

   Apart from this, we can observe that any constant function between ordered pca's is a morphism, and that any two constant maps are isomorphic.

2. The category **OPCA+** has products: given $\mathbb{A}$ and $\mathbb{B}$, we define $\mathbb{A} \times \mathbb{B}$ as $\mathbb{A} \times \mathbb{B} = (A \times B, \bullet, \leq)$ with $(a, b) \leq (a', b')$ iff $a \leq a'$ and $b \leq b'$, $(a, b) \bullet (a', b'){\downarrow}$ iff $aa'{\downarrow}$ and $bb'{\downarrow}$, in which case $(a, b) \bullet (a', b') = (aa', bb')$. The pairs $(k_A, k_B), (s_A, s_B)$ serve as k and s in the product.

3. Monos and epis are just injective and surjective maps, respectively. For, consider a map $f : \mathbb{A} \to \mathbb{B}$ that is not injective, say $f(a) = f(a')$. Then

we take two (different) maps $1 \to \mathbb{A}$ sending the unique element to a and a', respectively. Their composites with f are obviously equal.

If $f : \mathbb{A} \to \mathbb{B}$ is not surjective, then there is some element $b_0 \in \mathbb{B}$ that is outside the image of f. Consider the trivial structure $\mathbb{P}$ consisting of two elements p, q with $p \leq q$. Now define maps $g, h : \mathbb{B} \to \mathbb{P}$ by

$$g(b) = \begin{cases} q & \text{if } b_0 < b \\ p & \text{otherwise,} \end{cases} \qquad h(b) = \begin{cases} q & \text{if } b_0 \leq b \\ p & \text{otherwise.} \end{cases}$$

It is not hard to verify that these are indeed morphisms in our category, and that $gf = hf$, but not $g = h$.

4. Equalizers do not exist in **OPCA+**. The reason is simple: if we have two structures $\mathbb{A}$, $\mathbb{B}$, then we can take two different constant maps. There equalizer would have to have the empty set as underlying set, but no such ordered pca exists.

3.2 The Downset-monad

Now we describe a monad $(T, \delta, \cup)$ on **OPCA+**. On objects, we define

$$T\mathbb{A} = (\{\alpha \mid \alpha \in I\mathbb{A}, \alpha \neq \emptyset\}, \subseteq, \bullet).$$

So the underlying set of $T\mathbb{A}$ consists of all nonempty downsets in $\mathbb{A}$. It is ordered by inclusion, and partial application is defined by $\alpha \bullet \beta\downarrow$ iff $\forall a \in \alpha\ \forall b \in \beta\ ab\downarrow$, and if $\alpha \bullet \beta\downarrow$ then $\alpha \bullet \beta = \downarrow\{ab \mid a \in \alpha, b \in \beta\}$. It is not hard to verify that this gives again a ordered pca, with $\downarrow(k)$ and $\downarrow(s)$ serving as combinators. Also, there is a map $\delta : \mathbb{A} \to T\mathbb{A}$, given by $\delta(a) = \downarrow(a)$.

For a morphism $f : \mathbb{A} \to \mathbb{B}$, we put $Tf(\alpha) = \bigcup_{a \in \alpha} \downarrow(f(a))$. It is easily verified that this is well-defined. Finally, it is clear that composition and identities are preserved, so T is indeed an endofunctor. Actually, it is an endo-2-functor, since it preserves the ordering on morphisms (in fact it also reflects the order).

Now let $\cup : T^2\mathbb{A} \to T\mathbb{A}$ be the map given by union: $\cup\xi = \{a \in \mathbb{A} \mid \exists \alpha \in \xi : a \in \alpha\}$. The verifications that both δ and $\cup$ are natural transformations, and that the monad identities are satisfied are left to the reader.

Lemma 3.3 *If $f : \mathbb{A} \to T\mathbb{B}$ is a morphism in* **OPCA+**, *then f is equivalent to a morphism that preserves the ordering on the nose.*

Proof. Let u be a realizer such that $a \leq a' \Rightarrow u \bullet f(a) \leq f(a')$. Put $g(a) = \cup_{a' \leq a} f(a)$. This clearly preserves the ordering. Since $f(a) \subseteq g(a)$, $f \leq g$. And if $b \in g(a)$, that is, $b \in f(a')$ for some $a' \leq a$, then $u \bullet b \in f(a)$; hence $g \leq f$.

□

The theorem by Pitts (3.1) that we stated at the beginning of this section can now be strengthened as follows: let **Kl(T)** denote the Kleisli category for the monad $(T, \delta, \cup)$ (this is a 2-category, since the pre-ordering of the arrows is inherited from **OPCA+**). Let **RTripExact** denote the 2-category of realizability triposes of the form $I(\mathbb{A})^{(-)}$, with exact functors as arrows, and natural transformations pre-ordering those exact functors. Then we obtain:

Theorem 3.4 *Every map $f : \mathbb{A} \to T\mathbb{B}$ induces a Set-indexed functor from $I(\mathbb{A})^{(-)}$ to $I(\mathbb{B})^{(-)}$, that commutes with $\wedge, \top$ and $\exists$. Moreover, every such Set-indexed functor is, up to isomorphism, induced by a map $f : \mathbb{A} \to T\mathbb{B}$. Hence we have a 2-functor from the Kleisli category* **Kl(T)** *to* **RTripExact**. *This 2-functor is bijective on objects and a local equivalence: it induces equivalences on the Hom categories.*

Proof. Given $f : \mathbb{A} \to T\mathbb{B}$, define the tripos map $\bar{f} : I(\mathbb{A}) \to I(\mathbb{B})$ as $\bar{f}(\alpha) = \bigcup_{a \in \alpha} f(a)$.

Conversely, take $\phi : I(\mathbb{A}) \to I(\mathbb{B})$ with the mentioned properties. By 3.1 it follows that there is a map $\lambda : \mathbb{A} \to I^*\mathbb{B}$ such that ϕ is naturally isomorphic to $\bar{\lambda}$, and $\bigcap_{a,a' \in Dom(\bullet)} \lambda(a) \to (\lambda(a') \to \lambda(aa')) \neq \emptyset$. This map λ preserves the ordering up to a realizer: consider the object $X = \{(a', a) | a' \leq a\}$, and the two projections $\pi_1, \pi_2 \in I(\mathbb{A})^X$. Clearly $\pi_1 \vdash \pi_2$. Hence also $\lambda \circ \pi_1 \vdash \lambda \circ \pi_2$, so there is a realizer $c \in \bigcap_{a' \leq a}(\lambda(a') \to \lambda(a))$. Therefore, λ is a map of ordered pca's.

$\square$

This theorem shows, in effect, that our approach is an extension of Longley's, because Longley's 2-category of pca's is a full sub-2-category of $\mathbf{KL}(T)$.

A final observation for this section: just as a map $f : \mathbb{A} \to \mathbb{B}$ presents $\mathbb{A}$ as a projective internal ordered pca in $\mathbf{RT}[\mathbb{B}]$, a map $g : \mathbb{A} \to T\mathbb{B}$ presents $\mathbb{A}$ as a *separated* internal ordered pca in $\mathbf{RT}[\mathbb{B}]$.

3.3 The 2-category OPCA

For reasons that are about to become transparent, we introduce the following definition:

Definition 3.5 A morphism $f : \mathbb{B} \to \mathbb{A}$ is said to be *computationally dense* (cd) iff the following condition holds:

$$\forall a \in \mathbb{A} \exists b \in \mathbb{B} \forall b' \in \mathbb{B} : a \bullet f(b')\downarrow \Rightarrow bb'\downarrow \ \ \& \ f(bb') \leq a \bullet f(b') \qquad \text{(cd)}$$

The terminology is explained by the fact that the condition actually tells that any representable function from $\mathbb{B}$ to $\mathbb{A}$ (representable by some element in $\mathbb{A}$, that is), is bounded below by a function which is representable by some element in $\mathbb{B}$.

It is evident that the composition of two computationally dense maps is again such a map, and that the identity map is one, too, so we can form the lluf subcategory **OPCA** on the computationally dense maps. Moreover, the structure maps of the monad δ and $\cup$ are both cd, and if f is cd, then so is Tf. Therefore, the monad $(T, \delta, \cup)$ restricts to a monad on **OPCA**. We shall not distinguish notationally between the two uses of T; relying on context to make clear in which category we work.

Let us now explain what the relevance of computational density is. Consider a morphism $f : \mathbb{B} \to T\mathbb{A}$ in **OPCA** (by 3.3, we may assume that it preserves the ordering on the nose). First we will show that it induces a geometric morphism of triposes:

$$I(\mathbb{A}) \underset{f^{-1}}{\overset{\bar{f}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} I(\mathbb{B})$$

where the arrows $\bar{f}$ and f^{-1} are defined as

$$\bar{f}(\beta) = \bigcup_{b\in\beta} f(b), \qquad f^{-1}(\alpha) = \{b \in \mathbb{B} \mid f(b) \subseteq \alpha\}.$$

First, the existence of the left adjoint $\bar{f}$ that preserves finite limits follows from theorem 3.4. Second, f^{-1} is order-preserving. Suppose $a \in \phi \to \psi$. Use (cd) to find $b \in \mathbb{B}$ with $\forall b' \in \mathbb{B} : \downarrow(a) \bullet f(b')\downarrow \Rightarrow bb'\downarrow \;\; \& \; f(bb') \subseteq \downarrow(a) \bullet f(b')$. This b realises $f^{-1}(\phi) \to f^{-1}(\psi)$, since $f(b') \subseteq \phi \Rightarrow \downarrow(a) \bullet f(b')\downarrow$, so $bb'\downarrow \;\; \& \; f(bb') \subseteq \downarrow(a) \bullet f(b') \subseteq \psi$.

Finally, we have $\bar{f} \dashv f^{-1}$. The verification of this fact goes along the same lines as that of the previous facts. This completes the proof of the claim that we have an induced geometric morphism of triposes. Note in particular that for any map $g : \mathbb{B} \to \mathbb{A}$ in **OPCA**, composition with the structure map $\delta : \mathbb{A} \to T\mathbb{A}$ of the monad induces a geometric morphism.

The next step is to show, that, up to isomorphism, any geometric morphism of realizability triposes is induced by a morphism in **OPCA**.

Lemma 3.6 *Suppose we have a geometric morphism*

$$I(\mathbb{A}) \underset{f_*}{\overset{f^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} I(\mathbb{B}).$$

Then there is a map $f : \mathbb{B} \to T\mathbb{A}$ *such that* $\bar{f} \dashv\vdash f^*, f^{-1} \dashv\vdash f_*$.

Proof. As has already been shown by Pitts, putting $f(b) = f^*(\downarrow(b))$ is the only choice we have, since this gives $f^*(\beta) \dashv\vdash \bigcup_{b\in\beta} f(b) = \bar{f}(\beta)$, because f^*, as a left adjoint, preserves unions. Again from theorem 3.4, it follows that this is a morphism in **OPCA+**.

We know that $\bar{f}f^{-1}(\alpha) \subseteq \alpha$ and $\beta \subseteq f^{-1}\bar{f}(\beta)$. So we get $f^{-1}(\alpha) \vdash f_*(\alpha)$. Also, we find $f_*(\alpha) \vdash f^{-1}\bar{f}f_*(\alpha) \vdash f^{-1}(\alpha)$, hence $f^{-1} \dashv\vdash f_*$.

Next, we show that this f is computationally dense. Suppose that it isn't, that is, there is $\alpha \in T\mathbb{A}$ for which we have

$$\forall b \in \mathbb{B} \exists b' \in \mathbb{B} : \alpha \bullet f(b')\downarrow \;\; \& \; \neg(b \bullet b'\downarrow \;\; \& \; f(bb') \subseteq \alpha \bullet f(b'))$$

We may take a choice function $k : \mathbb{B} \to \mathbb{B}$, that satisfies

$$\forall b \in \mathbb{B} : \alpha \bullet f(k(b))\downarrow \quad \& \neg(b \bullet k(b)\downarrow \quad \& \; f(b \bullet k(b)) \subseteq \alpha \bullet f(k(b)).$$

Now define $D_\alpha = \{b \in \mathbb{B} \mid \alpha \bullet f(b)\downarrow\}$. Consider the functions $\phi, \psi : D_\alpha \to I\mathbb{A}$, given by $\phi(b) = f(b)$, $\psi(b) = \alpha \bullet f(b)$. Clearly, we have that any $a \in \alpha$ satisfies $a \in \bigcap_{b \in D_\alpha} \phi(b) \to \psi(b)$. Now f^{-1} preserves the ordering, from which it follows that there is an element $x \in \bigcap_{b \in D_\alpha} f^{-1}\phi(b) \to f^{-1}\psi(b)$. We find in particular that, taking $b = k(x)$, $\forall y \in \mathbb{B} : f(y) \subseteq f(k(x)) \Rightarrow xy\downarrow \quad \& \; f(xy) \subseteq \alpha \bullet f(k(x))$. If we take $y = k(x)$ we obtain a contradiction.

□

This establishes, that geometric morphisms $I(\mathbb{B})^{(-)} \to I(\mathbb{A})^{(-)}$, are, up to isomorphism, the same as ordered pca morphisms $\mathbb{A} \to T\mathbb{B}$ that are computationally dense. But the latter are precisely the morphisms from $\mathbb{A}$ to $\mathbb{B}$ in the Kleisli category $\mathbf{Kl}(T)$ for the monad T on **OPCA**.

Let **RTrip** denote the 2-category with as objects triposes of the form $I(\mathbb{A})^{(-)}$ for some ordered pca $\mathbb{A}$, and as arrows geometric morphisms of triposes. For two geometric morphisms $(f^*, f_*), (g^*, g_*)$ from $I(\mathbb{B})^{(-)}$ to $I(\mathbb{A})^{(-)}$, we say that $(f^*, f_*) \leq (g^*, g_*)$ iff for every set X and any $\phi : X \to I\mathbb{A}$, $f^*\phi \vdash g^*\phi$. This makes **RTrip** into a preorder-enriched category. Moreover, let **RTop** be the 2-category of toposes of the form $\mathbf{RT}[\mathbb{A}]$ for some ordered pca $\mathbb{A}$, with geometric morphisms commuting with the inclusion of **Set**, and natural transformations between them. It is known that these categories are equivalent when we forget about the 2-categorical structure. The following lemma shows that there is also a correspondence between natural transformations on the tripos-level and on the topos-level.

Lemma 3.7 *Let $\mathbb{A}, \mathbb{B}$ be ordered pca's, an let $f, g : \mathbb{A} \to T\mathbb{B}$ be two maps in* **OPCA**. *Then $\bar{f} \leq \bar{g}$ in* **RTrip** *iff there is a (necessarily unique) natural transformation $\eta : \bar{f} \to \bar{g}$ in* **RTop**.

Proof. The idea of the proof is, first to establish this for separated objects, and then to use the fact that every object can be covered by a separated object. Details are left to the reader.

□

Now we relate the preorder on Hom-sets in **OPCA** to the one on the Hom-Sets in **RTrip**.

Lemma 3.8 *Let $f, g : \mathbb{A} \to T\mathbb{B}$ be two maps in* **OPCA**, *inducing two geometric morphisms of triposes, $(\bar{f}, f^{-1})$ and $(\bar{g}, g^{-1})$. Then $f \leq g$ iff $(\bar{f}, f^{-1}) \leq (\bar{g}, g^{-1})$.*

Proof. If $f \leq g$ then there is an element $b \in \mathbb{B}$ with the property that $b \in \bigcap_{a \in \mathbb{A}} f(a) \to g(a)$. This implies that $b \in \bigcap_{\alpha \in I\mathbb{A}} \bar{f}(\alpha) \to \bar{g}(\alpha)$. Therefore $\bar{f}(\phi) \vdash \bar{g}(\phi)$ for any $\phi : X \to I\mathbb{A}$.

Conversely, assume $\bar{f}(\phi) \vdash \bar{g}(\phi)$ for any $\phi : X \to I\mathbb{A}$. In particular, taking X to be $\mathbb{A}$ and $\phi(a) = \downarrow(a)$, we find $\bar{f}(\phi)(a) = f(a)$, $\bar{g}(\phi)(a) = g(a)$, and there is an element $b \in \mathbb{B}$ such that $b \in \bigcap_{a\in\mathbb{A}} f(a) \to g(a)$, proving $f \leq g$.

□

We can wrap up by saying that there is a 2-functor from the opposite of the Kleisli 2-category $\mathbf{Kl}(T)$ to the 2-category **RTrip** of realizability triposes. This functor is, again, bijective on objects and a local equivalence.

4 Pseudo-algebras for T

In this section we relate properties of the category $\mathbf{PAss}(\mathbb{A})$ to monad-theoretic properties of $\mathbb{A}$. The first thing to notice is, that our monad is an instance of a so-called *KZ-doctrine* (see [7]). The verification of this comes down to observing that the following hold: $T\delta_{\mathbb{A}} \leq \delta_{T\mathbb{A}}$, $\cup \circ T\delta_{\mathbb{A}} = \cup \circ \delta_{T\mathbb{A}}$ and $T\delta_{\mathbb{A}} \circ \delta_{\mathbb{A}} = \delta_{T\mathbb{A}} \circ \delta_{\mathbb{A}}$. We will use some facts about KZ-doctrines to simplify some of the proofs below. Recall that a *pseudo-algebra* for the monad T is a map $\phi : T\mathbb{A} \to \mathbb{A}$ such that the two diagrams below commute up to 2-isomorphism:

$$\begin{tikzcd} & T\mathbb{A} \arrow[d, "\phi"] \\ \mathbb{A} \arrow[ur, "\delta"] \arrow[r, "Id"'] & \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} T^2\mathbb{A} \arrow[r, "T\phi"] \arrow[d, "\cup"'] & T\mathbb{A} \arrow[d, "\phi"] \\ T\mathbb{A} \arrow[r, "\phi"'] & \mathbb{A}. \end{tikzcd}$$

Similarly, we say that a map $f : \mathbb{A} \to \mathbb{B}$ is a pseudo-T-homomorphism if the diagram

$$\begin{tikzcd} T\mathbb{A} \arrow[r, "Tf"] \arrow[d, "\phi"'] & T\mathbb{B} \arrow[d, "\psi"] \\ \mathbb{A} \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$

commutes up to 2-isomorphism (where ϕ, ψ are the pseudo-algebra structures for $\mathbb{A}$ and $\mathbb{B}$ respectively).

The facts about KZ-doctrines of which we will make use are:

1. A pseudo-algebra is the same as a left adjoint reflection for the unit. Hence pseudo-algebras are unique up to isomorphism.

2. If $\phi : T\mathbb{A} \to \mathbb{A}$ is a pseudo-algebra, then a left adjoint for ϕ is automatically a pseudo-T-homomorphism.

3. If $T^2\mathbb{A} \to T\mathbb{A}$ is a free algebra, then the algebra map always has a left adjoint.

As a heuristics, one can think of a pseudo-algebra $\phi : T\mathbb{A} \to \mathbb{A}$ for T as a "complete" opca where ϕ plays the role of supremum map. For free algebras, the multiplication is a genuine supremum map, but in general ϕ is only a supremum map up to a realizer (and the underlying poset of $\mathbb{A}$ also has non-empty suprema up to a realizer).

Also, notice that if a pseudo-algebra exists, then it is automatically a computationally dense map. This is true, because $\phi \dashv \delta$ implies that $T\phi \dashv T\delta$. So ϕ induces a geometric morphism of triposes, and must therefore be a computationally dense map.

Now we turn to the categories of partitioned assemblies. First, we show that opca-maps from $\mathbb{A}$ to $\mathbb{B}$ are precisely finite limit-preserving functors from $\mathbf{PAss}(\mathbb{A})$ to $\mathbf{PAss}(\mathbb{B})$ that commute with the inclusion of $\mathbf{Sets}$.

Lemma 4.1

1. *An opca-map $h : \mathbb{A} \to \mathbb{B}$ induces a left exact functor $H : \mathbf{PAss}(\mathbb{A}) \to \mathbf{PAss}(\mathbb{B})$ that commutes with the inclusion of* $\mathbf{Sets}$.
2. *A left exact functor $H : \mathbf{PAss}(\mathbb{A}) \to \mathbf{PAss}(\mathbb{B})$ that commutes with the inclusion of* $\mathbf{Sets}$ *induces an opca-map $h : \mathbb{A} \to \mathbb{B}$.*
3. *The operations $h \mapsto H$ and $H \mapsto h$ are, up to 2-isomorphism, inverse to each other.*

Proof. We just remark that $h : \mathbb{A} \to \mathbb{B}$ gives H by $H(X, \epsilon_X) = (X, h \circ \epsilon_X)$. Conversely, every functor H satisfying the above property is, up to isomorphism, induced by its action on the generic object. Details of the proof are omitted, since there is a very similar theorem for the categories of assemblies in [8].

□

Remark. In fact, lemma 4.1 could be stated in terms of a 2-functor from $\mathbf{OPCA+}$ to the 2-category of categories of the form $\mathbf{PAss}(\mathbb{A})$, and lex functors that commute with the inclusion of $\mathbf{Sets}$. This functor then is a local equivalence.

Another point worth noticing is, that it follows now that two maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{A}$ are adjoint if and only if the induced functors between $\mathbf{PAss}(\mathbb{A})$ and $\mathbf{PAss}(\mathbb{B})$ are adjoint. This fact will be exploited later on.

Theorem 4.2 *The following are equivalent for an ordered pca $\mathbb{A}$:*

1. *$\mathbb{A}$ admits a pseudo-algebra structure*
2. *$\mathbf{PAss}(\mathbb{A})$ is regular*
3. *The embedding of $\mathbf{PAss}(\mathbb{A})$ into $\mathbf{Ass}(\mathbb{A})$ is a localization that commutes with the inclusion of* $\mathbf{Sets}$.

Proof. First, assume 1). As in 4.1, such a structure $\phi : T\mathbb{A} \to \mathbb{A}$ gives a functor $\phi : \mathbf{Ass}(\mathbb{A}) \simeq \mathbf{PAss}(T\mathbb{A}) \to \mathbf{PAss}(\mathbb{A})$, that is left adjoint to the embedding (which corresponds to the unit of the monad at $\mathbb{A}$)). The counit of the adjunction is an isomorphism, since it is so on the level of opca's. This proves 3).

Now assume that a localization as in 3) exists. This gives, again by the lemma, some opca-map $\phi : T\mathbb{A} \to \mathbb{A}$, that is left adjoint to the unit at $\mathbb{A}$, and hence a pseudo-algebra. Thus, 3) implies 1).

Next, assume 2). Because of the universal property of $\mathbf{Ass}(\mathbb{A})$ w.r.t. regular categories, there is a retraction $\phi : \mathbf{Ass}(\mathbb{A}) \to \mathbf{PAss}(\mathbb{A})$. It is straightforward to check that this commutes with the inclusion of **Sets** and that the adjointness holds, so we have 3).

Finally, assume 3) (again, the left adjoint is called ϕ). Because any parallel pair in $\mathbf{PAss}(\mathbb{A})$ has a coequalizer in $\mathbf{Ass}(\mathbb{A})$, and because ϕ preserves coequalizers, $\mathbf{PAss}(\mathbb{A})$ has coequalizers. Moreover, the fact that ϕ is left exact ensures that these coequalizers are pullback-stable. So $\mathbf{PAss}(\mathbb{A})$ is regular.

□

If $\mathbf{PAss}(\mathbb{A})$ is regular, then we can give the following characterization of the regular epimorphisms:

Lemma 4.3 *Let $\mathbf{PAss}(\mathbb{A})$ be regular, and let ϕ be the pseudo-algebra map that exists by theorem 4.2. Then a surjective map $f : (X, \epsilon_X) \to (Y, \epsilon_Y)$ is regular epi iff there is an element p with $p \bullet \epsilon_Y(y) \leq \phi(\downarrow\{\epsilon_X(x) | f(x) = y\})$.*

Proof. First, take a surjection $f : (X, \epsilon_X) \to (Y, \epsilon_Y)$ and p with the property that $p \bullet \epsilon_Y(y) \leq \phi(\downarrow\{\epsilon_X(x)|f(x) = y\})$. Suppose that there is another map $g : (X, \epsilon_X) \to (Z, \epsilon_Z)$ such that the underlying function g can be written as $g = hf$ for some $h : Y \to Z$. In other words, $f(x) = f(x')$ implies $g(x) = g(x')$. We show that the map h has a tracking.

Let c be an element tracking g, so $c \bullet \epsilon_X(x) \leq \epsilon_Z(g(x))$ for all $x \in X$. Take any $y \in Y$ and write α_y for the set $\downarrow\{\epsilon_X(x)|f(x) = y\}$. Now c inhabits $\alpha_y \to \epsilon_Z(h(y))$, so $\downarrow(c) \bullet \alpha_y \subseteq \downarrow(\epsilon_Z(h(y)))$. By the fact that ϕ preserves the ordering and application up to a realizer, we obtain a realizer c' with the property $c' \bullet \phi(\alpha_y) \leq \epsilon_Z(h(y))$, and hence (using p) also a realizer c'' such that $c'' \bullet \epsilon_Y(y) \leq \epsilon_Z(h(y))$.

On the other hand, let $f, g : (X, \epsilon_X) \to (Y, \epsilon_Y)$ be a parallel pair. We form the coequalizer (Z, ϵ_Z) by letting $q : Y \to Z$ be the underlying coequalizer in **Sets**, and $\epsilon_Z(z) = \phi(\downarrow\{\epsilon_Y(y)|q(y) = z\})$. If $(Z', \epsilon_{Z'})$ is isomorphic to (Z, ϵ_Z), then $p \bullet \epsilon_{Z'}(z) \leq \epsilon_Z(z)$ for some p, and hence $\epsilon_{Z'}$ is of the required form.

□

Before we state the next theorem, we recall that a diagram of the form

$$\begin{tikzcd}
(X, \epsilon_X) \arrow[r, "f"] \arrow[d, "\eta_X"'] & (Y, \epsilon_Y) \arrow[d, "\eta_Y"] \\
\nabla(X) \arrow[r, "\nabla(f)"'] & \nabla(Y)
\end{tikzcd}$$

is a pullback if and only if $(X, \epsilon_X) \cong (X, \epsilon')$ with $\epsilon'(x) = \epsilon_Y(f(x))$ for all $x \in X$. Following Menni (see [9]), we call such maps *pre-embeddings*.

Theorem 4.4 *For any opca $\mathbb{A}$, the following are equivalent:*

1. $\mathbf{PAss}(\mathbb{A})$ *is a regular completion;*

2. *There is a "cylinder" of adjoints $\psi \dashv \phi \dashv \delta : \mathbf{PAss}(\mathbb{A}) \to \mathbf{Ass}(\mathbb{A})$, with $\phi \circ \delta \sim Id$, and with ψ preserving finite limits and commuting with the inclusion of* **Sets***;*

3. $\mathbb{A}$ *admits a pseudo-algebra structure, and this pseudo-algebra has a left adjoint $\psi : \mathbb{A} \to T\mathbb{A}$;*

4. $\mathbb{A}$ *is equivalent to a free T-algebra.*

Proof. The equivalence between 2) and 3) needs no explication. First assume 1). Since $\mathbf{PAss}(\mathbb{A})$ is regular, we have the map $\phi : T\mathbb{A} \to \mathbb{A}$. By the characterization of completions, there are enough projectives and the projectives are closed under finite limits. We first explain why there is a *generic projective* object. Take the generic object, namely (A, Id), and cover it with a projective $e : (B, \epsilon_B) \to (A, Id)$. This means that for each $a \in \mathbb{A}$ there is a set $\beta_a = \downarrow\{\epsilon_B(b) | e(b) = a\}$. Also put $Irr = \cup_{a \in \mathbb{A}} \beta_a$. Just as the map ϕ can be thought of as supremum mapping, we think of Irr as the set of join-irreducible elements in $\mathbb{A}$, and of β_a as the join-irreducibles that are below a. Moreover, because the covering is regular epi, we have an isomorphism $(A, Id) \cong (A, \lambda a.\phi(\beta_a))$.

The fact that (B, ϵ_B) is projective now implies that, for some realizer r, if $b \in Irr$, and $b \leq \phi(\alpha)$ for some set $\alpha \in T\mathbb{A}$, then there is some $a \in \alpha$ such that $r \bullet b \leq a$ (this is just writing out what it means that every regular epi with codomain (B, ϵ_B) has a section). Note that this is, indeed, some kind of irreducibility.

From this one deduces that if an object (Y, ϵ_Y) has $\epsilon_Y(y) \in Irr$ for every $y \in Y$, then it is also projective.

This object (B, ϵ_B) is generic projective in the following sense: if (X, ϵ_X) is any object, then we have a map $\epsilon_X : (X, \epsilon_X) \to (A, Id)$. If we form the pullback

$$\begin{array}{ccc} (Q, \epsilon_Q) & \xrightarrow{h} & (B, \epsilon_B) \\ \downarrow & & \downarrow \\ (X, \epsilon_X) & \xrightarrow[\epsilon_X]{} & (A, Id) \end{array}$$

then the left-hand map is again regular epi.

The map ϵ_X is a pullback of $\nabla(\epsilon_X)$, hence the top map is also a pre-embedding. This means that, for any $q \in Q$, $\epsilon_Q(q) = \epsilon_B(h(q)) \in Irr$. From this we obtain that (Q, ϵ_Q) is also projective. We refer to coverings obtained in this way by *canonical coverings*.

Moreover, if (X, ϵ_X) already happened to be projective, then the left-hand map would split, presenting (X, ϵ_X) as a (regular) subobject of (Q, ϵ_Q). But regular monos are pre-embeddings in this context, so (X, ϵ_X) is pre-embedded in (B, ϵ_B). Hence every projective is a pullback of (B, ϵ_B).

Now the map $\psi : \mathbb{A} \to T\mathbb{A}$, defined by $a \mapsto \beta_a$ gives a functor $\psi : \mathbf{PAss}(\mathbb{A}) \to \mathbf{Ass}(\mathbb{A})$, by saying $\psi(X, \epsilon_X) = (X, \psi \circ \epsilon_X)$. Let us check that this is well-defined: take $f : (X, \epsilon_X) \to (Y, \epsilon_Y)$, and consider the diagram

$$\begin{array}{ccc} (P, \epsilon_P) & \longrightarrow & (Q, \epsilon_Q) \\ \downarrow & & \downarrow \\ (X, \epsilon_X) & \xrightarrow[f]{} & (Y, \epsilon_Y). \end{array}$$

Here, the vertical maps are canonical projective covers, and the top map arises because of the projectivity of (P, ϵ_P). The fact that this map has a tracking is just the same as the fact that $f : (X, \psi \circ \epsilon_X) \to (Y, \psi \circ \epsilon_Y)$ does.

Next, the composite $\phi \circ \psi$ is isomorphic to the identity, since $(A, Id) \cong (A, \lambda a.\phi(\beta_a))$. Moreover, $\psi \dashv \phi$. Indeed, if $\psi(a) \to \beta$ is inhabited (uniformly in $a \in \mathbb{A}$, and in $\beta \in T\mathbb{A}$), then so is $\phi\psi(a) \to \phi(\beta)$. But then $a \to \phi(\beta)$ is also inhabited. Conversely, if $a \to \phi(\alpha_a)$ is inhabited then we have a regular epi $f : (X, \epsilon_X) \to (A, Id)$, where $X = \{(a, b) | b \in \alpha_a\}$, and $\epsilon_X(a, b) = b$. Thus there is a map $g : (B, \epsilon_B) \to (X, \epsilon_X)$, such that the composite fg equals the projection $(B, \epsilon_B) \to (A, Id)$. Now it is easily deduced that the tracking element for g sends all elements in $\psi(a)$ to elements in α_a, and the adjointness is proved.

Finally, since the projectives are closed under finite limits, we can derive that ψ preserves finite limits.

Next, we prove the converse; so assume that ϕ has a left adjoint ψ, which, by the considerations that we saw before, may be taken to be induced by a function $\psi : \mathbb{A} \to T\mathbb{A}$. Now consider the generic object (A, Id) in $\mathbf{PAss}(\mathbb{A})$, and cover this object by (B, ϵ_B), where $B = \{(a, c) | c \in \psi(a)\}$ and $\epsilon_B(a, c) = c$. The projection is regular epi since the unit of the adjunction $\phi \dashv \psi$ is an isomorphism. We show that (B, ϵ_B) is (generic) projective. The fact that ϕ is right adjoint to ψ translates into the fact that the object (B, ϵ_B) has the property that for every regular epi $f : (X, \epsilon_X) \to (A, Id)$ there is a map $(B, \epsilon_B) \to (X, \epsilon_X)$, that makes the projection factor through f:

$$\begin{tikzcd} & (B, \epsilon_B) \arrow[dl, dashed] \arrow[d, two heads] \\ (X, \epsilon_X) \arrow[r, two heads, "f"] & (A, Id). \end{tikzcd}$$

Indeed, f regular epi means $\downarrow(a) \to \phi(\downarrow\{\epsilon_X(x) | f(x) = a\})$ inhabited, and by the adjunction, $\psi(a) \to \{\epsilon_X(x) | f(x) = a\}$ inhabited. This says precisely that there is a tracked function from $(B, \epsilon_B) \to (X, \epsilon_X)$.

Consider the pullback

$$\begin{tikzcd}
(Q,\epsilon_Q) \arrow[r] \arrow[d, two heads] & (B,\epsilon_B) \arrow[d, two heads] \\
(B,\epsilon_B) \arrow[r, "\epsilon_B"'] & (A, Id)
\end{tikzcd}$$

where the bottom map is a pre-embedding (and hence the top map, too). The left-hand map has a section, say m. Now if $Y \to X$ is any regular epi, and $f : (B,\epsilon_B) \to X$ any arrow, then the adjunction gives us a map as in the diagram:

$$\begin{tikzcd}
 & (Q,\epsilon_Q) \arrow[d, two heads] \arrow[ddl, dashed] \\
 & (B,\epsilon_B) \arrow[d, "f"] \\
Y \arrow[r, two heads] & X.
\end{tikzcd}$$

We obtain a map from (B,ϵ_B) to Y by using the section $m : (B,\epsilon_B) \to (Q,\epsilon_Q)$. Hence (B,ϵ_B) is projective. Now it is also easily established that (B,ϵ_B) is generic projective, as we in the proof of the other direction.

The implication from 4) to 3) is just the third fact about KZ-doctrines that we listed at the beginning of this section. It remains to show that 3) implies 4). So let ψ be left adjoint to ϕ, and consider the set $Irr = \{c \in \mathbb{A} | c \in \psi(a), a \in \mathbb{A}\}$. We endow this set with an opca-structure. Observe that we may assume that ψ preserves the ordering on the nose; because $T\mathbb{A}$ is free, lemma 3.3 is applicable. Let r be a realizer up to which ψ preserves application. Now put

$$c \bullet' c' \simeq r \bullet c \bullet c'$$

and order Irr as a subset of $T\mathbb{A}$. It is an easy exercise to verify that this is indeed an opca, that ϕ restricts to a map $\phi : T(Irr) \to \mathbb{A}$ and that ψ takes values in $T(Irr)$. We only have to show that these restricted maps form an equivalence of opca's. Since $\phi \circ \psi$ is isomorphic to the identity, it remains to show that $\psi \circ \phi$ is isomorphic to the identity on $T(Irr)$. The direction $\psi \circ \phi \leq 1$ is just the counit of the adjunction. By the second fact about KZ-doctrines, ψ is a pseudo-T-homomorphism, meaning that the square

$$\begin{tikzcd}
T\mathbb{A} \arrow[r, "T\psi"] \arrow[d, "\phi"'] & T^2\mathbb{A} \arrow[d, "\cup"] \\
\mathbb{A} \arrow[r, "\psi"'] & T\mathbb{A}
\end{tikzcd}$$

commutes up to isomorphism. Hence we can show that $1 \leq \cup \circ T\psi$. Recall that there is a realizer s that takes each $c \in Irr$ to an element in $\psi(c)$. (This is just expressing that a covering of a projective object has a section.) But $\cup \circ T\psi(\gamma) = \cup_{c\in\gamma}\psi(c)$, so s takes γ to $\cup_{c\in\gamma}\psi(c)$, uniformly in γ. This completes the proof.

□

Remark. If there exists a left adjoint to the pseudo-algebra map, then this left adjoint is automatically a computationally dense map, since it has a right adjoint.

5 Applications

In this section we discuss three applications of the machinery that we developed. First, we study relative realizability and local maps. This subject has been treated for ordinary pca's in [1]; we have a look at some facts that emerge when we consider ordered pca's. In particular, we see when an inclusion of ordered pca's gives rise to a local map of toposes. Then, we use this to relate the toposes $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[T\mathbb{A}]$, and we show that a conjecture of Menni is true. Finally, we slightly generalize the fact that the Effective topos is not equivalent to any realizability topos obtained from a total pca.

5.1 Local maps

Let $\mathbb{B}$ be some pca and let $\mathbb{A}$ be a sub-pca of $\mathbb{B}$, that is, $\mathbb{A}$ is a subset containing (some choice for) k and s that is closed under the partial application. In [1] the toposes $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[\mathbb{B}]$ are compared. In the previous section we saw that a geometric morphism from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$ is, up to isomorphism, the same as a map $f : \mathbb{A} \to T\mathbb{B}$ that is computationally dense. Note, however, that for ordinary pca's this requirement implies surjectivity of the map f, and from this it readily follows that there will never be a geometric morphism from $\mathbf{RT}[\mathbb{B}]$ to $\mathbf{RT}[\mathbb{A}]$, except for the trivial case where $\mathbb{A} = \mathbb{B}$. There is, however, a topos $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$, called the *relative realizability topos*, that has the property that there is a local localic geometric morphism $\mathbf{RT}[\mathbb{B}, \mathbb{A}] \to \mathbf{RT}[\mathbb{A}]$, and a logical functor $L : \mathbf{RT}[\mathbb{B}, \mathbb{A}] \to \mathbf{RT}[\mathbb{B}]$. (For more on local maps we refer to [5].) In a picture:

$$\begin{tikzcd} \mathbf{RT}[\mathbb{A}] \arrow[rr, bend left, "\overline{i}"] \arrow[rr, bend right, "i_*"'] & & \mathbf{RT}[\mathbb{B}, \mathbb{A}] \arrow[ll, "i^{-1}"' description] \arrow[r, "L"] & \mathbf{RT}[\mathbb{B}] \end{tikzcd}$$

The intermediate topos $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$ is constructed by taking the tripos $I(\mathbb{B})^{(-)}$ and taking the following preorder: $\phi \vdash' \psi$ iff $\exists a \in \mathbb{A} : a \in \bigcap_{x\in X}(\phi(x) \Rightarrow \psi(x))$. (All the other structure is exactly as in the tripos $I(\mathbb{B})^{(-)}$.) Now the maps $\overline{i}$, i_* and i^{-1} are defined on the tripos-level, as follows (for $\phi : X \to I(\mathbb{A})$, $\psi : X \to \mathbb{B}$):

$$\bar{i}(\phi)(x) = \downarrow(\phi(x)), \qquad i^{-1}(\psi)(x) = \psi(x) \cap \mathbb{A},$$

$$i_*(\phi)(x) = \bigcup_{\alpha \in I(\mathbb{B})} (\alpha \wedge (\mathbb{A} \cap \alpha \to \downarrow(\phi(x)))).$$

Remarks.

1. First of all, we have given this definition in such a way, that it also applies to ordered pca's. That is, way say that $\mathbb{A}$ is a *sub-opca* of $\mathbb{B}$ if it is a full sub-poset, closed under the partial application and contains (some choice of) k and s. It is completely straightforward to check that this still gives a local geometric morphism: one can copy the proof of theorem 3.1 in [1] almost literally.
2. Second, note that the functors $\bar{i}$ and i^{-1} are precisely the maps that are induced by the inclusion $\mathbb{A} \hookrightarrow \mathbb{B} \hookrightarrow T\mathbb{B}$ as in the previous section.
3. We also mention that the counit of the adjunction $i^{-1} \dashv i_*$ is an isomorphism, just as the unit of $\bar{i} \dashv i^{-1}$ is, so that $\mathbf{RT}[\mathbb{A}]$ is actually a retract of $\mathbf{RT}[\mathbb{B}, \mathbb{A}]$.

Now for our purposes it will be interesting to know when the functor L is an equivalence.

Proposition 5.1 *If $\mathbb{A}$ is a sub-opca of $\mathbb{B}$, the functor L is an equivalence if and only if $\forall b \in \mathbb{B} \exists a \in \mathbb{A} : i(a) \leq b$.*

Proof. ($\Rightarrow$:) If L is an equivalence, then i induces a geometric morphism, and therefore is computationally dense. ($\Leftarrow$:) Take $\phi, \psi : X \to I(\mathbb{B})$, and assume that we have $b \in \mathbb{B}$ with $b \in \bigcap_{x \in X} \phi(x) \to \psi(x)$. Pick $a \in \mathbb{A}$ with $i(a) \leq b$. Then $i(a) \in \bigcap_{x \in X} \phi(x) \to \psi(x)$.

□

Remarks.

1. In our opinion, this proposition can be taken as providing some evidence for the claim that ordered pca's really are a useful generalization of ordinary pca's, because it shows us that there are non-trivial inclusions of ordered pca's that induce topos morphisms, something which is impossible for pca's (see the first paragraph of this section).
2. If we have such a local localic map, induced by an inclusion $\mathbb{A} \hookrightarrow \mathbb{B}$ of ordered pca's, then it follows that $\mathbb{A}$ is actually a retract of $\mathbb{B}$ in the Kleisli category $\mathbf{Kl}(T)$. The converse need not be true.

3. We said before, that an inclusion of ordinary pca's would never yield a geometric morphism between the associated realizability toposes. It must be stressed, however, that the proof of this fact relies on classical logic, and does not remain true when we switch to an arbitrary base topos instead of **Set**. In fact, in [2] the notion of an *elementary subobject* is introduced. This definition is chosen in such a way, that if $\mathbb{B}$ is now a pca-object in an arbitrary topos $\mathcal{S}$, and $\mathbb{A}$ is a sub-pca of $\mathbb{B}$, then the requirement that $\mathbb{A}$ is an elementary subobject (rather than the maximal subobject) of $\mathbb{B}$ is enough to guarantee that there is a local map between the realizability toposes.

5.2 Iteration of T

In this section we study iteration of the endofunctor T. This gives rise to a sequence of ordered pca's, and, as we will see, to a sequence of the corresponding realizability toposes. It was already predicted by Menni that certain chains of realizability toposes could be obtained in this fashion.

Let us fix an ordered pca $\mathbb{A}$. In the category **OPCA**, we have a diagram

$$\mathbb{A} \xrightarrow{\delta} T\mathbb{A} \xrightarrow{\delta} T^2\mathbb{A} \xrightarrow{\cup} T\mathbb{A}$$

This composition equals the map $\delta : \mathbb{A} \to T\mathbb{A}$ (this is one of the monad identities), so in the category $Kl(T)$, $\mathbb{A}$ is a retract of $T\mathbb{A}$. Now the inclusion of $\mathbb{A}$ in $T\mathbb{A}$ is easily seen to satisfy the condition of proposition (3.7) of the previous section. This means that there is an induced local localic geometric morphism. On the tripos level, it looks like this:

$$\begin{tikzcd} I(\mathbb{A}) \arrow[rr, bend left, "D"] \arrow[rr, bend right, "P"'] & & I(T\mathbb{A}) \arrow[ll, "U" description] \end{tikzcd}.$$

Let us give a direct description of the functors in this diagram (take $\alpha \in I(\mathbb{A})$ and $\xi \in I(T\mathbb{A})$):

$$D(\alpha) = \downarrow(\{\downarrow(a) \mid a \in \alpha\}), \qquad P(\alpha) = \downarrow(\alpha),$$

$$U(\xi) = \bigcup_{\alpha\in\xi}\{a \mid a \in \alpha\}.$$

We used the notation U, D, and P as to remind the reader of the words "union", "discrete" and "principal", respectively.

On the level of toposes, we get the following, similar picture:

$$\begin{tikzcd} \mathbf{RT}[\mathbb{A}] \arrow[rr, bend left, "D"] \arrow[rr, bend right, "P"'] & & \mathbf{RT}[T\mathbb{A}] \arrow[ll, "U" description] \end{tikzcd}.$$

We have the following:

Theorem 5.2 *There is an equivalence* $\mathbf{RT}[T\mathbb{A}] \simeq ((\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}]})_{reg})_{ex}$.

Proof. We know that each $\mathbf{RT}[T\mathbb{A}]$ is the exact completion of its category of projectives, which is the same as the category of separated objects in $\mathbf{RT}[\mathbb{A}]$. But this latter category is the regular completion of the category of projectives of $\mathbf{RT}[\mathbb{A}]$.

□

Remark. In [12] it is remarked, that in some cases an there is another tripos that we can associated with an ordered pca: we can define $J(\mathbb{A}) \subseteq I(\mathbb{A})$ as those downsets in A that are closed under pushouts.

There is an inclusion map $i : J(\mathbb{A}) \hookrightarrow I(\mathbb{A})$, which induces an indexed map of preorders $i : J(\mathbb{A})^X \hookrightarrow I(\mathbb{A})^X$. Left adjoint to this map is composition with the operation Cl_p, which takes a downset to its closure under pushouts. From this it is not hard to establish that there is a geometric inclusion of triposes $J(\mathbb{A})^{(-)} \hookrightarrow I(\mathbb{A})^{(-)}$, and hence an inclusion of toposes (denote the topos represented by the tripos $J(\mathbb{A})^{(-)}$ by $\mathbf{RT}'[\mathbb{A}]$), $\mathbf{RT}'[\mathbb{A}] \hookrightarrow \mathbf{RT}[\mathbb{A}]$.

To complete the picture, we remark that the local localic map between $\mathbf{RT}[T\mathbb{A}]$ and $\mathbf{RT}[\mathbb{A}]$ restricts:

$$\begin{tikzcd}
\mathbf{RT}'[\mathbb{A}] \arrow[r, shift right, "P"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "i"'] \arrow[d, phantom, "\vdash"] & \mathbf{RT}'[T\mathbb{A}] \arrow[l, shift right, "U"'] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "i"'] \arrow[d, phantom, "\vdash"] & \mathbf{RT}'[\mathbb{A}] \arrow[l, shift right, "D"'] \arrow[d, shift right, "i"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{RT}[\mathbb{A}] \arrow[r, shift right, "P"'] \arrow[r, phantom, "\perp"] \arrow[u, shift right, "Cl_p"'] & \mathbf{RT}[T\mathbb{A}] \arrow[l, shift right, "U"'] \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] \arrow[u, shift right, "Cl_p"'] & \mathbf{RT}[\mathbb{A}] \arrow[l, shift right, "D"'] \arrow[u, shift right, "Cl_p"']
\end{tikzcd}$$

It is easiest to see why the functors U, P and D restrict if we consider them on the tripos-level (again, we use the same notation for the functors on the tripos- and on the topos-level). Note first that $P(\alpha)$ is trivially closed under pushouts, since it is principal. Second, if $\alpha \in I(\mathbb{A})$ is closed under pushouts, then the same holds for $D(\alpha)$, since if $\downarrow\{a\}, \downarrow\{b\} \in D(\alpha)$, then $\downarrow\{a\} \cup \downarrow\{b\} \subseteq \downarrow\{a \vee b\}$. Third, the map U also preserves the property of being closed under pushouts. Now the adjointness is immediate, and so is the commutation of the diagram.

We can iterate the downset-construction: starting with an arbitrary ordered pca $\mathbb{A} = \mathbb{A}_0$, we get a sequence $\mathbb{A}_0, \mathbb{A}_1, \mathbb{A}_2, \ldots$ when we put $\mathbb{A}_{n+1} = (T\mathbb{A}_n)$.

This immediately gives us a sequence $I(\mathbb{A}_0)^{(-)}, I(\mathbb{A}_1)^{(-)}, \ldots$ of triposes, and hence a sequence $\mathbf{RT}[\mathbb{A}_0], \mathbf{RT}[\mathbb{A}_1], \ldots$ of toposes.

On the other hand, the results in [9] show that there are sequences of toposes of the form $(\mathcal{C}_{reg(n)})_{ex}$, (for appropriate categories $\mathcal{C}$). With the previous results in mind, the following theorem should not be all too surprising:

Theorem 5.3 *For each* $n \in \mathbb{N}$, *there is an equivalence of categories* $\mathbf{RT}[\mathbb{A}_n] \simeq ((\mathbf{Proj}_{\mathbf{RT}[\mathbb{A}_0]})_{reg(n)})_{ex}$.

Proof. This goes by induction and is an immediate consequence of the facts that we established concerning $\mathbf{RT}[\mathbb{A}]$ and $\mathbf{RT}[T\mathbb{A}]$.

□

As a last observation, we mention the fact that there is also a chain of toposes coming from the hierarchy $J(\mathbb{A}), J(T\mathbb{A}), \ldots$. This chain is included in the one coming from $I(\mathbb{A}), I(T\mathbb{A}), \ldots$.

5.3

In a very short paper [6], Johnstone and Robinson gave a categorical proof of the fact that the Effective Topos is not equivalent to a realizability topos obtained from a total pca. Longley observed, that, for two pca's $\mathbb{A}$ and $\mathbb{B}$, $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{RT}[\mathbb{B}]$ iff there are functions $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{A}$ such that $fg \sim 1, gf \sim 1$. Using this, he showed that $\mathbb{A}$ is decidable iff $\mathbb{B}$ is. So if we are to prove the inequivalence of two realizability toposes, then it suffices to show that one of the underlying pca's is decidable, whereas the other is not. Now Kleene's pca $\mathbb{N}$ is decidable, but a total pca is never decidable.

We wish to give a variation on this proof. First, it can be shown that if $\mathbf{RT}[\mathbb{A}] \simeq \mathbf{RT}[\mathbb{B}]$, then there are bijective maps $f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{A}$, with f and g inverse. Then we have the following:

Lemma 5.4 *Let $\mathbb{A}$, $\mathbb{B}$ be pca's. Assume that $\mathbb{A}$ is total, and $\mathbb{B}$ has an element z such that for all $b \in \mathbb{B}$: $zb\downarrow$ and $zb \neq b$. Then $\mathbf{RT}[\mathbb{A}] \not\simeq \mathbf{RT}[\mathbb{B}]$.*

Proof. Assume that the toposes are equivalent, and take functions f, g as above and realizers $r \in \mathbb{B}$ with $rf(a)f(a') = f(aa')$, and $s \in \mathbb{B}$ with $bb'\downarrow \Rightarrow rg(b)g(b') \leq g(bb')$. Also, using the recursion theorem in $\mathbb{B}$, choose an element $e \in \mathbb{B}$ such that

$$e \bullet x \simeq z \bullet (r \bullet (r \bullet f(s) \bullet e) \bullet x).$$

Then:

$$\begin{aligned} e \bullet x &= fg(e \bullet x) \\ &= f(s \bullet g(e) \bullet g(x)) \\ &= r \bullet (r \bullet f(s) \bullet fg(e)) \bullet fg(x) \\ &= r \bullet (r \bullet f(s) \bullet e) \bullet x \end{aligned}$$

but, on the other hand, $e \bullet x \neq r \bullet (r \bullet f(s) \bullet e) \bullet x$ because of the property of the element z. Contradiction.

$\square$

Note that this proof is properly more general in that it doesn't depend on the decidability of the pca's involved (e.g. it also works for Kleene's pca of functions, which is not decidable).

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