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Realizability with a local operator of A.M. Pitts

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Dedicated to Glynn Winskel on the occasion of his 60th birthday

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ABSTRACT

We study a notion of realizability with a local operator $\mathcal{J}$ which was first considered by A.M. Pitts in his thesis [7]. Using the Suslin–Kleene theorem, we show that the representable functions for this realizability are exactly the hyperarithmetical (Δ^1_1) functions.

We show that there is a realizability interpretation of nonstandard arithmetic, which, despite its classical character, lives in a very non-classical universe, where the Uniformity Principle holds and König's Lemma fails. We conjecture that the local operator gives a useful indexing of the hyperarithmetical functions.

This paper is dedicated to Glynn Winskel. In the years 1994 and 1995, I worked under Glynn at Aarhus University, in the project BRICS (Basic Research In Computer Science). Supported by the Danish government, Glynn collected around him the most amazing group of people I have ever seen. Among the post-docs I remember were Dany Breslauer, Gian-Luca Cattani, Devdatt Dubhashi, Claudio Hermida, Ulrich Kohlenbach, Søren Riis, Vladimiro Sassone, Sergei Soloviev, Igor Walukiewicz. Although working on hugely different topics, we felt we were a very special group; a bit like Lars Iyer's *Essex postgraduates*.[1]

Glynn managed to create a fantastic atmosphere of academic freedom, broad-mindedness, tolerance, cosmopolitanism. I learned from him what a *semantics for a programming language* is; I learned what *concurrency* is, and a *model for concurrency*. But most of all I admired a great character and a great fighter for *abstract*, *pure* science.

0. Introduction

This short note collects a few results from an analysis of a notion of realizability with a local operator first identified by A.M. Pitts. The notions 'local operator' and the realizability to which it gives rise are defined in Sections 1 and 2, respectively. We refer to this as '$\mathcal{J}$-realizability'.

This $\mathcal{J}$-realizability was studied in [2] where it was established that all arithmetical functions are '$\mathcal{J}$-representable' (again, for a definition see Section 2). Here we sharpen this result and characterize the $\mathcal{J}$-representable functions as exactly the hyperarithmetical (Δ^1_1) functions.

We show that there is a $\mathcal{J}$-realizability interpretation of nonstandard arithmetic. This is in sharp contrast to ordinary Kleene realizability, where there cannot even exist a nonstandard model of intuitionistic $I\Sigma_1$: see [3,11].

Despite these 'classical' features of $\mathcal{J}$-realizability, it forms part of a very non-classical universe, in which for example the Uniformity Principle holds and König's Lemma fails.

We conjecture that Pitts' local operator gives a neat indexing of the hyperarithmetical functions, which could be fruitful in developing 'recursion theory with hyperarithmetical functions' (a topic touched upon in Chapter 16 of the classic [8]).

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[1] Lars Iyer, *Exodus*.

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The paper starts out as concretely as possible, in an effort to be accessible to any reader who is familiar with realizability and recursion theory. More general and conceptual, topos-theoretic comments are therefore relegated to a final section, which can be skipped without detriment to the reader's understanding of the technical material presented before.

1. Notation and preliminaries

We assume a recursive coding of finite sequences; the code of a sequence $\sigma = (a_0, \ldots, a_{n-1})$ is written $\langle a_0, \ldots, a_{n-1}\rangle$; we have a recursive function lh giving the length of a coded sequence, and recursive projections $(\cdot)_i$, such that the following equations hold:

$$\big(\langle a_0, \ldots, a_{n-1}\rangle\big)_i = a_i \quad 0 \leqslant i < n$$
$$\big\langle (s)_0, \ldots, (s)_{\mathsf{lh}(s)-1}\big\rangle = s$$

For subsets A, B of $\mathbb{N}$ we write $A \to B$ for the set of indices of partial recursive functions which map A into B (and in particular, are defined on every element of A). We write $A \wedge B$ for the set $\{\langle a, b\rangle \mid a \in A,\ b \in B\}$.

The partial recursive function with index e is denoted ϕ_e. We employ λ-notation: the expression $\lambda x.t$ denotes a standard index (obtained by the S-m-n theorem) for the (partial) function $x \mapsto t$.

Definition 1.1. A function $\mathcal{F} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is *monotone* if the set

$$\bigcap_{A,B\subseteq\mathbb{N}} (A \to B) \to (\mathcal{F}A \to \mathcal{F}B)$$

is nonempty.

The set of monotone functions is preordered as follows: we write $\mathcal{F} \leqslant \mathcal{G}$ if the set $\bigcap_{A\subseteq\mathbb{N}} \mathcal{F}A \to \mathcal{G}A$ is nonempty.

Remark 1.2. The use of 'monotone' in Definition 1.1 seems nonstandard. In the context of realizability however, the set $A \to B$ is thought of as the set of realizers for the *implication* $A \to B$; and a monotone function in our sense is a function which preserves the implication ordering.

Definition 1.3. A function $\mathcal{J} : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a *local operator* if the following sets are nonempty:

$$E_1(\mathcal{J}) = \bigcap_{A,B\subseteq\mathbb{N}} (A \to B) \to (\mathcal{J}A \to \mathcal{J}B)$$
$$E_2(\mathcal{J}) = \bigcap_{A\subseteq\mathbb{N}} A \to \mathcal{J}A$$
$$E_3(\mathcal{J}) = \bigcap_{A\subseteq\mathbb{N}} \mathcal{J}\mathcal{J}A \to \mathcal{J}A$$

So, every local operator is a monotone function. Examples of local operators are: the function which maps every set to $\mathbb{N}$ (the *trivial* local operator) and the function which maps $\emptyset$ to $\emptyset$ and every nonempty set to $\mathbb{N}$ (the $\neg\neg$-operator).

It is left to the reader to verify that from elements of $E_1(\mathcal{J}), E_2(\mathcal{J}), E_3(\mathcal{J})$ we can recursively obtain an element of

$$E_4(\mathcal{J}) = \bigcap_{A,B\subseteq\mathbb{N}} \mathcal{J}A \wedge \mathcal{J}B \to \mathcal{J}(A \wedge B)$$

The following theorem was proved in [7] and [1].

Theorem 1.4 *(Hyland–Pitts). For any monotone function $\mathcal{F}$ there is a least (w.r.t. the preorder on monotone functions) local operator $L(\mathcal{F})$ with the property that $\mathcal{F} \leqslant L(\mathcal{F})$.*

An explicit formula for $L(\mathcal{F})$ is

$$L(\mathcal{F})A = \bigcap \big\{ B \subseteq \mathbb{N} \,\big|\, \{0\} \wedge A \subseteq B \text{ and } \{1\} \wedge \mathcal{F}B \subseteq B \big\}$$

For more on local operators, the reader is referred to [2].

In this paper, we shall deal with only one monotone function $\mathcal{F}$ and its associated local operator $L(\mathcal{F})$. This function was defined by A.M. Pitts in [7]:

$$\mathcal{F}A = \bigcup_{n\in\mathbb{N}} ({\uparrow}n \to A)$$

where ${\uparrow}n$ is short for $\{m \in \mathbb{N} \mid n \leqslant m\}$. Henceforth we write $\mathcal{J}$ for this $L(\mathcal{F})$.

Pitts proved the following facts:

Lemma 1.5.

i) $\mathcal{J}\emptyset = \emptyset$
ii) $\mathcal{J}\{0\} \cap \mathcal{J}\{1\} = \emptyset$
iii) $\mathcal{J}$ *preserves inclusions.*

From items i) and ii) it follows that $\mathcal{J}$ is not the $\neg\neg$-operator.
We reserve the letters a, b, c, d, e for chosen elements of the following sets:

$$\mathsf{a} \in \bigcap_{A \subseteq \mathbb{N}} A \to \mathcal{J}A$$

$$\mathsf{b} \in \bigcap_{A,B \subseteq \mathbb{N}} (A \to B) \to (\mathcal{J}A \to \mathcal{J}B)$$

$$\mathsf{c} \in \bigcap_{A \subseteq \mathbb{N}} \mathcal{F}A \to \mathcal{J}A$$

$$\mathsf{d} \in \bigcap_{A \subseteq \mathbb{N}} \mathcal{J}\mathcal{J}A \to \mathcal{J}A$$

$$\mathsf{e} \in \bigcap_{A,B \subseteq \mathbb{N}} \mathcal{J}A \wedge \mathcal{J}B \to \mathcal{J}(A \wedge B)$$

The following lemma was proved in [2].

Lemma 1.6. *For any total recursive function F there is a partial recursive function G (an index for which can be obtained recursively in an index for F), such that for every coded sequence $s = \langle a_0, \dots, a_{n-1}\rangle$ and every n-tuple $x_0, \dots, x_{n-1}$ such that $x_0 \in \mathcal{J}\{a_0\}, \dots, x_{n-1} \in \mathcal{J}\{a_{n-1}\}$, we have*

$$G\big(\langle x_0, \dots, x_{n-1}\rangle\big) \in \mathcal{J}\big\{F(s)\big\}$$

The following corollary is easy, and just stated for easy reference:

Corollary 1.7. *There are partial recursive functions G and H such that for $x_0 \in \mathcal{J}\{a_0\}, \dots, x_{n-1} \in \mathcal{J}\{a_{n-1}\}$ we have*

$$\begin{array}{ll} G\big(\langle x_0, \dots, x_{n-1}\rangle\big) \in \mathcal{J}\{0\} & \text{if for some } i < n,\ a_i = 0 \\ G\big(\langle x_0, \dots, x_{n-1}\rangle\big) \in \mathcal{J}\{1\} & \text{otherwise} \\ H\big(\langle x_0, \dots, x_{n-1}\rangle\big) \in \mathcal{J}\{i\} & \text{if } i < n \text{ is least such that } a_i = 0 \\ H\big(\langle x_0, \dots, x_{n-1}\rangle\big) \in \mathcal{J}\{n\} & \text{if there is no such } i < n \end{array}$$

2. $\mathcal{J}$-assemblies and $\mathcal{J}$-realizability

The category of $\mathcal{J}$*-assemblies* has as objects pairs (X, E) where X is a set and E a function which assigns to every $x \in X$ a nonempty set $E(x) \subseteq \mathbb{N}$. A *morphism* of $\mathcal{J}$-assemblies $(X, E) \to (Y, F)$ is a function $f : X \to Y$ such that the set

$$\bigcap_{x \in X} E(x) \to \mathcal{J}F\big(f(x)\big)$$

is nonempty; any element of this set is said to *track* the function f.

Morphisms can be composed: given $f : (X, E) \to (Y, F)$ and $g : (Y, F) \to (Z, G)$, tracked by n and m respectively, then

$$\lambda v.\phi_{\mathsf{d}}\big(\phi_{\phi_{\mathsf{b}}(m)}\big(\phi_n(v)\big)\big)$$

tracks gf, as is easy to check.

The category of $\mathcal{J}$-assemblies is cartesian closed: the product of $\mathcal{J}$-assemblies (X, E) and (Y, F) can be given as $(X \times Y, G)$ where $G(x, y) = E(x) \wedge F(y)$. The exponent $(Y, F)^{(X,E)}$ has as underlying set the set of morphisms from (X, E) to (Y, F); and assigns to such a morphism the set of its trackings. Moreover, the category has a *natural numbers object*: the object $N = (\mathbb{N}, E)$ with $E(n) = \{n\}$.

For a $\mathcal{J}$-assembly (X, E), a *subobject* is given by a function $R : X \to \mathcal{P}(\mathbb{N})$ such that the set $\bigcap_{x \in X} R(x) \to \mathcal{J}E(x)$ is nonempty; this data determines a $\mathcal{J}$-assembly (X', R), where $X' = \{x \in X \mid R(x) \neq \emptyset\}$, and a monomorphism $(X', R) \to (X, E)$.

The category of $\mathcal{J}$-assemblies can be used for interpreting first-order logic. Suppose we have a first-order language with function symbols and relation symbols. Let (X, E) be a $\mathcal{J}$-assembly; assume that n-ary function symbols f of the language are interpreted as morphisms $[f]: (X, E)^n \to (X, E)$, and n-ary relation symbols R by subobjects $[R]$ of $(X, E)^n$ (thought of as maps $[R]: X^n \to \mathcal{P}(\mathbb{N})$).

We now define, for a formula $\varphi(v_1, \dots, v_n)$ of the language and elements $x_1, \dots, x_n$ of X, what it means that a natural number e *$\mathcal{J}$-realizes* $\phi(x_1, \dots, x_n)$:

- e $\mathcal{J}$-realizes $t = s(\vec{x})$ iff $e \in \mathcal{J}E(x_1) \wedge \cdots \wedge \mathcal{J}E(x_n)$ and $[t](\vec{x}) = [s](\vec{x})$
- e $\mathcal{J}$-realizes $R(\vec{x})$ iff $e \in [R](\vec{x})$
- e $\mathcal{J}$-realizes $(\phi \wedge \psi)(\vec{x})$ iff $(e)_0$ $\mathcal{J}$-realizes $\phi(\vec{x})$ and $(e)_1$ $\mathcal{J}$-realizes $\psi(\vec{x})$
- e $\mathcal{J}$-realizes $(\phi \vee \psi)(\vec{x})$ iff either $(e)_0 = 0$ and $(e)_1$ $\mathcal{J}$-realizes $\phi(\vec{x})$, or $(e)_0 \neq 0$ and $(e)_1$ $\mathcal{J}$-realizes $\psi(\vec{x})$
- e $\mathcal{J}$-realizes $(\phi \to \psi)(\vec{x})$ iff $(e)_0 \in \mathcal{J}E(x_1) \wedge \cdots \wedge \mathcal{J}E(x_n)$ and for all m such that m $\mathcal{J}$-realizes $\phi(\vec{x})$, $\phi_{(e)_1}(m)$ is defined and is an element of $\mathcal{J}\{k \mid k \ \mathcal{J}\text{-realizes } \psi(\vec{x})\}$
- e $\mathcal{J}$-realizes $\exists x \phi(\vec{x})$ iff for some $a \in X$, $(e)_0 \in \mathcal{J}E(a)$ and $(e)_1$ $\mathcal{J}$-realizes $\phi(a, \vec{x})$
- e $\mathcal{J}$-realizes $\forall x \phi(\vec{x})$ iff $(e)_0 \in \mathcal{J}E(x_1) \wedge \cdots \wedge \mathcal{J}E(x_n)$ and for all $y \in X$ and all $k \in E(y)$, $\phi_{(e)_1}(k)$ is defined and is an element of $\mathcal{J}\{m \mid m \ \mathcal{J}\text{-realizes } \phi(y, \vec{x})\}$

Finally we say that a *sentence* (a formula without free variables) is *true* if it has a $\mathcal{J}$-realizer. This gives a semantics for which intuitionistic first-order logic is sound.

In particular, this can be applied to the natural numbers object N and the language of arithmetic. It was proved in [2] that an arithmetical sentence is true under $\mathcal{J}$-realizability (i.e., has a $\mathcal{J}$-realizer) precisely if it is classically true.

This theorem was based on considering *$\mathcal{J}$-decidable* subsets of $\mathbb{N}$, and *$\mathcal{J}$-representable* functions $\mathbb{N} \to \mathbb{N}$.

Definition 2.1. A subset $A \subseteq \mathbb{N}$ is called *$\mathcal{J}$-decidable* if there is a total recursive function F such that $F(n) \in \mathcal{J}\{0\}$ if $n \in A$, and $F(n) \in \mathcal{J}\{1\}$ if $n \notin A$.

A function $f : \mathbb{N} \to \mathbb{N}$ is *$\mathcal{J}$-representable* if there is a total recursive function F such that for all $n \in \mathbb{N}$, $F(n) \in \mathcal{J}\{f(n)\}$.

In [2] it was shown that every arithmetical subset of $\mathbb{N}$ is $\mathcal{J}$-decidable; the following theorem sharpens this result.

Recall that a subset A of $\mathbb{N}$ is Π^1_1 if it can be defined in the language of second-order arithmetic by a formula $A = \{x \mid \forall X \psi(X, x)\}$ where $\forall X$ is the only second-order quantifier in $\forall X \psi(X, x)$. A set is Σ^1_1 if its complement is Π^1_1; and a set is *hyperarithmetical* or Δ^1_1, if it is both Π^1_1 and Σ^1_1. A function $f : \mathbb{N} \to \mathbb{N}$ is hyperarithmetical if

$$\mathrm{graph}(f) = \{\langle n, f(n)\rangle \mid n \in \mathbb{N}\}$$

is a hyperarithmetical set.

Theorem 2.2. *The $\mathcal{J}$-decidable sets are precisely the hyperarithmetical sets, and the $\mathcal{J}$-representable functions are precisely the hyperarithmetical functions.*

Proof. Recall that $\mathcal{F}A = \bigcup_{n \in \mathbb{N}} \uparrow n \to A$, so $\mathcal{F}A$ is defined by an arithmetical formula in A. By the explicit formula for $\mathcal{J} = L(\mathcal{F})$ given in Theorem 1.4, we see that $\mathcal{J}A$ is defined by a formula

$$\mathcal{J}A = \{x \mid \forall B(\psi(A, B) \to x \in B)\}$$

with $\psi(A, B)$ arithmetical in A. It follows that if A is arithmetical, then $\mathcal{J}A$ is a Π^1_1-set. In particular, $\mathcal{J}\{0\}$ is Π^1_1. Hence, if $A \subseteq \mathbb{N}$ is $\mathcal{J}$-decided by the recursive function F in the sense of Definition 2.1, then $A = F^{-1}(\mathcal{J}\{0\})$, so also Π^1_1. Since the complement of A is $F^{-1}(\mathcal{J}\{1\})$ hence also Π^1_1, it follows that A is hyperarithmetical.

For the converse, in order to show that every hyperarithmetical set is $\mathcal{J}$-decidable, we consider the set

$$C = \{e \mid \phi_e \text{ is total and for all } n \in \mathbb{N},\ \phi_e(n) \in \mathcal{J}\{0\} \cup \mathcal{J}\{1\}\}$$

and the C-indexed collection of subsets of $\mathbb{N}$:

$$C_e = (\phi_e)^{-1}(\mathcal{J}\{0\})$$

Recall from Lemma 1.5 that $\mathcal{J}\{0\} \cap \mathcal{J}\{1\} = \emptyset$, so the collection $\{C_e \mid e \in C\}$ consists precisely of the $\mathcal{J}$-decidable sets. We need to show that it contains all Δ^1_1-sets.

This, in fact, is a straightforward application of the Suslin–Kleene theorem (see [6,5]). We have to check that our collection $\{C_e \mid e \in C\}$ is a so-called *SK-class* [6] or an *effective σ-ring* [5]. This means that we must exhibit partial recursive functions τ_1, τ_2 and σ for which the following hold:

i) For all n, $\tau_1(n)$ is defined and $C_{\tau_1(n)} = \{n\}$

ii) For all $e \in C$, $\tau_2(e)$ is defined and $C_{\tau_2(e)} = \mathbb{N} - C_e$
iii) For every e such that ϕ_e is total and ϕ_e takes values in C, $\sigma(e)$ is defined and

$$C_{\sigma(e)} = \bigcup_{n \in \mathbb{N}} C_{\phi_e(n)}$$

The Suslin–Kleene theorem asserts that there is an indexing $\{G_x \mid x \in G\}$ of the Δ^1_1-sets, which is the minimal SK-class (in an effective sense, which need not concern us here). So if we have proved i)–iii), it follows that $\{C_e \mid e \in C\}$ contains all the Δ^1_1-sets.

For i) let

$$\chi_n(x) = \begin{cases} 0 & \text{if } x = n \\ 1 & \text{otherwise} \end{cases} \quad \text{and let} \quad \tau_1(n) = \lambda x.\phi_{\mathsf{a}}\big(\chi_n(x)\big)$$

For ii) let c be such that $\phi_c(0) = 1$ and $\phi_c(1) = 0$. Let $\tau_2(e) = \lambda x.\phi_{\phi_{\mathsf{b}}(c)}(\phi_e(x))$.

For iii) let G be a recursive function as in Corollary 1.7. Now if ϕ_e is total and takes values in C, and $x \in \mathbb{N}$ is arbitrary, we have:

if $x \in \bigcup_{n \in \mathbb{N}} C_{\phi_e(n)}$ then

$$G\big(\langle \phi_{\phi_e(0)}(x), \ldots, \phi_{\phi_e(n)}(x)\rangle\big) \in \mathcal{J}\{0\}$$

for n large enough;
if $x \notin \bigcup_{n \in \mathbb{N}} C_{\phi_e(n)}$ then

$$G\big(\langle \phi_{\phi_e(0)}(x), \ldots, \phi_{\phi_e(n)}(x)\rangle\big) \in \mathcal{J}\{1\}$$

always.

So if

$$\chi(x) = \left\{\begin{array}{ll} 0 & \text{if } x \in \bigcup_n C_{\phi_e(n)} \\ 1 & \text{else} \end{array}\right\} \quad \text{and} \quad \psi(e,x) = \lambda n.G\big(\langle \phi_{\phi_e(0)}(x), \ldots, \phi_{\phi_e(n)}(x)\rangle\big)$$

then $\psi(e,x) \in \mathcal{F}\mathcal{J}\{\chi(x)\}$. So, let $\sigma(e) = \lambda x.\phi_{\mathsf{d}}(\phi_{\mathsf{c}}(\psi(e,x)))$.

For the statement about the $\mathcal{J}$-representable functions: clearly, if f is $\mathcal{J}$-representable then graph(f) is a $\mathcal{J}$-decidable subset of $\mathbb{N}$, hence hyperarithmetical by the first part of the proof. Conversely, if graph(f) is $\mathcal{J}$-decidable we can find an index for a function which $\mathcal{J}$-represents f by using the function H from Corollary 1.7 in a way similar to what we have done in the first part, since $f(x)$ is the least y such that $\langle x, y\rangle \in \text{graph}(f)$. □

3. A $\mathcal{J}$-realizability interpretation of nonstandard arithmetic

In [10], the first nonstandard model of Peano Arithmetic was constructed. Since the construction does not appear to be well-known and because elements of it are essential for what follows, we outline it here.

Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all arithmetical functions $\mathbb{N} \to \mathbb{N}$. We construct a strictly increasing function ψ such that for all $i, j \in \mathbb{N}$ we have one of three possibilities: $\alpha_i\psi(n) < \alpha_j\psi(n)$ for almost all n, or $\alpha_i\psi(n) = \alpha_j\psi(n)$ for almost all n, or $\alpha_i\psi(n) > \alpha_j\psi(n)$ for almost all n.

In order to achieve this, one constructs a sequence $A_0 \supset A_1 \supset \cdots$ of infinite sets; each A_k must have the property that for all $i, j \leqslant k$, $\alpha_i < \alpha_j$ on A_k or $\alpha_i = \alpha_j$ on A_k or $\alpha_i > \alpha_j$ on A_k. This is done as follows: let $A_0 = \mathbb{N}$. Suppose inductively, that A_k has been constructed and has the required property. Suppose that the restrictions of $\alpha_0, \ldots, \alpha_k$ to A_k are ordered as $\beta_1 < \cdots < \beta_l$. Now A_k can be written as a finite union

$$\begin{aligned} A_k = {} & \big\{x \in A_k \bigm| \alpha_{k+1}(x) < \beta_1(x)\big\} \\ & \cup \big\{x \in A_k \bigm| \alpha_{k+1}(x) = \beta_1(x)\big\} \\ & \cup \big\{x \in A_k \bigm| \beta_1(x) < \alpha_{k+1}(x) < \beta_2(x)\big\} \\ & \cup \cdots \\ & \cup \big\{x \in A_k \bigm| \beta_l(x) < \alpha_{k+1}(x)\big\} \end{aligned}$$

Let A_{k+1} be the first set in this list which is infinite. This completes the construction of the sequence $A_0 \supset A_1 \supset \cdots$.

Finally let ψ be defined by: $\psi(0) = 0$ and $\psi(k+1)$ is the least element of A_{k+1} which is $> \psi(k)$.

The underlying set of Skolem's model is the set $\mathcal{N}$ of equivalence classes of arithmetical functions, where two such functions α and β are equivalent if $\alpha\psi(n) = \beta\psi(n)$ for n large enough. We have an embedding $\iota : \mathbb{N} \to \mathcal{N}$ which sends n

to (the equivalence class of) the constant function with value n. We can extend an arithmetical function $\alpha : \mathbb{N} \to \mathbb{N}$ to $\mathcal{N}$ by putting $\alpha([\beta]) = [\alpha\beta]$; this is well-defined on equivalence classes, so $\mathcal{N}$ is a structure for the language of arithmetic; and ι is an elementary embedding since we can prove for any formula $\varphi(v_1, \dots, v_n)$ in the language of arithmetic and any n-tuple $[\beta_1], \dots, [\beta_n]$ of elements of $\mathcal{N}$, that $\mathcal{N} \models \varphi([\beta_1], \dots, [\beta(n)])$ if and only if $\mathbb{N} \models \varphi(\beta_1\psi(k), \dots, \beta_n\psi(k))$ for almost all k.

Now it is not hard to see that the whole construction, which needs an enumeration of all arithmetical functions and checking whether or not an arithmetical set is infinite, can be done recursively in a truth function for arithmetic, which is hyperarithmetical (see, e.g., [8, 16-XI]). Therefore, the function ψ can be assumed to be $\mathcal{J}$-representable.

We can now endow the set $\mathcal{N}$ with the structure of a $\mathcal{J}$-assembly, by putting

$$E\big([\alpha]\big) = \big\{e \bigm| \text{for some } \beta \in [\alpha],\ e\ \mathcal{J}\text{-represents } \beta\psi\big\}$$

For any arithmetical β, the map $[\alpha] \to [\beta\alpha]$ is well-defined and tracked, so the $\mathcal{J}$-assembly $\mathcal{N}$ is also a structure for the language of arithmetic. And again, we have an embedding $i : N \to \mathcal{N}$ of $\mathcal{J}$-assemblies, which is just ι on the level of sets.

By a straightforward application of the proof method in [2] for the theorem that the $\mathcal{J}$-realizable sentences of arithmetic are exactly the classically true ones, one now obtains the following theorem.

Theorem 3.1. *The map i is an elementary embedding. For a formula $\varphi(v_1, \dots, v_n)$ and numbers $a_1, \dots, a_n$ the following four assertions are equivalent:*

i) *$\varphi(a_1, \dots, a_n)$ is true in the classical model $\mathbb{N}$*
ii) *$\varphi(a_1, \dots, a_n)$ has a $\mathcal{J}$-realizer (in the sense of the assembly N)*
iii) *$\varphi(i(a_1), \dots, i(a_n))$ has a $\mathcal{J}$-realizer (in the sense of the assembly $\mathcal{N}$)*
iv) *$\varphi(i(a_1), \dots, i(a_n))$ is true in the classical model $\mathcal{N}$*

Moreover, the equivalence ii) $\Leftrightarrow$ iii) is effective in realizers.

If $\alpha_1, \dots, \alpha_k$ are arithmetical functions then the following are equivalent:

i) *$\varphi([\alpha_1], \dots, [\alpha_n])$ is true in the classical model $\mathcal{N}$*
ii) *$\varphi([\alpha_1], \dots, [\alpha_n])$ has a $\mathcal{J}$-realizer*
iii) *$\varphi(\alpha_1\psi(k), \dots, \alpha_n\psi(k))$ is true in $\mathbb{N}$ for almost all k*

The model $\mathcal{N}$ is in fact very classical: let St (the subobject of standard numbers) denote the image of $i : N \to \mathcal{N}$. Since the condition 'α is bounded' is arithmetical in α, we have:

Proposition 3.2. *The statement $\forall y(\mathrm{St}(y) \vee \neg\mathrm{St}(y))$ has a $\mathcal{J}$-realizer.*

Nevertheless, the universe of $\mathcal{J}$-assemblies also has non-classical features. Just like in the category of ordinary assemblies, König's Lemma fails, and Cantor space and Baire space are isomorphic:

Proposition 3.3. *In the category of $\mathcal{J}$-assemblies, the objects 2^N and N^N are isomorphic. Hence, König's Lemma fails: there is a continuous but unbounded function $2^N \to N$.*

Proof. This follows from the result (see [8, Corollary 16-XLI(b)]) that, analogous to the ordinary Kleene tree, there is a recursive, finitely-branching, infinite tree which has no infinite hyperarithmetical branch. The stated isomorphism now follows in a way similar to [12, 3.2.26] (see also [1, 13.1–4]). $\square$

4. General comments and further work

Just as ordinary Kleene realizability is the standard notion of truth in an elementary topos, the *effective topos* $\mathcal{E}ff$ of J.M.E. Hyland [1], $\mathcal{J}$-realizability is the standard notion of truth in a topos, a *subtopos* of the effective topos. Let us denote this subtopos by $\mathcal{E}ff_{\mathcal{J}}$. The topos $\mathcal{E}ff_{\mathcal{J}}$ shares some features with $\mathcal{E}ff$: it is the free exact completion over the regular category of $\mathcal{J}$-assemblies. Every object is covered by a $\mathcal{J}$-assembly. The subobject classifier Ω is the object $(\mathcal{P}(\mathbb{N}), =)$ where $[A = B]$ is the set $(A \to_{\mathcal{J}} B) \wedge (B \to_{\mathcal{J}} A)$. It is immediate that $\langle \mathsf{a}, \mathsf{a}\rangle$ is an element of $[A = A]$ for all A, and this implies that the *Uniformity Principle* holds:

Proposition 4.1. *For any object X which is a subquotient of N, the natural map $X \to X^{\Omega}$ is an isomorphism. In particular, this holds for the objects N, N^N, $\mathcal{N}$ and $\mathcal{N}^{\mathcal{N}}$.*

Analogies between $\mathcal{E}\!f\!f_{\mathcal{J}}$ and $\mathcal{E}\!f\!f$ can also be drawn on the basis of an analysis of the (partial) hyperarithmetical functions and the indexing to which the local operator $\mathcal{J}$ gives rise: write $F = \psi_e$ if for every n: $n \in \mathrm{dom}(F)$ if and only if $\phi_e(n) \in \mathcal{J}\{m\}$ for some (necessarily unique) m, and $\phi_e(n) \in \mathcal{J}\{F(n)\}$ if $n \in \mathrm{dom}(F)$.

One sees that $\mathrm{dom}(\psi_e)$ is a Π^1_1-set; this is in accordance with the philosophy of 'recursion theory with hyperarithmetical functions', that if the latter are analogous to recursive functions, the analogues of r.e. sets are the Π^1_1-sets [8, p. 402].

We conjecture that (a subcollection of) the Π^1_1-sets form a *dominance* in $\mathcal{E}\!f\!f_{\mathcal{J}}$ [9,12] and that there is a model of Synthetic Domain Theory in this topos.

Finally, let us remark that the nonstandard model given here, should be compared with the model defined in [4, Section 3]. In both cases it is a model in a sheaf topos over $\mathcal{E}\!f\!f$, and there is an obvious similarity between (the monotone function generating) Pitts' local operator and the Fréchet filter in $\mathcal{E}\!f\!f$. But our Proposition 3.2 contrasts with what Moerdijk claims to hold in his model (Proposition 3.1 in [4]).

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