All realizability is relative

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(*Received*)

Abstract

We introduce a category of basic combinatorial objects, encompassing PCAs and locales. Such a basic combinatorial object is to be thought of as a pre-realizability notion. To each such object we can associate an indexed preorder, generalizing the construction of triposes for various notions of realizability. There are two main results: first, the characterization of triposes which arise in this way, in terms of ordered PCAs equipped with a filter. This will include "Effective Topos-like" triposes, but also the triposes for relative, modified and extensional realizability and the dialectica tripos. Localic triposes can be identified as those arising from ordered PCAs with a trivial filter. Second, we give a classification of geometric morphisms between such triposes in terms of maps of the underlying combinatorial objects. Altogether, this shows that the category of ordered PCAs with non-trivial filters serves as a framework for studying a wide variety of realizability notions.

1. *Introduction*

In the area of research known as realizability, we have the interesting phenomenon that there are many different realizability definitions, but no definition of realizability. What this means is, that we have many instances of realizability interpretations (many of which are variations on Kleene's original definition [**8**]) but that there is no clear answer to the question of what constitutes a notion of realizability. Moreover, it is not clear how to compare various notions of realizability. A parallel problem is the following: given a realizability definition we have a corresponding tripos, and hence a realizability topos; what is a good framework to study these toposes in? What would be a good category of realizability toposes and what properties would it have? (Admittedly, equating syntactical realizability definitions with triposes/toposes is a simplification, since there are realizability definitions which do not have a categorical counterpart, and there are also definitions which have more than one. But in this paper, we will ignore these issues.)

Most of the realizability notions that have been invented over the years, such as arithmetical, modified and extensional realizability, seem to have in common that there is a domain of elements, called realizers, and a class of (possibly partial) endofunctions on this domain. These functions are thought of as realizable or computable functions.

In this paper, we take this picture quite naively and introduce the notion of a *basic combinatorial object*, which is roughly a poset equipped with a collection of (possibly partial) monotone functions. These combinatorial objects can be thought of as "pre-realizability notions". Examples of combinatorial objects will include partial combinatorial algebras and ordered variants thereof, but also (complete) lattices.

Every such combinatorial object gives rise to a **Set**-indexed preorder. In fact, the category of basic combinatorial objects (**BCO**) is fully embedded into the category of **Set**-indexed preorders. This means that we can now set up a correspondence between categorical properties of a certain combinatorial object and logical features of the associated indexed preorder.

In particular, there is a monad on the category **BCO** which takes downsets on the underlying poset of a combinatorial object. On the level of indexed preorders, this monad has a nice interpretation: it freely adds existential quantification.

Because in the end we are interested in realizability triposes/toposes, we wish to know when the indexed preorder associated to a combinatorial object is a tripos (so that it interprets all higher order intuitionistic logic). To this end, we prove the following characterization theorems: first, a free algebra for the downset monad gives a tripos precisely when it is an *ordered PCA equipped with a filter*. This filter is just a sub-ordered PCA in a strict sense, and the computable functions are now precisely the functions representable by elements of this filter. Second, an arbitrary algebra gives a tripos precisely when it is an ordered PCA equipped with a filter such that the additional condition that the partial application preserves suprema in the first variable (up to a realizer) is satisfied.

We recover two extreme examples of triposes: when our combinatory object is a locale, then the filter is trivial and the only computable function is the identity. And on the other extreme our combinatory object can be a Partial Combinatory Algebra, in which case the filter consists of all elements of the PCA. Other well-known triposes for various types of realizability turn out to be more in the middle of the spectrum.

These results show that if we are interested in realizability triposes and toposes, then the notion of an ordered PCA with a filter is forced upon us. It should be remarked that this result is not completely original, since Carboni, Freyd, and Scedrov proved the following special case: starting with the natural numbers and a class of partial endofunctions, they build a "realizability universe"; then they show that this universe is a topos precisely when the class of partial endofunctions contains all partial recursive ones (and in particular is a PCA). So, the characterization result that we prove here extends this to all combinatorial objects.

This characterization can be seen as the "object part" of our work: we have shown which combinatorial objects should be the focus of our study, namely the ones which give rise to triposes. Next we concentrate on the "morphism part" of the correspondence. Which morphisms of combinatorial objects give rise to geometric morphisms between the associated triposes? The main result here extends a characterization theorem for geometric morphisms between toposes coming from partial combinatory algebras (see [**2**, **3**]), and involves a notion of density. We prove the correspondence between dense maps of algebras and geometric morphisms of triposes. This result shows how the study of the category of realizability toposes and geometric morphisms may be reduced to that of the (much simpler) category of basic combinatorial objects and dense maps, thereby justifying our claim that the latter category is a suitable framework for realizability. It also provides a possible answer to a question posed in [**10**], namely what constitutes a homomorphism of realizabilities.

As applications of this characterization, we have a look at structure morphisms from and into the category of sets, and we give a simple characterization of those triposes which are localic as those arising from a PCA with a trivial filter. This gives a precise meaning to the intuition that localic triposes are not realizability triposes, for the only realizable function over a locale will be the identity function.

The structure of this paper is as follows. The first section is devoted to the study of the category of basic combinatorial objects. Then we investigate the indexed preorders associated to combinatorial objects and see how we can use the logic of indexed preorders to reason about combinatorial objects. In section 4, we introduce the downset monad and characterize its algebras. Before going to the main theorems in section 6, we provide the key example of an ordered PCA with a filter. We also show how this captures various notions of realizability, such as modified realizability, extensional realizability and relative realizability. In section 7, we characterize which morphisms of combinatorial objects induce geometric morphisms between the associated triposes. Finally, some possible extensions of the framework, sidelines and open problems are discussed in section 8.

All of the results presented go through when we replace the category of sets by an arbitrary elementary topos. Some of them, however, require a mild choice condition, namely that the basic combinatorial objects have enough global elements. This is guaranteed, for example, when the terminal object is projective.

2. *The category* **BCO**

In this section we set up the category around which our investigations are centered. This category will be called the category of basic combinatorial objects, reflecting the idea that the objects form the basic building blocks for various indexed preorders relevant for realizability. We will see that this category is enriched in preorders and explore some constructions and properties of the category, such as (internal) products. Designated truth-values will be introduced; these will play an important role in the characterization theorems in section 6. Finally, we have a brief look at a certain type of comma objects.

Basic Combinatorial Objects. We are interested in partially ordered sets $(\Sigma, \leq)$ equipped with a class of partial endofunctions $\mathcal{F}_\Sigma$. We will think -and speak- of these partial endofunctions as "realizable", or as "computable" functions. The tuple $(\Sigma, \leq, \mathcal{F}_\Sigma)$ is called a *basic combinatorial object* (BCO for short) if the following conditions are met:

(i) $\forall f \in \mathcal{F}_\Sigma\, \forall a \in dom(f)\, \forall b[b \leq a \Rightarrow b \in dom(f)\, \&\, f(b) \leq f(a)]$.
So, the maps in $\mathcal{F}$ are monotone and have downwards closed domain.

(ii) $\exists i \in \mathcal{F}\, \forall a \in \Sigma . a \in dom(i)\, \&\, i(a) \leq a$.
There is a "weak identity".

(iii) $\forall f, g \in \mathcal{F}_\Sigma\, \exists h \in \mathcal{F}_\Sigma\, \forall a \in dom(f) : f(a) \in dom(g) \Rightarrow a \in dom(h)\, \&\, h(a) \leq g(f(a))$.
"Weak composition".

In order to keep statements like the ones above readable we will usually omit references to elements having to be in the domain of partial endofunctions. For example, the identity condition could be written $\exists i \in \mathcal{F}_\Sigma . \forall a \in \Sigma . i(a) \leq a$.

Every poset can be viewed as a BCO by taking $\mathcal{F}_\Sigma$ to be the class consisting of only the identity function. Also, any PCA fits into this framework by giving it the discrete ordering and by taking the class of representable functions (i.e. the functions of the form $a \bullet -$) for $\mathcal{F}_\Sigma$.

Even though it is perfectly possible for a (po)set to have various non-equivalent BCO-structures on it, we will often just write Σ for $(\Sigma, \leq, \mathcal{F}_\Sigma)$, since there will rarely be a situation in which this will cause confusion.

A *morphism* $\phi : (\Sigma, \leq \mathcal{F}_\Sigma) \to (\Theta, \leq, \mathcal{F}_\Theta)$ between two BCOs is a function $\phi : \Sigma \to \Theta$ on the underlying sets subject to the requirements that:

(i) There exists $u \in \mathcal{F}_\Theta$ such that for all $a \leq a'$ in Σ we have $u(\phi(a)) \leq \phi(a')$;

(ii) For all $f \in \mathcal{F}_\Sigma$ there exists $g \in \mathcal{F}_\Theta$ with $g\phi(a) \leq \phi(f(a))$ for all $a \in dom(f)$.

The following diagram serves as a heuristics for the second condition:

$$\begin{tikzcd} \Sigma \arrow[r, "\phi"] \arrow[d, "\forall f \in \mathcal{F}_\Sigma"'] & \Theta \arrow[d, dotted, "\exists g \in \mathcal{F}_\Theta"] \\ \Sigma \arrow[r, "\phi"] & \Theta. \end{tikzcd} \tag{2·1}$$

Of course, the vertical maps in this diagram can be partial maps, and the conditions on the domains of these are left implicit. This diagram is not supposed to commute on the nose, but up to an inequality. We will often refer to the second condition by saying that morphisms of BCOs preserve realizable functions. There is an obvious dual condition, namely that of reflection of realizable functions. This dual condition (or rather an up-to-isomorphism variation of it) will play an essential role in the study of geometric morphisms in section 7.

It is now easily seen that basic combinatorial objects and their morphisms form a category, which we will call **BCO**. Note that the conditions on morphisms are not needed to show that the axioms for a category hold. Rather, they are needed for the enrichment discussed below.

Enrichment. The category **BCO** is enriched in preorders: for two parallel morphisms $\phi, \psi : \Sigma \to \Theta$, we define

$$\phi \vdash \psi \Leftrightarrow \exists g \in \mathcal{F}_\Theta \forall a \in \Sigma . f\phi(a) \leq g(a).$$

The weak composition and identity requirements on $\mathcal{F}_\Theta$ ensure that this is indeed transitive and reflexive. Note that this preorder is, in general, not the pointwise preorder, unless $\mathcal{F}_\Theta$ only contains functions f with the property that $f(x) \leq x$. The requirements on morphisms ensure that composition becomes functorial, so that we have an enriched category. We will say that two parallel morphisms ϕ, ψ are *isomorphic* if $\phi \vdash \psi$ and $\psi \vdash \phi$ both hold, in which case we write $\phi \cong \psi$ or $\phi \dashv\vdash \psi$, depending on whether we are in a categorical or in a logical mood.

The enrichment enables us to talk about *adjunctions* or *equivalences* between BCOs. For example, $\phi : \Sigma \to \Theta$ is called right adjoint to $\psi : \Theta \to \Sigma$ if $\psi\phi \vdash 1$ and $1 \vdash \phi\psi$. Also, Σ and Θ are called equivalent if there is an adjunction $\phi \dashv \psi$ between them with $\psi\phi \cong 1$ and $\phi\psi \cong 1$. We will be mostly interested in BCOs up to this notion of equivalence, just as we are interested in realizability toposes and -triposes up to equivalence.

Saturation. For some practical purposes it is inconvenient that $\mathcal{F}_\Sigma$ has only weak closure properties. To overcome this, define a collection $Sat(\mathcal{F}_\Sigma)$ as follows:

$$f \in Sat(\mathcal{F}_\Sigma) \Leftrightarrow \exists f' \in \mathcal{F}_\Sigma . \forall a \in dom(f). f'(a) \leq f(a).$$

We call $Sat(\mathcal{F}_\Sigma)$ the *saturation* of $\mathcal{F}_\Sigma$. Then we have:

LEMMA 2·1. *With the definition of $Sat(\mathcal{F}_\Sigma)$ as above:*

(i) *The class $Sat(\mathcal{F}_\Sigma)$ contains $\mathcal{F}_\Sigma$, and is closed under restricting the domain, i.e. $f \in Sat(\mathcal{F}_\Sigma), A \subseteq dom(f) \Rightarrow f|_A \in Sat(\mathcal{F}_\Sigma)$.*

(ii) *The class $Sat(\mathcal{F}_\Sigma)$ is closed under composition and contains the identity.*
(iii) *There is an equivalence of BCOs $(\Sigma, \leq, \mathcal{F}_\Sigma) \simeq (\Sigma, \leq, Sat(\mathcal{F}_\Sigma))$.*
(iv) *$Sat(\mathcal{F}_\Sigma)$ is the largest extension of $\mathcal{F}$ with the above properties.*

Proof. Easy exercise. □

This tells us that there is no harm in assuming that our BCOs are saturated and we will often do this without further mention.

Well-foundedness. A less trivial normalization problem concerns the morphisms in the category **BCO**: can we assume that every morphism is equivalent to one that is strictly order-preserving? The answer here is, that this depends on the nature of the codomain.

First say that a partial map $f \in \mathcal{F}_\Sigma$ is a *subidentity function* if it satisfies $f(a) \leq a$ for all a in its domain. So, each such f and each element $a \in \Sigma$ gives a descending chain $a \geq f(a) \geq ff(a) \geq \ldots$. We say that f is *bounded* if there exists a natural number n such that $f^n(a) = f^{n+1}(a)$ for all $a \in \Sigma$. Then Σ is called *well-founded* if each subidentity function is bounded.

Lemma 2·2. *If Σ is well-founded, then every morphism $\phi : \Theta \to \Sigma$ is equivalent to a morphism that preserves the order on the nose.*

Proof. Let $u \in \mathcal{F}_\Sigma$ be a function such that $u(\phi(a)) \leq \phi(a')$ for all $a \leq a'$. Pick n such that $u^n(t) = u^{n+1}(t)$ for all $t \in \Sigma$. Then define a new function $\phi' : \Theta \to \Sigma$ as $\phi'(a) = u^n(\phi(a))$. Then ϕ' preserves the order on the nose and is equivalent to ϕ. □

Although the condition of well-foundedness looks strong and is stronger than the requirement that each morphism can be normalized, I do not know any non-artificial examples of BCOs that are non-well-founded. Some of the results in section 6 will require a normalization for certain morphisms.

Finite Limits. The category **BCO** has a terminal object, namely the one-element poset equipped with the identity function. Moreover, **BCO** has binary products, with all structure taken coordinatewise. In particular, the collection of realizable functions of $\Sigma \times \Theta$ is given by

$$\mathcal{F}_{\Sigma\times\Theta} = \{f \times g | f \in \mathcal{F}_\Sigma, g \in \mathcal{F}_\Theta\}.$$

Together, this gives that **BCO** has all finite products. (In fact, it has all products.) Equalizers are, in general, not present but are fortunately not needed for our purposes.

A BCO Σ is said to have a *top element* $\top$ if the map $\Sigma \to 1$ has a right adjoint $\top : 1 \to \Sigma$. This is generally not the same as a top element for Σ *qua poset*. Rather, it means that there exists a realizable function f such that $f(a) \leq \top$ for all $a \in \Sigma$. Similarly, if the diagonal $\Delta : \Sigma \to \Sigma \times \Sigma$ has a right adjoint, then Σ is said to have finite products and, again, this is in general not the same as having finite meets as a poset. If a BCO has a top element as well as finite products, we say that it has (internal) finite limits.

Designated Truth-Values. If Σ is a BCO with a top element $\top$, then we write

$$TV(\Sigma) = \{a \in \Sigma | \exists f \in \mathcal{F}_\Sigma . f(\top) \leq a\}$$

and call the elements of $TV(\Sigma)$ the *designated truth-values* of Σ. Note that $TV(\Sigma)$ is upwards closed. Every $v \in TV(\Sigma)$ has the property that for any $a \in \Sigma$ there exists an

$f \in \mathcal{F}_\Sigma$ such that $f(a) \leq v$. Therefore, if $v \in TV(\Sigma)$, the map $v : 1 \to \Sigma$ defines a right adjoint to the map $\Sigma \to 1$.

Glueing. As an aside we sketch a construction which generalizes one used by Pitts to describe the Effective Monad. We assume that our BCOs have finite limits. Fix a BCO Θ. For a map $\phi : \Sigma \to \Theta$, we construct a new BCO $\Theta//\phi$ (the notation will be clarified in a moment). The underlying poset of $\Theta//\phi$ is the product $\Sigma \times \Theta$. A partial map $h : \Sigma \times \Theta \rightharpoonup \Sigma \times \Theta$ is realizable if there exist realizable $f \in \mathcal{F}_\Sigma, g \in \mathcal{F}_\Theta$ such that for all $(x, y) \in \Sigma \times \Theta$:

$$(f(x), g(\phi(x) \wedge y)) \leq h(x, y).$$

Here, $\wedge$ is of course the internal product map.

The object $\Theta//\phi$ comes equipped with two projections:

$$\pi_\Sigma(x, y) = x, \qquad \pi_\Theta(x, y) = \phi(x) \wedge y.$$

Moreover, there is a map $i_\Sigma : \Sigma \to \Theta//\phi$, given by

$$i_\Sigma(x) = (x, \phi(x)).$$

It is not hard to verify that the composite $\pi_\Theta \circ i_\Sigma : \Sigma \to \Theta$ is isomorphic to ϕ. Therefore we have a pseudo factorization system on the category **BCO**.

The notation $\Theta//\phi$ stems from the fact that the object just constructed can be viewed as a bicomma object, as in the diagram

$$\begin{tikzcd}
\Theta//\phi \arrow[r, "\pi_\Sigma"] \arrow[d, "\pi_\Theta"'] & \Sigma \arrow[d, "\phi"] \\
\Theta \arrow[r, "1"] & \Theta.
\end{tikzcd}$$

This diagram does not commute, but we have $1 \circ \pi_\Theta \vdash \phi \circ \pi_\Sigma$; the square is weakly universal with this property.

The construction is a 2-functor on $\mathbf{BCO}/\Theta$. In fact, it is a pseudo monad; the unit is given by $i_\Sigma : \Sigma \to \Theta//\phi$, and the multiplication $\mu : \Theta//\pi_\Theta \to \Theta//\phi$ is given by $((x, y), y') \mapsto (x, y \wedge y')$. The (pseudo) algebras for this monad may be called pseudo fibrations, just as algebras for $1/-$ are ordinary fibrations.

The construction is mainly of interest because it sheds some conceptual light on Pitts' Iteration Theorem; this will be the content of a subsequent paper.

3. *Indexed preorders from BCOs.*

Let $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ be a BCO. We construct a **Set**-indexed category $[-, \Sigma]$ as follows: on objects, $[-, \Sigma]$ is the assignment $X \mapsto Hom(X, \Sigma)$, the set of all functions from X to Σ. This set is preordered, by defining for $\phi, \psi : X \to \Sigma$,

$$\phi \vdash_X \psi \quad \Leftrightarrow \quad \exists f \in \mathcal{F}_\Sigma . \forall x \in X . f(\phi(x)) \leq \psi(x).$$

On arrows, $[-, \Sigma]$ acts by precomposition.

It is beneficial to think of a set X as a type, and of elements of Σ as (non-standard) truth-values. A function $\phi : X \to \Sigma$ is then a predicate with a free variable of type X. Moreover, the preorder on $[X, \Sigma]$ becomes a logical entailment relation, and the reindexing functors can be thought of as relabelling of free variables.

By definition, the indexed preorder $[-,\Sigma]$ is *canonically presentable*, meaning that there is a representing object such that the indexed preorder is given by taking homsets into this object. This has as a consequence, that the reindexing functors compose on the nose, and not just up to natural isomorphism.

The construction of $[-,\Sigma]$ from Σ is functorial. In fact, we have:

PROPOSITION 3·1. *The assignment $\Sigma \mapsto [-,\Sigma]$ is the object part of a 2-embedding of the 2-category* **BCO** *into the 2-category of* **Set***-indexed preorders.*

(Of course, the 2-category of **Set**-indexed preorders has **Set**-indexed preorders for its objects, morphisms of indexed preorders (indexed functors) as maps, and indexed natural transformations as 2-cells.)

Proof. We first show the functoriality. Let $\phi : \Sigma \to \Theta$ be a map in **BCO**. We show that postcomposing with ϕ constitutes a morphism of indexed preorders $\phi \circ - : [-,\Sigma] \to [-,\Theta]$. To this end, let X be a set and take two functions $\alpha, \beta : X \to \Sigma$ with $\alpha \vdash_X \beta$. That is, for some $f \in \mathcal{F}_\Sigma$, we have $f\alpha(x) \leq \beta(x)$, all $x \in X$. Pick $g \in \mathcal{F}_\Theta$ with $g\phi(a) \leq \phi(f(a))$ for all $a \in dom(f)$. Then in particular we get $ug\phi(\alpha(x)) \leq u\phi(f(\alpha(x))) \leq \phi(\beta(x))$, where u is the element up to which ϕ preserves the ordering. Therefore, $\phi \circ \alpha \vdash_X \phi \circ \beta$, as witnessed by the composite ug.

To see that this extends to 2-cells, consider $\phi \vdash \psi$, say via some f with $f\phi(a) \leq \psi(a)$. To see that this induces a transformation from $\phi\circ-$ to $\psi\circ-$, take a set X and a predicate $\alpha : X \to \Sigma$. Then $\phi \circ \alpha \vdash_X \psi \circ \alpha$ via the same f, so there is a transformation as required.

Next, we observe that $[-,\Sigma]$ is full (well, up to isomorphism of arrows). First of all, it is standard to show (by considering the identity on Σ) that each morphism $[-,\Sigma] \to [-,\Theta]$ is, up to isomorphism, of the form $\phi \circ -$, where $\phi : \Sigma \to \Theta$ is a function between the underlying generic objects. We must verify that ϕ is a morphism of BCOs. So take a map $f \in \mathcal{F}_\Sigma$, and define $D = \{(a, f(a)) | a \in dom(f)\}$. Obviously we have $\pi_1 \vdash_D \pi_2$, where the π_i are the projections into Σ. Because $\phi \circ -$ is a functor, this entails $\phi \circ \pi_1 \vdash \phi \circ \pi_2$, which means that some $g \in \mathcal{F}_\Theta$ has $g(\phi(a)) \leq \phi(f(a))$ for all $a \in dom(f)$. The condition that ϕ preserves the order up to a realizer is treated in a similar fashion.

Finally, $[-,\Sigma]$ is full on the level of 2-cells: if we have two morphisms $\phi, \psi : \Sigma \to \Theta$, and there exists a transformation $\phi \circ - \Rightarrow \psi \circ -$ between the induced indexed functors, then by looking at the fibre over Σ and the identity predicate in there, we obtain a function $g \in \mathcal{F}_\Theta$ with $g(\phi(x)) \leq \psi(x)$ for all $x \in \Sigma$. This means that $\phi \vdash \psi$. □

Using Internal Logic. It will be convenient to reason about BCOs and their associated indexed preorders using the internal logic of indexed preorders, thereby avoiding reasoning with tracking functions all the time. Since the correctness of this reasoning is a basic fact from categorical logic, we will not justify this but refer the reader to standard text on the logic of indexed categories, such as [**5**]. Here we just give some examples to illustrate this way of reasoning and the associated notation. Let $\alpha, \beta : X \to \Sigma$ be two functions. We will write $\alpha(x) \vdash_x \beta(x)$ instead of $\alpha \vdash_X \beta$. This extends to several variables: for sets X, Y, Z and functions $\alpha : X \times Y \to \Sigma$ and $\beta : X \times Y \times Z \to \Sigma$, we write $\alpha(x, y) \vdash_{x,y,z} \beta(x, y, z)$ for $\alpha\pi_{12} \vdash_{X\times Y\times Z} \beta$. Now if we also had $\beta(x, y, z) \vdash_{x,y,z} \gamma(y)$, then we could derive $\alpha(x, y) \vdash_{x,y} \gamma(y)$.

A particular case that we will use a lot is the case where $X = \Sigma$. For example, saying that $f \in \mathcal{F}_\Sigma$ is equivalent to the statement $a \vdash_{a\in dom(f)} f(a)$. The identity and composition axioms for $\mathcal{F}_\Sigma$ can be restated by saying that $a \vdash_a a$ and that $a \vdash_a f(a)$ and $b \vdash_b g(b)$ imply $a \vdash_a gf(a)$.

Indexed Finite Limits. Our next concern is to translate possible additional structure on a basic combinatorial object to logical features of the associated indexed preorder. For finite limits, which we discuss in this section, this is almost a triviality; in the next sections on cocompletions and tripos characterizations we will have results which are less immediate.

When does $[-, \Sigma]$ have indexed finite meets? Well, to say that an indexed category has indexed finite products is to say that it is a Cartesian object in the category of indexed categories. Since the embedding of **BCO** into the category of indexed preorders is full on the level of 2-cells and fully faithful up to isomorphism on the level of 1-cells, this embedding preserves and reflects Cartesian objects. Therefore, $[-, \Sigma]$ has indexed finite meets if and only if Σ is a Cartesian object in the category **BCO**. But that just means that all diagonal maps $\Sigma \to \Sigma^n$, $n \geq 0$, have a right adjoint, so that Σ has internal finite products, generated by maps $\top : 1 \to \Sigma$ and $\wedge : \Sigma \times \Sigma \to \Sigma$.

It is important to note that in the above situation it is not guaranteed that the meet map $\wedge : \Sigma \times \Sigma \to \Sigma$ preserves the ordering on the nose in either variable.

If a BCO Σ has internal finite products, then the collection of designated truth-values has some extra structure: since the designated truth-values are precisely the elements $a \in \Sigma$ for which $\top \vdash_1 a$, this collection is upwards closed and is closed under taking finite products. For, if $a \leq a'$ and $\top \vdash a$ then also $\top \vdash a'$. Moreover, if $a, b \in TV(\Sigma)$ then $a \wedge b \in TV(\Sigma)$ because $\top \vdash a, \top \vdash b$ implies $\top \vdash a \wedge b$.

Because of these closure properties, we call $TV(\Sigma)$ a *filter* in Σ.

Closed Structure? At first sight, it might seem that the category **BCO** has exponents; the obvious candidate for a function space Σ^Θ is the poset $\mathbf{BCO}[\Theta, \Sigma]$, ordered pointwise and equipped with the set of endofunctions $\{f \circ - | f \in \mathcal{F}_\Sigma\}$. This is a well-defined BCO, but the correspondence between maps $\Omega \times \Theta \to \Sigma$ and $\Omega \to \Sigma^\Theta$ that exists on the level of posets fails to extend to the level of BCOs. It would be nice to have a left adjoint to $(-)^\Theta$, which would give a tensor product on **BCO**, but I couldn't find it. See also the related problem in section 4 under the caption "Frobenius Condition".

4. *Completions*

In this section we discuss an important monad on the category **BCO**, which takes downsets in a BCO. An algebra for this monad may be seen as a "complete" BCO, just like an algebra for the downset monad on posets is a complete sup-lattice. In fact, the category of algebras for our monad will contain the category of complete sup-lattices. First some formal, 2-categorical, aspects of this monad are explored, after which we will see that on the level of indexed preorders it is nothing but the free cocompletion monad. Thus, the monad freely adds existential quantification to an indexed preorder. Also a minor variation, which adds existential quantification along surjections, is introduced. Finally, we prove that algebras also admit universal quantification along surjections.

Downset Monad. Let us take a BCO $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ and write

$$\mathcal{D}\Sigma = \{A \subseteq \Sigma | \text{ A is downward closed }\}.$$

This set is ordered by subset inclusion. Also, there is a canonical choice for a class of realizable functions on this poset, namely

$$\mathcal{F}_{\mathcal{D}\Sigma} = \{F : \mathcal{D}\Sigma \rightharpoonup \mathcal{D}\Sigma | \exists f \in \mathcal{F}_\Sigma . \forall A \in dom(F) \forall a \in A . f(a) \in F(A)\}.$$

So, a function F is realizable when there is some $f \in \mathcal{F}_\Sigma$ that uniformly realizes all $A \mapsto F(A)$. It is straightforward to check that $\mathcal{D}\Sigma = (\mathcal{D}\Sigma, \subseteq, \mathcal{F}_{\mathcal{D}\Sigma})$ is again a BCO.

The downset construction is 2-functorial. For a map $\phi : \Sigma \to \Theta$ of BCOs, we define $\mathcal{D}\phi$ by $\mathcal{D}\phi(A) = \downarrow\{\phi(a)|a \in A\}$, the downward closure of the image of A under ϕ. Finally, if $\phi \vdash \psi$ for morphisms $\phi, \psi : \Sigma \to \Theta$, say via a realizer g, then it is not hard to show that the same g realizes $\mathcal{D}\phi \vdash \mathcal{D}\psi$.

Lemma 4·1. *The 2-functor $\mathcal{D}$ is a 2-monad.*

Proof. This is an extension of the fact that taking downsets is a 2-monad on the category of posets.

The unit of the monad is given by taking principle downsets. More explicitly, on a BCO Σ, the unit $\downarrow(-) : \Sigma \to \mathcal{D}\Sigma$ sends a to $\downarrow(a) = \{a' \in \Sigma | a' \leq a\}$.

Let us verify that this is a morphism of BCOs. The fact that $\downarrow(-)$ is strictly order-preserving is clear. So it remains to be seen that realizable functions are preserved. So take $f \in \mathcal{F}_\Sigma$. We have to find $G \in \mathcal{F}_{\mathcal{D}\Sigma}$ such that for all $a \in dom(f)$ we have $G(\downarrow(a)) \subseteq \downarrow(f(a))$. Take G to be the function defined by $G(A) = \cup\{\downarrow(f(a))|a \in A\}$. Then G is realizable, $G(\downarrow(a)) = \downarrow(f(a))$ and we are done.

The multiplication of the monad is given by taking unions. More explicitly, the map $\bigcup : \mathcal{D}\mathcal{D}\Sigma \to \mathcal{D}\Sigma$ sends $\mathcal{U}$, a downset of downsets, to its union $\bigcup\mathcal{U} = \{a \in \Sigma | \exists A \in \mathcal{U}. a \in A\}$. Again it is clear that this preserves the ordering. To see that it also preserves realizable functions, take $F \in \mathcal{F}_{\mathcal{D}\mathcal{D}\Sigma}$. By definition, there exists some G in $\mathcal{F}_{\mathcal{D}\Sigma}$ such that for all $\mathcal{U}$ in $dom(F)$ we have $G(U) \in F(\mathcal{U})$, all $U \in \mathcal{U}$. Now the function G has the required property that $G(\bigcup\mathcal{U}) \subseteq \bigcup F(\mathcal{U})$. $\square$

Incidentally, the unit not only preserves realizable functions (i.e. is a morphism of BCOs) but also reflects them. More on this in section 6.

If a BCO Σ carries an algebra structure we will usually write $\bigvee : \mathcal{D}\Sigma \to \Sigma$ for it. Such an algebra structure should be thought of as a supremum map. But beware: this need not be a supremum map for the underlying poset of Σ, the reason being that, unlike in the case of posets, the supremum map need not be order-preserving. Still, we will often refer to algebras as "complete" BCOs, stretching the analogy with complete sup-lattices.

We will be more interested in *pseudo-algebras*, where the defining diagrams for an algebra commute up to natural isomorphism as opposed to on the nose. These may be thought of as BCOs which are complete "up to a realizer". These really are the more natural objects to consider, first, because we are generally interested in BCOs (and indexed preorders) up to equivalence, and second, because they yield better characterization theorems.

$\mathcal{D}$ as a KZ-monad. The downset monad on BCOs has a strong 2-categorical property that will prove very useful. Like various other completion monads it is a Kock-Zöberlein Monad, which means that a pseudo- algebra structure is the same thing as a left adjoint to the unit. For more on KZ-monads see [**9**]. To show that we have a KZ-monad, it is enough to prove the following proposition.

Proposition 4·2. *For a BCO Σ, pseudo-algebra structures on Σ are in one-to-one* *correspondence with left adjoints to the unit. In particular, if a pseudo-algebra structure exists, it is unique up to isomorphism.*

Proof. By definition of a pseudo-algebra structure $\bigvee : \mathcal{D}\Sigma \to \Sigma$, we have that the composite $\bigvee \circ \downarrow(-)$ is isomorphic to the identity on Σ. It is therefore sufficient to show that there is a 2-cell from the identity on $\mathcal{D}\Sigma$ to $\downarrow(-) \circ \bigvee$. So we have to exhibit a function $f \in \mathcal{F}_\Sigma$ such that for all $A \in \mathcal{D}\Sigma$ and all $a \in A$ we have $f(a) \in\downarrow(\bigvee A)$, i.e. $f(a) \leq \bigvee A$. Because $\bigvee$ preserves the ordering up to a realizer we can pick some $g \in \mathcal{F}_\Sigma$ for which $g \bigvee U \leq \bigvee V$ whenever $U \subseteq V$. In particular, if $a \in A$ then $\downarrow(a) \subseteq A$, so that $g \bigvee \downarrow(a) \leq \bigvee A$. But because of the isomorphism $\bigvee \downarrow(-) \cong 1$ there is a realizable h with $h(a) \leq \bigvee \downarrow(a)$ for all $a \in \Sigma$. Now take f to be gh. $\square$

Existential Quantification and Characterization of Algebras. On the level of indexed preorders, the monad has the following interpretation. Let Σ be a BCO. Then the effect of applying the downset monad to Σ is that of freely adding existential quantification to $[-, \Sigma]$. That means that $[-, \mathcal{D}\Sigma]$ has indexed coproducts (left adjoints to reindexing functors) satisfying the Beck-Chevalley condition, and that the embedding $[-, \Sigma] \to [-, \mathcal{D}\Sigma]$ is universal with this property. So, $\mathcal{D}$ may be viewed as a monad on the category of indexed preorders (of suitable type, of course). It should be noted that this free construction which turns an indexed preorder into a cocomplete one is not the same as the well-known family construction (see [**5**]) which forms the free cocomplete indexed category on an arbitrary indexed category. More specifically, the result of applying the family construction to an indexed preorder need not be a preorder again.

We now characterize the pseudo-algebras for the monad $\mathcal{D}$.

PROPOSITION 4·3. *A BCO Σ carries a pseudo-algebra structure if and only if the associated indexed preorder $[-, \Sigma]$ has* **Set***-indexed coproducts.*

Proof. First we remark that for canonically presented indexed preorders, the Beck-Chevalley condition trivially holds, so that we can concentrate on left adjoints to reindexing functors.

Assume that $\bigvee : \mathcal{D}\Sigma \to \Sigma$ is a pseudo-algebra. Take a function $f : X \to Y$ of sets. The left adjoint to composition with f can now be defined on an element $\alpha : X \to \Sigma$ by

$$\exists_f(\alpha)(y) = \bigvee_{f(x)=y} \alpha(x).$$

To show that this is indeed a left adjoint, denote by $\alpha_f : Y \to \mathcal{D}\Sigma$ the map sending y to $\{\alpha(x) | f(x) = y\}$. Thus, $\exists_f(\alpha) = \bigvee \circ \alpha_f$. To show the equivalence of $\exists_f(\alpha) \vdash_Y \beta$ and $\alpha \vdash_X \beta \circ f$, observe that

$$\begin{aligned} \exists_f(\alpha) \vdash_Y \beta &\Leftrightarrow \bigvee \alpha_f \vdash_Y \beta \\ &\Leftrightarrow \alpha_f \vdash_X \downarrow \beta \\ &\Leftrightarrow \alpha \vdash_Y \beta \circ f. \end{aligned}$$

On the other hand, assume that $[-, \Sigma]$ has existential quantification. Let M be the set $M = \{(a, U) | a \in U \in \mathcal{D}\Sigma\}$. This gives two projections $\pi_1 : M \to \Sigma$ and $\pi_2 : M \to \mathcal{D}\Sigma$, and we can form $\bigvee = \exists_{\pi_2} \pi_1$. This will be the underlying function of the algebra map. We must show that $\bigvee$ is a morphism of BCOs, or, equivalently, that composition with

$\bigvee$ is a morphism of indexed preorders $[-,\mathcal{D}\Sigma] \to [-,\Sigma]$. Suppose that we have $\alpha \vdash_X \beta$ via $f \in \mathcal{F}_\Sigma$ for some $\alpha, \beta : X \to \mathcal{D}\Sigma$. Form the sets $P = \{(x, a, f(a)) | x \in X, a \in \alpha(x)\}$ and $Q = \{(x, \alpha(x), f[\alpha(x)]) | x \in X\}$. There is a function $g : P \to Q$ given by $g(x, a, f(a)) = (x, \alpha(x), f[\alpha(x)])$. The second and third projections $\pi_2, \pi_3 : P \to \Sigma$ clearly satisfy $\pi_2 \vdash_P \pi_3$, via f. Hence $\exists_g \pi_2 \vdash_Q \exists_g \pi_3$. Spelling out what $\exists_g \pi_2$ and $\exists_g \pi_3$ are, we get $\exists_g \pi_2(x, \alpha(x), f[\alpha(x)]) = \bigvee \alpha(x)$ and $\exists_g \pi_3(x, \alpha(x), f[\alpha(x)]) = \bigvee f[\alpha(x)]$. Thus there exists some $h \in \mathcal{F}_\Sigma$ such that for all $x \in X$, $h \bigvee(\alpha(x)) \leq \bigvee f[\alpha(x)]$, which shows that $\bigvee \alpha \vdash \bigvee \beta$.

The adjointness $\bigvee \dashv \downarrow(-)$ follows from similar arguments. □

Nonempty Downsets. The monad $\mathcal{D}$ admits a minor variation by replacing the full downsets by the nonempty downsets. Write $\mathcal{D}_i$ for this monad. The proofs above can be used almost verbatim to show that $\mathcal{D}_i$ is also a KZ-monad, and that pseudo-algebras are characterized by the property that their associated indexed preorders have left adjoints to reindexing functors along *surjections*.

Preservation of Structure. The monad $\mathcal{D}$ preserves the property of having finite meets. Explicitly, if $\wedge : \Sigma \times \Sigma \to \Sigma$ is the meet map, then conjunction in $\mathcal{D}\Sigma$ can be defined, for $A, B \in \mathcal{D}\Sigma$, by

$$A \wedge B = \{u \in \Sigma | \exists a \in A \exists b \in B. u \leq a \wedge b\}.$$

Obviously, the embedding $\downarrow(-) : \Sigma \to \mathcal{D}\Sigma$ preserves this structure.

Being expressible as a supremum, there is always a top element in $\mathcal{D}\Sigma$, namely Σ itself. If Σ already had a top element $\top$, then it is easily seen that $\downarrow(\top) \dashv\vdash \Sigma$, so that $\downarrow(-)$ preserves the top element.

Next, we look at implication. Suppose that $[-,\Sigma]$ has implication, given by a map $\Rightarrow : \Sigma \times \Sigma \to \Sigma$. (We have not investigated yet to what kind of structure on the BCO Σ this corresponds, but that will be done in section 6.) One can now define an operation $\Rightarrow^*$ on $\mathcal{D}\Sigma$ which induces implication on $[-,\mathcal{D}\Sigma]$, by putting, for $A, B \in \mathcal{D}\Sigma$,

$$A \Rightarrow^* B = \bigcap_{a \in A} \bigcup_{b \in B} \downarrow (a \Rightarrow b)$$

where the $\Rightarrow$ on the right hand side of the equation is the implication in Σ. Again, it is immediate that the embedding $\downarrow(-)$ preserves implication.

In short, we have:

Proposition 4·4. *If $[-,\Sigma]$ has indexed finite limits, then so does $[-,\mathcal{D}\Sigma]$ and the embedding $\downarrow(-) : [-,\Sigma] \to [-,\mathcal{D}\Sigma]$ preserves them. If, moreover, $[-,\Sigma]$ has indexed Heyting implication, then so does $[-,\mathcal{D}\Sigma]$, and the embedding preserves it.*

Frobenius Condition. Whenever an indexed preorder has both meets and existential quantification we can ask whether the Frobenius condition

$$\exists_f(\alpha \wedge \beta f) \dashv\vdash \exists_f \alpha \wedge \beta$$

holds. If our indexed preorder is of the form $[-,\Sigma]$, then finite meets are encapsulated in a monoid structure on Σ with multiplication $\wedge : \Sigma \times \Sigma \to \Sigma$. The question now reduces to: does $\wedge$ preserve suprema in both variables separately (up to a realizer)? Just as for posets, this is of course not true in general, and may be viewed as a special property of the object.

It would be nice to have a tensor product on BCOs that captures this bilinearity, but I couldn't find an extension of the tensor product of complete sup-lattices (see [**7**]) and hence failed to characterize indexed preorders with Frobenius as monoids in the category of $\mathcal{D}$-algebras.

It is also worth noting that in the case of a monoid in complete sup-lattices, one can define implication: since $- \wedge a$ preserves all suprema, it has a right adjoint $a \to -$ by the adjoint functor theorem. This result does not carry over to our more general setting, for much of the same reason as why the product functor $- \times \Sigma$ on **BCO** does not have a right adjoint.

Universal Quantification. As an aside we show that the monad $\mathcal{D}$ also adds universal quantifications along surjections. The structural map for this is simply given by intersection: $\bigcap : \mathcal{D}\mathcal{D}\Sigma \to \mathcal{D}\Sigma$.

More generally, take a pseudo-algebra $\bigvee : \mathcal{D}\Sigma \to \Sigma$ and define an infimum map $\bigwedge : \mathcal{D}\Sigma \to \Sigma$ as

$$\bigwedge A = \bigvee \{b \in \Sigma | b \leq a \text{ for all } a \in A\}.$$

Of course, this is the common way to define infima from suprema in a poset.

PROPOSITION 4·5. *The map $\bigwedge$ defines universal quantification along surjections.*

Proof. Take $f : X \to Y$ and functions $\alpha : X \to \Sigma$, $\beta : Y \to \Sigma$. We have to show that $\beta \circ f \vdash_X \alpha$ is equivalent to $\beta \vdash_Y \forall_f \alpha$.

So assume first that $\beta f \vdash_X \alpha$, i.e. that there exists $g \in \mathcal{F}_\Sigma$ such that $\forall x \in X$ we have $g(\beta f(x)) \leq \alpha(x)$. For fixed $y \in Y$, write

$$A_y = \{u \in \Sigma | u \leq \alpha(x) \text{ for all } x \in f^{-1}(y)\},$$

so that $\bigwedge_{f(x)=y} \alpha(x) = \bigvee A_y$. If such $y \in Y$ is given, then for any x with $f(x) = y$ we have $g(\beta(y)) = g\beta f(x) \leq \alpha(x)$, uniformly in x. This means that $g(\beta(y)) \in A_y$. Therefore,

$$g(\beta(y)) \vdash \bigvee \downarrow g(\beta(y)) \vdash \bigvee A_y = \bigwedge_{f(x)=y} \alpha(x),$$

also uniformly in x, which shows that $\beta \vdash_Y \forall_f \alpha$.

For the converse we assume $\beta \vdash_Y \forall_f \alpha$. Fix $x \in X$ and write $y = f(x)$. Then $A_y \subseteq \downarrow (\alpha(x))$, whence $\bigvee A_y \vdash \alpha(x)$, i.e. $\bigwedge_{f(x)=y} \alpha(x) \vdash \alpha(x)$. This is uniform in x, so we obtain $\beta(f(x)) = \beta(y) \vdash \alpha(x)$, as required. □

5. *Examples*

In this section we discuss two key examples of BCOs, namely locales and (ordered) PCAs. It will be seen that both can be viewed as extreme cases of the more general concept of an ordered PCA equipped with a filter. We explain what the associated indexed preorders are and what the effect of applying the downset monad is. Finally it is shown how various other types of realizability triposes fit in.

Locales. Any poset can be viewed as a BCO by taking only the identity function as realizable. That means that there is a 2-embedding of the category of posets in the category of BCOs. Moreover, the downset monad on the category of posets extends to the downset monad on **BCO**. That implies that if Σ is a poset viewed as a BCO, an algebra structure is just a sup-lattice structure on Σ. This is how we view complete sup-lattices, and in particular locales, as a subcategory of the category of $\mathcal{D}$-algebras.

If Σ is a locale, then the associated indexed preorder $[-, \Sigma]$ is usually called the tripos for the locale H. Because the only realizable function is the identity, all structure is defined pointwise.

A morphism between two locales, viewed as BCOs, is just a poset morphism; nothing forces meets or suprema to be preserved. In the section on geometric morphisms, we will characterize in elementary terms when such a morphism is a locale map. In particular, this characterizes which BCO maps between locales induce geometric morphisms between the associated triposes.

We can apply the downset monad $\mathcal{D}$ to a locale Σ, to obtain a new locale $\mathcal{D}\Sigma$. Similarly, $\mathcal{D}_i\Sigma$, the poset of nonempty downsets in Σ is a locale. The only difference between the two is that $\mathcal{D}\Sigma$ has a new bottom element, namely the empty set, whereas the bottom element in $\mathcal{D}_i\Sigma$ is $\{\perp\}$, the set containing only the bottom element of Σ. In fact, as BCOs, these are equivalent. Indeed, consider the embedding $i : \mathcal{D}_i\Sigma \to \mathcal{D}\Sigma$ and the retraction r which identifies $\emptyset$ and $\{\perp\}$. Because $ri = 1$ and $ir \simeq 1$, we find the desired equivalence.

Ordered PCAs. Any ordered PCA $(\Sigma, \leq, \bullet)$ can be viewed as a BCO by taking the same poset and by defining taking all functions of the form $a \bullet -$ to be realizable. (For the definition and examples of ordered PCAs, see [**3**, **11**].) It takes a small lemma to see that under this identification, the usual notion of a morphism of ordered PCAs is the same as that for BCOs.

LEMMA 5·1. *Let Σ, Θ be ordered PCAs. Then map of BCOs $\phi : \Sigma \to \Theta$ is a map of ordered PCAs precisely when it preserves finite limits.*

Proof. From the definition of a morphism of BCOs it follows that for a morphism ϕ, there exists an element $u \in \Theta$ such that for all $a \leq a' \in \Sigma$ we have $u\phi(a) \leq \phi(a')$. This is precisely the condition on morphisms of ordered PCAs that the order is preserved up to a realizer. Moreover, if ϕ is a morphism of BCOs then realizable functions are to be preserved, i.e. for every $a \in \Sigma$ there is a $d \in \Theta$ such that for all b with $ab\downarrow$ it holds that $d\phi(b) \leq \phi(ab)$. The corresponding condition on morphisms of ordered PCAs is seemingly stronger, namely that there exists $q \in \Theta$ such that for all $a, b \in \Sigma$ with $ab\downarrow$ we have $q\phi(a)\phi(b) \leq \phi(ab)$. We have to exhibit an element q with this property. Consider the assignment $a \wedge b \mapsto ab$. This is a realizable function, so there exists an element $d \in \Theta$ for which $d\phi(a \wedge b) \leq \phi(ab)$ for all a, b with $ab\downarrow$. Using the fact that ϕ preserves finite limits, we get a realizer e with $e(\phi(a) \wedge \phi(b)) \leq \phi(ab)$. Now put $q = \lambda xy.e(x \wedge y)$. □

This shows that there is a full embedding of the category of Ordered PCAs into the category of BCOs. The image of this embedding will be characterized in section 6.

The (nonempty) downset monad on ordered PCAs has been studied in [**2**] with the purpose of classifying geometric morphisms between realizability toposes. We see now that the downset monad on BCOs extends the one on ordered PCAs as well as the ones on posets.

For an ordered PCA Σ, the indexed preorder $[-, \mathcal{D}\Sigma]$ is called the realizability tripos for Σ. The fact that $\mathcal{D}$ is a monad now tells us that realizability triposes of this form are *free* triposes, obtained by making a certain indexed preorder cocomplete. This is really the counterpart of the well-known result that realizability toposes are exact completions.

To complete the parallel: the total category of the indexed preorder $[-, \Sigma]$ is the category of Partitioned Assemblies, of which the realizability topos is the exact completion.

Filters. We can unify the above two examples by considering the notion of a filter in an ordered PCA. This is a subset $\Phi \subseteq \Sigma$ such that Φ is closed under application and contains (some choice of) the combinators k and s. So, a filter is nothing but a sub-ordered PCA of Σ. To the pair (Φ, Σ) we associate a BCO by taking Σ for the underlying poset and by taking the realizable functions to be those of the form $a \bullet -$ for *a in the filter* Φ. Thus

$$\mathcal{F}_\Sigma = \{a \bullet - | a \in \Phi\}.$$

We see from this definition that it makes no difference for the resulting BCO if we take Φ to be upwards closed. This justifies the terminology "filter".

Consequently, the indexed preorder that arises is given, in the fibre over X, by

$$\alpha \vdash_X \beta \Leftrightarrow \exists a \in \Phi. \forall x \in X. a \bullet \alpha(x) \leq \beta(x).$$

Readers familiar with the definition of the tripos for relative realizability will certainly recognize this preorder. In fact, the relative realizability tripos for a pair $\Phi \subseteq \Sigma$ is obtained by applying $\mathcal{D}$ to the above indexed preorder.

Of course, when we let $\Phi = \Sigma$ then we are back in the situation of ordered PCAs. To see a locale Σ as an ordered PCA with a filter, define, for $a, b \in \Sigma$, an application function by $a \bullet b = a \wedge b$. (This works for any meet-semilattice.) Any element will serve as k and s. The filter Φ now consists of only the top element of the locale, so that the only realizable function is the identity.

This shows, how ordered PCAs and locales are two ends of a spectrum of possibilities. The next section will show that this example is in fact universal, in the sense that every BCO which gives rise to a tripos is in fact an ordered PCA with a filter.

We will now discuss a few variations on ordinary realizability. From the definitions and results so far it is clear that *extensional realizability* fits into our framework. Explicitly, the generic element is $\mathcal{DD}_i(\mathbb{N})$ (see [**11**] for an exposition of extensional realizability and triposes for it). We already saw how relative realizability fits in. Therefore, we devote the remainder of this section to two other examples, namely modified realizability and the dialectica interpretation.

Modified Realizability. Let us show that the modified realizability tripos fits into our framework, i.e. that it is of the form $[-, \Sigma]$ for some ordered PCA Σ with a filter. For an account of the modified realizability tripos, see [**12**].

The generic element of the modified realizability tripos is

$$\Sigma = \{(U_a, U_p) | U_a \subseteq U_p \subseteq \mathbb{N}, 0 \in U_p\}.$$

Here, a coding of partial recursive functions is chosen in such a way that $0 \bullet x = x$ for all $x \in \mathbb{N}$. The entailment in the fibre over 1 is given by:

$$(U_a, U_p) \vdash_1 (V_a, V_p) \Leftrightarrow \exists n \in \mathbb{N} : n \in (U_a \Rightarrow V_a) \cap (U_p \Rightarrow V_p).$$

where the $\Rightarrow$ on the right-hand side of the equation is the ordinary $A \Rightarrow B = \{n | \forall a \in A : na \in V\}$. In the fibre over an arbitrary X, we require that the realizer n works uniformly in all $x \in X$.

The generic element may be endowed with an ordered PCA-structure. The ordering is defined pairwise:

$$(U_a, U_p) \leq (V_a, V_p) \text{ if and only if } U_a \subseteq V_a \text{ and } U_p \subseteq V_p$$

and we define

$$(U_a, U_p) \bullet (V_a, V_p) \simeq (U_a V_a, U_p V_p)$$

where the juxtaposition of sets on the righthand side is shorthand for $U_p V_p = \{ab | a \in U_p, b \in V_p\}$; so this is just pairwise application in the ordered PCA $\mathcal{D}\mathbb{N}$.

Next, there are the combinators k, s, which may be taken to be $(\{k\}, \{0, k\})$ and $(\{s\}, \{0, s\})$.

Note that, as an ordered PCA, Σ is trivial, since it has a least element $(\emptyset, \{0\})$. But the designated truth-values $\Phi \subseteq \Sigma$ are those (U_a, U_b) for which $U_a \neq \emptyset$.

The implication $\Rightarrow: \Sigma \times \Sigma \to \Sigma$ is given by

$$(U_a, U_p) \Rightarrow (V_a, V_p) = ((U_a \Rightarrow V_a) \cap (U_p \Rightarrow V_p), U_p \Rightarrow V_p)$$

Now we derive that $(U_a, U_p) \bullet (V_a, V_p) \leq (W_a, W_p)$ if and only if $(U_a, U_p) \leq ((V_a, V_p) \Rightarrow (W_a, W_p))$.

$$\begin{aligned}
(U_a, U_p) \bullet (V_a, V_p) &\leq (W_a, W_p) \\
\Leftrightarrow \quad & (U_a V_a, U_p V_p) \leq (W_a, W_p) \\
\Leftrightarrow \quad & U_a V_a \subseteq W_a \;\&\; U_p V_p \subseteq W_p \\
\Leftrightarrow \quad & U_a \subseteq V_a \Rightarrow W_a \;\&\; U_p \subseteq V_p \Rightarrow W_p \\
\Leftrightarrow \quad & U_a \subseteq V_a \Rightarrow W_a \;\&\; U_p \subseteq V_p \Rightarrow W_p \;\&\; U_a \subseteq V_p \Rightarrow W_p \\
\Leftrightarrow \quad & U_a \subseteq ((V_a \Rightarrow W_a) \cap (V_p \Rightarrow W_p)) \;\&\; U_p \subseteq V_p \Rightarrow W_p \\
\Leftrightarrow \quad & (U_a, U_p) \leq ((V_a \Rightarrow W_a) \cap (V_p \Rightarrow W_p), V_p \Rightarrow W_p) \\
\Leftrightarrow \quad & (U_a, U_p) \leq ((V_a, V_p) \Rightarrow (W_a, W_p))
\end{aligned}$$

From this, it follows that the tripos may be recaptured as

$$\alpha \vdash_X \beta \Leftrightarrow \exists (U_a, U_p) \in \Phi \, \forall x \in X : (U_a, U_p) \bullet \alpha(x) \subseteq \beta(x).$$

This shows that the modified realizability tripos arises in the canonical way from an ordered PCA with a filter.

Dialectica Tripos. We show that the dialectica tripos can also be incorporated. For a description of this tripos we refer to [**1**].

The dialectica tripos has a generic object

$$\Sigma = \{(X, Y, A) | X, Y \subseteq \mathbb{N}, A \subseteq X \times Y, 0 \in A \cap Y\}$$

and the preorder in the fibre over 1 is given by

$$\begin{aligned}
(X, Y, A) \vdash (X', Y', A') \Leftrightarrow \;\; & \exists f, F \in \mathbb{N} : f \in (X \Rightarrow X'), \\
& F \in (X \times Y' \Rightarrow Y), \\
& A(x, F(x, y)) \text{ implies } A'(fx, y)
\end{aligned}$$

and in the fibre over M we require this uniformly in all $m \in M$. We order the generic element by putting

$$(X, Y, A) \leq (X', Y', A') \Leftrightarrow X \subseteq X', Y' \subseteq Y, A \subseteq A'.$$

So, $(\Sigma, \leq)$ is the underlying poset. If ϕ is a partial endofunction on Σ, then say that ϕ is realizable if there exist f, F such that if $\phi(X, Y, A) = (P, Q, B)$, then $f[X] = P$, $F[X \times Q] = Y$, and $A(x, F(x, q))$ implies $B(fx, q)$. Clearly such ϕ satisfies $(X, Y, A) \vdash \phi(X, Y, A)$, uniformly in (X, Y, A). Conversely, if we have $(X_i, Y_i, A_i) \vdash_I (P_i, Q_i, B_i)$, then we can find a $\phi \in \mathcal{F}$ such that $\phi(X_i, Y_i, A_i) \leq (P_i, Q_i, B_i)$, for all i. Namely, let f, F be the required realizers for $(X_i, Y_i, A_i) \vdash_I (P_i, Q_i, B_i)$, and put $\phi(X_i, Y_i, A_i) = (X_i', Y_i', A_i')$, where $X_i' = \{f\}X_i, Y_i' = \{a \in \mathbb{N} | \forall x \in X : F(x, a) \in Y_i\}, A_i' = \{(fx, a) | A(x, F(x, a))\}$.

This makes clear that the dialectica tripos arises from a BCO. From the results in the following section it will follow that the object Σ is an ordered PCA, although it is not easy to give an explicit description.

6. *Tripos characterizations and ordered PCAs*

Our main goal in this section is to give a characterization of those BCOs Σ for which $[-, \mathcal{D}\Sigma]$ is a tripos. We will also derive a characterization of when $[-, \Sigma]$ is a tripos. Since finite limits are an obvious necessary condition, we will assume in this section that all BCOs possess these. We will also assume that the operation $\wedge : \Sigma \times \Sigma \to \Sigma$ preserves the order in both variables on the nose. (This can be guaranteed, for example, by assuming that our BCOs are well-founded.) The sensitive reader is warned in advance that the characterization theorems are not entirely constructive. Sufficient choice conditions will be discussed after the statements of the theorems.

We start by introducing some notation and by proving some technical lemmas which will facilitate the oncoming proofs.

Let $\Sigma = (\Sigma, \leq, \mathcal{F}_\Sigma)$ be a BCO. We define $\mathcal{F}^2_\Sigma$ to be the class of partial functions $f : \Sigma \times \Sigma \rightharpoonup \Sigma$ for which there exists a $g \in \mathcal{F}_\Sigma$ such that $g(a \wedge b) \leq f(a, b)$ for all $a, b \in dom(f)$. Note that, from our assumption that $\wedge$ preserves the order in both variables, it follows that each $f \in \mathcal{F}^2_\Sigma$ does so as well. The following closure properties are easily derived:

LEMMA 6·1. *$\mathcal{F}^2_\Sigma$ contains both projections $\pi_1, \pi_2 : \Sigma \times \Sigma \to \Sigma$ and the meet map $\wedge : \Sigma \times \Sigma \to \Sigma$.*

Proof. We have $a \wedge b \vdash_{a,b} a$, which says precisely that the first projection is in $\mathcal{F}^2_\Sigma$. Because of $a \wedge b \vdash_{a,b} a \wedge b$, the meet map is in $\mathcal{F}^2_\Sigma$. $\square$

LEMMA 6·2. *Let $f, g \in \mathcal{F}_\Sigma$. Then the map $(a \wedge b) \mapsto (f(a) \wedge g(b))$ is in $\mathcal{F}_\Sigma$. For any map $h \in \mathcal{F}^2_\Sigma$, the composite $h \circ (f \times g)$ is in $\mathcal{F}^2_\Sigma$.*

Proof. We have $a \vdash_a f(a)$ and $b \vdash g(b)$, hence $a \wedge b \vdash_{a,b} f(a) \wedge g(b)$. Hence $a \wedge b \mapsto f(a) \wedge g(b)$ is in $\mathcal{F}_\Sigma$.

Given $h \in \mathcal{F}^2_\Sigma$, pick $k \in \mathcal{F}_\Sigma$ with $k(x \wedge y) \leq h(x, y)$ for all $(x, y) \in dom(h)$. Then both $a \wedge b \vdash_{a,b} f(a) \wedge g(b)$ and $f(a) \wedge g(b) \vdash h(f(a), g(b))$ hold, so that $a \wedge b \mapsto h(f(a), g(b))$ is realizable. $\square$

Heyting Implication. We can now relate Heyting structure on $[-, \mathcal{D}\Sigma]$ to closure properties of $\mathcal{F}_\Sigma$.

PROPOSITION 6·3. *The following are equivalent:*

(i) $[-, \mathcal{D}\Sigma]$ *has Heyting implication*

(ii) *There is a map* $App \in \mathcal{F}^2_\Sigma$ *such that for each* $h \in \mathcal{F}^2_\Sigma$ *there is a map* $\hat{h} \in \mathcal{F}_\Sigma$ *such that* $(a,b) \in dom(h)$ *implies that* $App(\hat{h}(a), b)$ *is defined and* $App(\hat{h}(a), b) \leq h(a,b)$.

Of course, this just says that we can carry out the process known as "Currying". This proposition relies on a mild choice condition, namely that every surjection from a subobject of Σ onto 1 splits.

Proof. First assume (1). Let $\Rightarrow : \mathcal{D}\Sigma \times \mathcal{D}\Sigma \to \mathcal{D}\Sigma$ be the implication map. For the map App, we take a tracking m for $(A \Rightarrow B) \wedge A \vdash B$ and put $App(x,y) = m(x \wedge y)$. So App is in $\mathcal{F}^2_\Sigma$, and has the property that for each $c \in A \Rightarrow B$ and $a \in A$, $App(c,a) \in B$. In particular:

$$c \in \downarrow(a) \Rightarrow \downarrow(b) \text{ implies } App(c,a) \leq b.$$

Now take $h \in \mathcal{F}^2\Sigma$; then $a \wedge b \vdash_{a,b} h(a,b)$. Therefore $\downarrow(a) \wedge \downarrow(b) \vdash \downarrow(h(a,b))$, so $\downarrow(a) \vdash \downarrow(b) \Rightarrow \downarrow(h(a,b))$. Thus we have a $\hat{h}$ with $\hat{h}(a) \in \downarrow(b) \Rightarrow \downarrow(h(a,b))$ for all $(a,b) \in dom(h)$. Hence $App(\hat{h}(a), b) \leq h(a,b)$. This proves (2).

Next assume (2). We define, for $A, B \in \mathcal{D}\Sigma$,

$$A \Rightarrow B = \downarrow\{c \in \Sigma | \forall a \in A : App(c,a) \in B\}.$$

We have to show that this indeed gives rise to Heyting implication in each fibre. Let X be a set, and let $\alpha, \beta, \gamma : X \to \mathcal{D}\Sigma$ be functions. First suppose that $\alpha \wedge \beta \vdash_X \gamma$. So, $\exists g \in \mathcal{F}_\Sigma \, \forall x \, \forall a \in \alpha(x) \, \forall b \in \beta(x) : g(a \wedge b) \in \gamma(x)$. By the closure property (2), we find a $\hat{g}$, such that for all $a \in \alpha(x), \forall b \in \beta(x) : App(\hat{g}(a), b) \leq g(a \wedge b)$. Hence $\hat{g}$ is a tracking for $\alpha \vdash_X \beta \Rightarrow \gamma$.

Conversely if $\alpha \vdash_X \beta \Rightarrow \gamma$, then we have a map $h \in \mathcal{F}$ sending each $a \in \alpha(x)$ to an element of $\beta(x) \Rightarrow \gamma(x) = \{c | \forall b \in \beta(x) : App(c,b) \in \gamma(x)\}$. Now the map $a \wedge b \mapsto App(h(a), b)$ is in $\mathcal{F}_\Sigma$ since both h and App are, and is a tracking for $\alpha \wedge \beta \vdash_X \gamma$. This proves (1). □

If we are in the situation that the equivalent conditions of the above proposition hold, we may call a function of the form $App(a, -)$ *representable*. In particular, we get a partial application on Σ defined by $a \bullet b = App(a,b)$.

Recall that a designated truth-value of Σ is an element $a \in \Sigma$ such that $\top \vdash a$. Now the next lemma states that representable maps are given by designated truth-values of Σ.

LEMMA 6·4. *An element* a *of* Σ *is a designated truth-value of* Σ *if and only if* $App(a, -)$ *is in* $\mathcal{F}_\Sigma$.

Proof. If a is a designated truth-value, then $\top \vdash a$. Then, uniformly in b, we have $b \vdash_b \top \wedge b \vdash_b a \wedge b$. Also, the map m in the proof of proposition 6·3 gives $a \wedge b \vdash_{a,b} m(a \wedge b)$. By transitivity we get $b \vdash_b m(a \wedge b) = App(a,b)$, which means that $App(a, -)$ is in $\mathcal{F}_\Sigma$.

For the converse, suppose that $App(a, -) = m(a \wedge -)$ is in $\mathcal{F}_\Sigma$. That means that $b \vdash_b m(a \wedge b)$, uniformly in b. Take a tracking function t for the sequent $\downarrow(a) \vdash \downarrow(\top) \Rightarrow \downarrow(a)$; so $t(a) \in \downarrow(\top) \Rightarrow \downarrow(a)$, and hence $m(t(a) \wedge \top) \leq a$ (by definition of m). Taking all this together, we find (using lemma 6·2) that $\top \vdash m(a \wedge \top) \vdash m(t(a) \wedge \top) \vdash a$, and a is a designated truth-value. □

Tripos Characterizations. We now derive some further properties of the applicational structure.

PROPOSITION 6·5. *Suppose that $\mathcal{E}(-,\Sigma)$ satisfies the equivalent conditions of proposition* 6·3. *Then there are combinators $k, s \in \Sigma$ that make $(\Sigma, \leq, \bullet, k, s)$ into an ordered PCA. Moreover, k, s are designated truth-values of Σ.*

Proof. Consider the propositional scheme

$$A \Rightarrow (B \Rightarrow A).$$

There is an element k (which is a designated truth-value) with the property $\forall a \in A \, \forall b \in B : k \bullet a \bullet b \in A$. Thus for $a, b \in \Sigma : k \bullet a \bullet b \leq a$. For s, take a designated truth-value for the scheme

$$(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C).$$

Take any $x, y \in \Sigma$, and put $A = \emptyset$. Thus $x \in A \Rightarrow (B \Rightarrow C)$, and $y \in A \Rightarrow B$. Hence $s \bullet x \bullet y\downarrow$. If, in addition, it holds that $xz(yz)\downarrow$, then we can put $A = \downarrow(z), B = \downarrow(yz), C = \downarrow(xz(yz))$. Then $y \in A \Rightarrow B = \downarrow(z) \Rightarrow (yz)$, and $x \in A \Rightarrow (B \Rightarrow C)$. Hence $sxyz\downarrow$ and $sxyz \leq xz(yz)$.

The condition that $\wedge$ is strictly order-preserving in both variables implies that the application map also strictly preserves the order, i.e. that

$$ab\downarrow \,\&a' \leq a, b' \leq b \Rightarrow a'b'\downarrow \,\&a'b' \leq ab,$$

which is the last axiom for ordered PCAs that we had to verify. □

PROPOSITION 6·6. *In the situation of proposition* 6·3, *the collection of designated truth-values of Σ is an ordered PCA. In fact, it is a filter in the ordered PCA Σ.*

Proof. We have to show that the designated truth-values are closed under application. Suppose that a, b are designated truth-values, i.e. that $\top \vdash a$ and $\top \vdash b$. Hence also $\top \vdash a \wedge b$. Since we have $a \wedge b \vdash a \bullet b$, transitivity gives $\top \vdash a \bullet b$, whence $a \bullet b$ is a designated truth-value. It was already observed in the previous proposition that k and s were designated truth-values. It is now evident that the designated truth-values constitute a sub-ordered PCA of Σ. □

For now, let us denote the set of designated truth-values of Σ by Φ.

LEMMA 6·7. *Let $f \in \mathcal{F}_\Sigma$. Then there is some $a \in \Phi$ such that for all $b \in dom(f)$: $ab \leq f(a)$.*

Proof. If $f \in \mathcal{F}_\Sigma$ then, uniformly in all $b \in dom(f)$, $b \vdash_b f(b)$, whence $\downarrow(b) \vdash \downarrow(f(b))$. Then $\top \vdash \downarrow(b) \Rightarrow \downarrow(f(b))$, so there is some $p \in \downarrow(b) \Rightarrow \downarrow(f(b))$, which is a designated truth-value. This p satisfies $App(p, b) \leq f(b)$ for all b, so p represents f. □

COROLLARY 6·8. *Let Σ satisfy the conditions of theorem* 6·3. *Then the preorder on $[-,\Sigma]$ can be given by (in the fibre over X):*

$$\alpha \vdash_X \beta \Leftrightarrow \exists a \in \Phi \, \forall x \in X : a \bullet \alpha(x) \leq \beta(x).$$

Consequently, the preorder on $[-,\mathcal{D}\Sigma]$ can be given by

$$\alpha \vdash_X \beta \Leftrightarrow \exists a \in \Phi \, \forall x \in X \, \forall b \in \alpha(x) : a \bullet b \in \beta(x).$$

Proof. This follows immediately from the previous lemma identifying maps in $\mathcal{F}_\Sigma$ with representable functions. □

Putting all of this together, we can characterize when $[-, \mathcal{D}\Sigma]$ is a tripos:

THEOREM 6·9. *Let Σ be a BCO. Then the following are equivalent:*

(i) *Σ carries an ordered PCA-structure together with a filter Φ of designated truth-values, and the preorder on $[-, \Sigma]$ is given as in corollary* 6·8*;*

(ii) *$[-, \mathcal{D}\Sigma]$ is a tripos.*

Proof. One direction is immediate from proposition 6·3, since every tripos has implication. For the other direction we can also be brief, since all the tripos structure can be defined exactly as for an ordinary tripos from an ordered PCA, with the only difference that we restrict the collection of realizers to the designated truth-values. All constructions go through, because of the combinatorial completeness of Φ. □

This theorem shows that the notion of an ordered PCA with a filter is unavoidable; since a filter is a sub-ordered PCA, it has the remarkable interpretation that all triposes obtained from BCOs are relative realizability triposes.

Variation. We now look at the analogous statement for the non-empty downset monad. The main difference is that $\mathcal{D}_i\Sigma$ need not have a bottom element.

THEOREM 6·10. *Let Σ be a BCO. If Σ has a least element $\perp$ then the following are equivalent:*

(i) *Σ carries an ordered PCA-structure together with a filter Φ of designated truth-values, and the preorder is given as in corollary* 6·8

(ii) *$[-, \mathcal{D}_i\Sigma]$ is a tripos.*

Proof. The proof that $[-, \mathcal{I}_i\Sigma]$ has implication is completely the same as for theorem 6·9. For universal quantification we need only observe that the intersection of an arbitrary family of downsets always contains the bottom element, so that the usual definition works. But we know from [**13**] that implication and universal quantification already give us a tripos.

The other direction is the same as for theorem 6·9 □

As a corollary, we get the result that, on the level of indexed preorders, the operation $\mathcal{D}_i$ preserves triposes.

COROLLARY 6·11. *If $[-, \Sigma]$ is a tripos, then so is $[-, \mathcal{D}_i\Sigma]$.*

Proof. $\mathcal{E}(-, \Sigma)$ has implication, and therefore has $\mathcal{E}(-, \mathcal{D}_i\Sigma)$ has implication, too, given by

$$\alpha \Rightarrow \beta = \bigcap_{a \in \alpha} \bigcup_{b \in \beta} \downarrow(a \Rightarrow b),$$

where the $\Rightarrow$ on the right hand side is the implication in $[-, \Sigma]$. Hence, Σ has an ordered PCA-structure. Furthermore, $[-, \Sigma]$ has an indexed least element, which may be taken to be induced by a global element $\perp_1 : 1 \to \Sigma$, so the conditions of theorem 6·10 are met.

Finally, we may take the bottom element of $[-, \mathcal{D}_i\Sigma]$ to be $\{\perp\}$, where $\perp$ is the bottom element of $[-, \Sigma]$. □

By iterating $\mathcal{D}_i$, this result gives rise to various hierarchies of triposes.

When is $[-,\Sigma]$ a tripos?

We have seen some necessary conditions for this: if $[-,\Sigma]$ is a tripos, then so is $[-,\mathcal{D}_i\Sigma]$. Therefore, Σ is an ordered PCA. Moreover, Σ must be a $\mathcal{D}$-algebra. This, however, is not sufficient: we need to know when $[-,\Sigma]$ has implication. This is taken care of by the following lemma:

LEMMA 6·12. *Let Σ be an ordered PCA with a filter Φ, giving rise in the canonical way to a tripos $[-,\mathcal{D}\Sigma]$. Write $\Rightarrow$ for the implication on $\mathcal{D}\Sigma$. Assume also that Σ is a $\mathcal{D}$-algebra. Then the following are equivalent.*

(i) $[-,\Sigma]$ *has implication, given by a map $a \to b = \bigvee(\downarrow a \Rightarrow \downarrow b)$;*

(ii) *There exists $u \in \Phi$ such that $\forall ab \in \Sigma : u \bullet \bigvee(\downarrow a \Rightarrow \downarrow b) \in \downarrow a \Rightarrow \downarrow b$;*

(iii) *The counit of the adjunction $\bigvee \dashv \downarrow(-)$ is an isomorphism at objects of the form $\downarrow a \Rightarrow \downarrow b$;*

(iv) *There exists $v \in \Phi$ such that for any family c_i in Σ, if $c_i \bullet a \leq b$ for all i, then $v \bullet \bigvee c_i \bullet a \leq b$.*

Proof. $(1 \Rightarrow 3)$ Assume that $[-,\Sigma]$ has implication. We know that implication is then preserved by $\downarrow(-) : \Sigma \to \mathcal{D}\Sigma$. This gives condition 3.

$(3 \Rightarrow 1)$ Defining implication on Σ as in condition 1, we find that

$$\begin{aligned} a \wedge b \vdash c &\Leftrightarrow \downarrow a \wedge \downarrow b \vdash \downarrow c \\ &\Leftrightarrow \downarrow a \vdash \downarrow b \Rightarrow \downarrow c \\ &\Leftrightarrow \bigvee \downarrow a \vdash \bigvee(\downarrow b \Rightarrow \downarrow c) \\ &\Leftrightarrow a \vdash b \to c \end{aligned}$$

which shows that we indeed have a well-defined Heyting implication on Σ.

$(2 \Leftrightarrow 3)$ This is immediate.

$(3 \Leftrightarrow 4)$ Suppose that $c_i a \leq b$ for all i. Then $c_i \in \downarrow a \Rightarrow \downarrow b$ for all i. Hence $\bigvee_i c_i \vdash \bigvee(\downarrow a \Rightarrow \downarrow b)$. Thus $u \bullet \bigvee_i c_i \bullet a \leq b$. The converse is similar. □

Put in words, this says that we can define implication on an ordered PCA with a filter when this ordered PCA is a $\mathcal{D}$-algebra and when the partial application preserves suprema in the first variable (up to a realizer).

Now the final characterization is:

THEOREM 6·13. *Let Σ be a BCO with internal finite products. Then the following are equivalent:*

(i) $[-,\Sigma]$ *is a tripos;*

(ii) *Σ is a $\mathcal{D}$-algebra and carries an ordered PCA structure with a filter Φ, such that the preorder on $[-,\Sigma]$ is given as in Corollary 6·8 and such that the application map preserves suprema up to a realizer in the first variable, as in the last condition of lemma 6·12.*

Proof. We have seen that the conditions are necessary. If we can show that Σ has universal quantification along all maps, then we are done, since the entire tripos structure can be defined from the implication and the universal quantification. Consider a map $f : X \to Y$ and a predicate $\alpha : X \to \Sigma$, and define

$$A_y = \{u \in \Sigma | f(x) = y \Rightarrow \forall a \in \Sigma . ua \leq \alpha(x)\}.$$

Now universal quantification along f is defined by

$$\forall_f \alpha(y) = \bigvee A_y.$$

(In the case that f is surjective, this reduces to the definition given in section 4.) The verification of the adjointness $- \circ f \dashv \forall_f$ is now standard. □

7. *Geometric Morphisms*

In this section we determine which morphisms of finitely complete BCOs give rise to geometric morphisms between the associated indexed preorders. For ordered PCAs, this problem was solved using the downset monad; the results presented here are an extension of those in [**3**, **2**]. This extension is meaningful, since we will classify geometric morphisms between any two triposes coming from BCOs, so that we can compare various realizability triposes. In this section we will always assume that our BCOs have internal finite limits (as is the case, for example, when they are ordered PCAs with a filter).

Geometric Morphisms. Let Σ and Θ be BCOs which are finitely complete. Then a *geometric morphism* from Σ to Θ consists of a pair of adjoint maps $\phi_\circ : \Sigma \to \Theta$, $\phi^\circ : \Theta \to \Sigma$ with $\phi^\circ \dashv \phi_\circ$, such that the inverse image part ϕ° preserves finite limits. Usually, we will write $\phi = (\phi_\circ, \phi^\circ) : \Sigma \to \Theta$ for such a geometric morphism. A *geometric transformation* between two such geometric morphisms ϕ, ψ is of course a 2-cell $\phi^\circ \leq \psi^\circ$ between the inverse image maps; for standard reasons, such a transformation induces a 2-cell between the direct image maps.

We have already seen an example of a geometric morphism: namely the embedding of Σ into $\mathcal{D}\Sigma$ where Σ is a complete ordered PCA: the algebra map is the inverse image map.

Another example arises from locales; if Σ, Θ are locales and $\phi : \Sigma \to \Theta$ a poset morphism which preserves the locale structure then ϕ will automatically have a right adjoint in the category of meet-semilattices (see [**7**, **6**]). Hence ϕ is the inverse image part of a geometric morphism in the category of BCOs.

Computational Density. We are interested in a characterization of those maps of BCOs which have a right adjoint. To this end, we study a reflection principle, that was introduced in [**2**] for morphisms of ordered PCAs. We will now extend this to morphisms of arbitrary BCOs.

DEFINITION 7·1. *Let $\phi : \Sigma \to \Theta$ be a morphism of BCOs. Then ϕ is called* computationally dense *if there exist an $h \in \mathcal{F}_\Sigma$ such that for every realizable $g \in \mathcal{F}_\Theta$ there exists a realizable $f \in \mathcal{F}_\Sigma$ such that for all a with $\phi(a) \in dom(g)$ we have $h\phi(f(a)) \leq g(\phi(a))$.*

In words, the map ϕ reflects realizable maps, but only up to the realizable h. For the sake of brevity we omit the adverb “computationally”, speaking simply about dense maps. Also, we refer to the map h in the definition as the *witness* for the density of ϕ. Such a witness need of course not be unique.

We will first prove some elementary properties from this definition.

LEMMA 7·2.

(i) *The identity is dense;*
(ii) *dense maps compose;*
(iii) *if $\phi : \Sigma \to \Theta$ is dense then so is $\mathcal{D}\phi : \mathcal{D}\Sigma \to \mathcal{D}\Theta$;*
(iv) *the unit map $\downarrow(-) : \Sigma \to \mathcal{D}\Sigma$ is dense. Also, the multiplication $\bigcup : \mathcal{D}\mathcal{D}\Sigma \to \mathcal{D}\Sigma$ is dense.*

Proof.

(i) Trivial.
(ii) Let $\phi : \Sigma \to \Theta$ and $\psi : \Theta \to \Gamma$ be given. Let h_1 be a witness for ϕ and h_2 be a witness for ψ. Take a map $k \in \mathcal{F}_\Gamma$. Because ψ is dense, this gives a map $g \in \mathcal{F}_\Theta$ with $h_2\psi g(b) \leq k\psi(b)$. Because ϕ is dense there is $f \in \mathcal{F}_\Sigma$ with $h_1\phi f(a) \leq g\phi(a)$. Put $b = \phi(a)$ to obtain
$$h_2\psi h_1\phi f(a) \leq h_2\psi g\phi(a) \leq k\psi\phi(a).$$
Because ψ preserves realizable functions there exists h_3 with $h_3\psi(x) \leq \psi h_0(x)$. Therefore we have
$$h_1h_3\psi\phi f(a) \leq h_1\psi h_0\phi f(a) \leq k\psi\phi(a),$$
which shows that h_1h_3 is a witness for the density of the composite $\psi\phi$.
(iii) Let ϕ be dense, witnessed by h. Let G be a realizable function of $\mathcal{D}\Theta$, i.e. there exists $g \in \mathcal{F}_\Theta$ with $g(a) \in G(A)$ for all $A \in dom(G), a \in A$. Pick $f \in \mathcal{F}_\Sigma$ such that $h\phi f(a) \leq g\phi(a)$. Now the function $F(A) = \downarrow\{f(a)|a \in A\}$ is realizable, and we have $h\phi f(a) \leq g\phi(a) \in G\{\phi(a)|a \in A\} = G(\mathcal{D}\phi)(A)$, which proves that H, defined by $H(U) = \downarrow\{h(u)|u \in U\}$ is a witness for the density of $\mathcal{D}\phi$.
(iv) Easy.

□

In particular, we see that BCOs and dense maps form a category, which we denote by $\mathbf{BCO}_d$. The downset monad restricts to this category. We also remark that this category inherits finite products. We will also write $\mathbf{BCO}_{dl}$ for the category of finitely complete BCOs and dense morphisms preserving all finite limits.

The first relation between density and geometric morphisms is the content of the following theorem.

THEOREM 7·3. *A morphism ϕ is dense if and only if $\mathcal{D}\phi$ has a right adjoint.*

We will split the proof in a couple of lemmas.

LEMMA 7·4. *Let $\psi : \Sigma \to \mathcal{D}\Theta$ be a dense map. Then the induced map $\psi^\circ : \mathcal{D}\Sigma \to \mathcal{D}\Theta$ has a right adjoint.*

Proof. Define a right adjoint $\psi_\circ$ by
$$\psi_\circ(U) = \{a \in \Sigma | h(\psi(a)) \subseteq U\}.$$
The fact that this is order-preserving is a straightforward extension of the case for ordered PCAs, as is the verification that the required adjointness holds. □

LEMMA 7·5. *Up to natural isomorphism, any geometric morphism $\psi : \mathcal{D}\Theta \to \mathcal{D}\Sigma$ is induced by a dense map $\phi : \Sigma \to \mathcal{D}\Theta$.*

Proof. Again analogous to the ordered PCA case. □

LEMMA 7·6. *Let $\phi : \Sigma \to \Theta$ be a map. Then the composite $\downarrow(-) \circ \phi : \Sigma \to \mathcal{D}\Theta$ is dense if and only if ϕ is.*

Proof. Easy. □

Proof. (Of Theorem 7·3.) If $\phi : \Sigma \to \Theta$ is dense then so is the composite $\downarrow(-) \circ \phi : \Sigma \to \mathcal{D}\Theta$. Hence the induced map $\mathcal{D}\Sigma \to \mathcal{D}\Theta$ has a right adjoint. Conversely, if $\downarrow(-) \circ \phi : \Sigma \to \mathcal{D}\Theta$ is dense then so is ϕ. □

So, dense maps are precisely those which induce geometric morphisms between the free algebras. Note that we have actually shown the following corollary.

COROLLARY 7·7. *There is a natural isomorphism*

$$\mathbf{BCO}_{dl}(\Sigma, \mathcal{D}\theta) \cong Geom(\mathcal{D}\Theta, \mathcal{D}\Sigma).$$

Here, the category $Geom(\mathcal{D}\Theta, \mathcal{D}\Sigma)$ stands for the category of geometric morphisms and natural transformations. This result is a generalization of the characterization of geometric morphisms between triposes of the form $[-, \mathcal{D}\Sigma]$, where Σ is an ordered PCA (see [**2**, **3**]).

Dense maps between algebras. In case $\phi : \Sigma \to \Theta$ is a dense map between algebras we can "bring down" the geometric morphism $(\phi_\circ, \phi^\circ) : \mathcal{D}\Theta \to \mathcal{D}\Sigma$. More concretely, ϕ then has a right adjoint ϕ_*, defined by

$$\phi_*(u) = \bigvee \phi_\circ \downarrow(u).$$

Now it is easily verified that this gives an adjointness $\phi \dashv \phi_*$. In particular, this shows that $\phi_\circ$ is isomorphic to $\mathcal{D}\phi_*$.

To summarize:

THEOREM 7·8. *Let $\phi : \Sigma \to \Theta$ be a map of algebras. Then ϕ is dense if and only if it has a right adjoint.*

Proof. The preceding discussion shows sufficiency of density. The other direction follows from the fact that $\mathcal{D}$ preserves adjunctions. □

It is also worth noting that every algebra structure map, being a left adjoint, is itself dense.

As a simple illustration look at a map $\phi : \Sigma \to \Theta$, where Σ, Θ are in fact locales and where ϕ preserves meets and arbitrary joins (i.e. is a frame map). Because Σ and Θ only have the identity function realizable, the density condition for ϕ now trivializes. Hence we got back to the well-known fact that ϕ has a right adjoint.

The main consequence of interest of Theorems 7·8 and 6·13 is that we have now reduced the study of realizability triposes and geometric morphisms between them to the study of the category of complete ordered PCAs with a filter and dense maps between them (as mentioned earlier, it is automatic that a morphism of ordered PCAs preserves finite meets).

Structure Maps. When $[-, \Sigma]$ is a tripos, then we write $\mathbf{Set}[\Sigma]$ for the topos constructed out of it (see [**13**, **4**]). It is well-known that localic toposes come equipped with a structural geometric morphism into **Set**, whereas toposes like the Effective Topos contain **Set** as a subtopos. The notion of density will shed some light on these geometric morphisms, and will also provide some characterizations of when a tripos $[-, \Sigma]$ is localic.

First of all, writing $\Omega = \mathcal{D}1$ for the free algebra on one generator, we have the familiar fact that $[-, \Omega]$ is the subobject-tripos, with **Set** as its topos.

Now take an arbitrary BCO Σ and consider the adjunction

$$\begin{tikzcd} \Sigma \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & 1. \arrow[l, bend left, "\top"] \end{tikzcd}$$

The fact that $! : \Sigma \to 1$ has a right adjoint means that it is dense, so we get a geometric morphism (which is in fact an inclusion) $\Omega \to \mathcal{D}\Sigma$. On the level of toposes, this means that $\mathbf{Set}[\mathcal{D}\Sigma]$ has **Set** as a subtopos, thus providing a simple explanation of this for realizability toposes.

Now let Σ be an algebra. When do we have a geometric morphism $\mathbf{Set}[\Sigma] \to \mathbf{Set}$? Well, this would correspond to a dense map $\phi : 1 \to \Sigma$. Because this has to preserve finite limits, we may just as well assume that $\phi = \top$, the top element. Therefore the question reduces to: when is $\top : 1 \to \Sigma$ dense? By definition of density this means that there exists an element $h \in \mathcal{F}_\Sigma$ such that for all $f \in \mathcal{F}_\Sigma$ we have

$$h(\top) \leq f(\top).$$

Remember that elements of the form $f(\top)$ are precisely the designated truth-values of Σ. Therefore, the existence of such h amounts to giving a bottom element of the poset $TV(\Sigma)$. Specializing to triposes, we have:

PROPOSITION 7·9. *Let $[-, \Sigma]$ be a tripos. Then the following are equivalent:*
(i) $[-, \Sigma]$ *is localic;*
(ii) *there is a geometric morphism of toposes* $\mathbf{Set}[\Sigma] \to \mathbf{Set}$*;*
(iii) *there is a geometric morphism of BCOs* $\Sigma \to \Omega$*;*
(iv) *the poset of designated truth-values of* Σ *has a least element;*
(v) $TV(\Sigma)$ *is trivial as an ordered PCA.*

Proof. The equivalence between (1) and (2) is well-known. The equivalence between (2) and (3) follows from theorem 7·8. That (3) is equivalent to (4) has been shown above. Finally, an ordered PCA is by definition trivial if it has a least element, so the equivalence between (4) and (5) is also immediate. □

Of course, if the equivalent conditions of the proposition hold, this does not mean that Σ is itself a locale; Σ is merely equivalent, in the category of BCOs, to a locale. More explicitly, let $\Sigma/TV(\Sigma)$ be the poset obtained from Σ by identifying all designated truth values. Because $TV(\Sigma)$ is upwards closed in Σ and has a least element, this is well-defined. Now make this quotient poset into a BCO by saying that only the identity function is realizable. It is easily verified that there is a retraction of BCOs

$$\begin{tikzcd} \Sigma \arrow[r, bend left, "q"] & \Sigma/TV(\Sigma). \arrow[l, bend left, "i"] \end{tikzcd}$$

To show that this is in fact an equivalence, we have to see that $iq(a) \dashv\vdash a$ for all $a \in \Sigma$. To this end, let ω denote the realizable function represented by the least element of $TV(\Sigma)$.

Clearly, we have $\omega(a) \leq a$ for all $a \in \Sigma$ (in fact, all realizable functions are subidentities). Therefore ω realizes both $a \vdash iq(a)$ and $iq(a) \vdash a$.

Relative Realizability Revisited. As said before, if $[-, \Sigma]$ is a tripos, then Σ is an ordered PCA, $TV(\Sigma)$ is a sub-ordered PCA of Σ, and $[-, \Sigma]$ may be viewed as the relative realizability tripos for the pair $(\Sigma, TV(\Sigma))$. The inclusion of $TV(\Sigma)$ into Σ is easily seen to be dense, and hence there is a geometric morphism $[-, \Sigma] \to [-, TV(\Sigma)]$. In the extreme case that Σ is localic, then this reduces to the structure map.

Fibrations. In section 2 we described the construction of a bicomma object from a map $\phi : \Sigma \to \Theta$. It is not hard to show that the projection $\pi_\Theta : \Theta//\phi \to \Theta$ is a dense map. As an application, take a map $\phi : \Sigma \to \Theta$ (not necessarily dense). This presents Σ as an internal BCO in the category $\mathbf{Set}[\Theta]$. If we assume that Σ and Θ are triposes, then Pitts' iteration theorem says that $\mathbf{Set}[\Theta][\Sigma]$ is again of the form $\mathbf{Set}[\Gamma]$ for some tripos Γ. The point is now that the construction of Γ is functorial in ϕ, and that Γ may be taken to be $\Theta//\phi$. Moreover the density of the projection π_Θ explains the structure map $\mathbf{Set}[\Theta] \to \mathbf{Set}[\Theta//\phi]$.

8. *Further Thoughts*

The work presented here should be thought of as a first and tentative step towards a framework for realizability. Although we have shown that many notions of realizability fit in quite naturally, we cannot exclude the possibility that refinements will be needed to incorporate others. In particular, we have not explained how to deal with topologies, i.e. with subtriposes.

In my opinion, there are several lines that are worthy of further exploration. First, the category **BCO** should be investigated in more detail. It would be nice to know which 2-categorical limits and colimits exist, so that we can carry this over to the topos-level. Furthermore, one would hope for structural and classificatory results describing arbitrary BCOs in terms of (co)limits of BCOs of certain types, thereby obtaining structure theorems for realizability toposes. A useful type of structure theorem would explain the interaction between the "localic" and the "combinatorial" parts of a BCO.

Secondly, there is a possible generalization of the notion of basic combinatorial object where the realizable functions are replaced by relations. This will encompass all triposes over **Set**. The main problem here is to prove the generalization of the tripos characterization theorems. It is not unthinkable, however, that the ordered PCAs which play an essential role in those theorems could be replaced by combinatorial structures where the application is many-valued.

Thirdly, it may be worthwile to see whether certain constructions on complete suplattices can be generalized to our setting. In particular, it would be interesting to see if there is a good tensor product on the category of $\mathcal{D}$-algebras.

Fourthly, since all results in this paper are constructive or need only very mild assumptions on the base topos, it would be useful to obtain concrete presentations of the category of BCOs in a topos of the form $\mathbf{Set}[\Sigma]$, extending the well-known description of $Loc(Sh(\Sigma))$ where Σ is a locale (see [**7**]). Combining this with the bicomma construction, this would make for a conceptual account of Pitts' iteration theorem as well as a better understanding of "change of base" for realizability toposes.

Finally, it would be nice to know whether there is a factorization system on the category **BCO** involving the dense maps. The factorization system involving the pseudo fibrations comes close but is not quite what we need: every pseudo fibration is dense, but not vice versa.

Acknowledgements. Since part of this work originated while I was doing my Ph. D. research, I wish to thank my supervisor J. van Oosten for various helpful and stimulating discussions on the subject. My gratitude also goes to R. Blute, P. Scott, P. Selinger and M. Weber for their interest and useful suggestions.

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