Well-foundedness in Realizability

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Introduction

Let $<$ be a binary relation on a set X. In ZFC, the following three statements are equivalent:

1) There are no infinite $<$-descending sequences in X: i.e. no sequences $(x_n)_{n \in \mathbb{N}}$ such that $x_{n+1} < x_n$ for all n.

2) Every subset Y of X that contains an element, contains an $<$-minimal element.

3) $(X, <)$ satisfies the principle of induction over $<$: if $Y \subseteq X$ is such that for all x, $\forall y < x(y \in Y)$ implies $x \in Y$, then $Y = X$.

However, the implication from 1) to 2) is equivalent to the principle of Dependent Choices (as the reader can show for himself) and cannot be proved in ZF.

A relation $<$ on X is called *well-founded* if any of these statements holds.

Intuitionistically, the 3 statements are far from equivalent (hence, in formalizing an intuitionistic notion of well-foundedness, care is needed). Notion 1) is, intuitionistically, too weak to be of any use; whereas notion 2) is far too strong (by a well-known argument, as soon as an inhabited relation $<$ satisfies 2), classical logic is forced on us). Hence we focus on notion 3), which is also usually taken as part of an axiomatization of intuitionistic set theory.

The behaviour of well-founded induction for primitive recursive well-founded relations on the natural numbers in formal arithmetic, has been studied by many people. Classical results in the area are:

- Kreisel et al ([9]): Peano Arithmetic PA extended with the full induction scheme over such relations is complete, and equivalent to true arithmetic.
- Kreisel ([8]): There is a primitive recursive linear order for which Peano Arithmetic proves the induction *scheme*, yet which is not a well-order.
- Friedman and Scedrov ([2]): Any primitive recursive relation for which HAH (higher order intuitionistic arithmetic) proves the induction scheme, is in fact well-founded (and has ordinal $< \varepsilon_0$ if this induction is already provable in HA).

In connection with realizability it is known ([13]) that the induction scheme over primitive recursive well-founded relations $<$ proves its own realizability (the actual proof, l.c. 3.2.23, seems to need that $<$ is a total order, but this assumption is redundant).

The result discussed in the present note is of a nature similar to Friedman's. In realizability toposes, of which the effective topos $\mathcal{E}\!f\!f$ (see [3] for details) is the best studied example, one has the phenomenon that the global sections functor $\Gamma : \mathcal{E}\!f\!f \to \mathbf{Set}$ *preserves and reflects* well-founded relations.

Let us spell this out in more elementary terms. We have a set X with a realizability relation for equality: for every $x, y \in X$ a set $[x = y]$ of realizers of the equality of x and y is given (subject to a few natural conditions). A binary relation R on X is given by, for each $x, y \in X$, a set $R(x, y)$ of realizers of the relation between x and y (which system has to be compatible with the equality realizers). $\Gamma(X)$ is the quotient set $X'/\sim$ where $X' = \{x \in X \mid [x = x] \neq \emptyset\}$ and $x \sim y$ iff $[x = y] \neq \emptyset$. $\Gamma(R)$ is the binary relation on $\Gamma(X)$ defined by: $([x], [y]) \in \Gamma(R)$ iff $R(x, y) \neq \emptyset$. We then have:

> The binary relation R on X is internally well-founded in $\mathcal{E}\!f\!f$, if and only if $\Gamma(R)$ is a (classically) well-founded relation on $\Gamma(X)$

This result can be formulated entirely in the internal logic of $\mathcal{E}\!f\!f$. Call $Y \subseteq X$ *progressive* if for all $x \in X$, the statement $\forall y(R(y, x) \to y \in Y)$ implies $x \in Y$. Hence, R is well-founded iff X has no nontrivial progressive subsets. Our result has then the following equivalent formulation:

> R is internally well-founded if and only if X has no nontrivial $\neg\neg$-stable progressive subsets

We shall work in a slightly more general context than the one adopted here: by $\mathbf{Set}$, we shall mean an arbitrary topos with natural numbers object in which we have an internal partial combinatory algebra A; instead of $\mathcal{E}\!f\!f$ we shall work in $\mathbf{RT}(A)$, the realizability topos constructed over $\mathbf{Set}$ w.r.t. A. Moreover, we shall (for some applications) need to formulate the above result "in parameters" (see Corollary 1.2).

1 Well-founded relations in realizability toposes

We assume the reader is familiar with the construction of the *Realizability Topos* $\mathbf{RT}(A)$ based on a partial combinatory algebra A. For a treatment of the paradigmatic case, see [3]. We recall that this topos comes equipped with an adjoint pair of functors $\nabla : \mathbf{Set} \to \mathbf{RT}(A)$ and $\Gamma : \mathbf{RT}(A) \to \mathbf{Set}$, making $\mathbf{Set}$ a subtopos of $\mathbf{RT}(A)$. The pair $(\Gamma \dashv \nabla)$ with ∇ full and faithful, is called a *geometric inclusion.* In this situation, the internal logic of $\mathbf{RT}(A)$ comes equipped with a modal operator j which satisfies the axioms:

$$\begin{gathered} p \to j(p) \\ j(j(p)) \to j(p) \\ (p \to q) \to (j(p) \to j(q)) \end{gathered}$$

Such an operator is often called an *internal topology* or a *local operator*. For a good introduction into the logic with such j, consult [1]. Here we just recall a few notions: a subset $A' \subset A$ is j-*dense* if $\forall x : A.j(x \in A')$ holds, and $A' \subseteq A$ is j-*closed* if $\forall x : A\,(j(x \in A') \to x \in A')$ holds. An object X is a j-*sheaf* if for any j-dense $A' \subset A$, any map from A' to X can be uniquely extended to a map from A to X. In the situation above, **Set** is equivalent to the category of j-sheaves in $\mathbf{RT}(A)$ for a unique j. If **Set** is Boolean, j will be the operator $\neg\neg$.

The following definitions make sense in any topos, and are important in Algebraic Set Theory, Synthetic Domain Theory and the study of W-types in toposes, as will become apparent in section 3.

Consider pairs $(X, <)$ where $<$ is a binary relation on the object X. A subobject P of X is called *progressive* with respect to $<$, if

$$\mathrm{Prog}(P) \qquad \forall x : X\,(\forall y : X(y < x \to y \in P) \to x \in P)$$

holds. $(X, <)$ is *well-founded* if X has no nontrivial progressive subobjects w.r.t. $<$, that is:

$$\mathrm{WF} \qquad \forall P : \mathcal{P}(X)(\mathrm{Prog}(P) \to \forall x : X.x \in P)$$

If j is a local operator (internal topology), we denote by $\mathcal{P}_j(X)$ the object of j-closed subobjects of X. We say that $(X, <)$ is j-*well-founded* if X has no nontrivial j-closed progressive subobjects:

$$\mathrm{WF}_j \qquad \forall P \in \mathcal{P}_j(X)(\mathrm{Prog}(P) \to \forall x : X.x \in P)$$

Theorem 1.1 *Let j be the topology in $\mathbf{RT}(A)$ for which* **Set** *is the sheaf subtopos. Then every j-well-founded object is well-founded. Moreover, the sheafification functor $\Gamma : \mathbf{RT}(A) \to$* **Set** *preserves and reflects well-founded objects.*

Proof. Suppose $X = (W, =)$ with $=: W \times W \to \mathcal{P}(A)$. We may suppose that $E(w) = [w = w]$ is inhabited for each $w \in W$, so there is an equivalence relation $\sim$ on W defined by: $w \sim w'$ iff $[w = w']$ is inhabited.

Now $\mathcal{P}_j(X)$ can be represented as $\nabla(\{U \subseteq W \mid U \text{ is closed under } \sim\})$. It is easy to see, that $(X, <)$ is j-well-founded, if for each $U \subseteq W$ which is closed under $\sim$ it holds, that:

> if for all $x \in W$, $\{y \in W \mid [y < x] \text{ is inhabited }\} \subseteq U$ implies $x \in U$, then $U = W$

Note at once that this is equivalent to saying that $(\Gamma X, \Gamma(<))$ is well-founded in **Set**, so the second assertion in the theorem follows from the first.

Now suppose $(X, <)$ is j-well-founded. By the second recursion theorem in the partial combinatory algebra A there is for each $e \in A$ an element r_e, uniformly in e, such that for all $n \in A$,

$$r_e n \simeq en(\Lambda y \Lambda q.r_e y)$$

Now let $P : W \to \mathcal{P}(A)$ represent an element of $\mathcal{P}(X)$. We claim that for each e,

$$\text{if } e \Vdash \mathrm{Prog}(P) \text{ then } r_e \Vdash \forall x : X.P(x)$$

(we write $\Vdash$ for the realizability relation).
To this end assume $e \Vdash \mathrm{Prog}(P)$. Define the following subset U of W:

$$(1)\;\; U \;=\; \begin{array}{l} \{x \in W \mid \forall y \in W\,([y < x] \text{ is inhabited } \Rightarrow \\ \qquad\qquad \forall m \in E(y)\,(r_e m \in P(y)))\} \end{array}$$

Clearly, U is closed under $\sim$.

The assumption $e \Vdash \mathrm{Prog}(P)$ means:

$$\begin{array}{ll}(2) & \forall x \in W \forall n \in E(x)\,(en \text{ is defined and} \\ & \forall b(b \Vdash (\forall y : X.y < x \to y \in P) \Rightarrow enb \Vdash x \in P))\end{array}$$

Now let $x \in W$ and suppose that $\{y \in W \mid [y < x] \text{ is inhabited}\}$ is a subset of U. This means:

$$(3)\quad \begin{array}{l} \forall yv \in W \forall m \in A\,(([y < x] \text{ and } [v < y] \text{ are inhabited and} \\ \qquad\qquad\qquad\qquad m \in E(v)) \Rightarrow r_e m \in P(v)) \end{array}$$

Then for all $y \in W$ with $[y < x]$ inhabited, we have

$$(4)\;\; \Lambda m \Lambda v.r_e m \Vdash \forall v : X.v < y \to v \in P$$

Hence by (2), applied to such y, we have that for each $n \in E(y)$,

$$(5)\;\; en(\Lambda m \Lambda v.r_e m) \Vdash y \in P$$

That is, $r_e n \Vdash y \in P$. Hence, $x \in U$.

From the j-well-foundedness of $(X, <)$ we conclude that $U = W$. So for all $x \in W$ we have

$$\Lambda m \Lambda v.r_e m \Vdash (\forall y : X.y < x \to y \in P)$$

We conclude that for all $n \in E(x)$, en is defined and that

$$en(\Lambda m \Lambda v.r_e m) \Vdash x \in P$$

in other words,

$$r_e n \Vdash x \in P$$

So $r_e \Vdash \forall x : X.x \in P$, as desired. $\square$

The above proof is entirely constructive and holds in the presence of arbitrary parameters. This means that it internalizes in $\mathbf{RT}(A)$ in the following sense. Let $(X, <)$ be as before. Let $\mathcal{P}_{\mathrm{wf}}(X)$ be the object of well-founded subobjects of X (that is: those subsets $Y \subset X$ which are well-founded w.r.t. the restriction of $<$), and $\mathcal{P}_{\mathrm{jwf}}$ the object of j-well-founded subobjects of X.

Corollary 1.2 $\mathcal{P}_{\mathrm{wf}}(X) = \mathcal{P}_{\mathrm{jwf}}(X)$. *In particular,* $\mathcal{P}_{\mathrm{wf}}(X)$ *is a* j*-closed subobject of* $\mathcal{P}(X)$*.*

This is of importance in Algebraic Set Theory and the theory of W-types in topoi (section 3), where one defines the "object of well-founded X-labelled trees".

If the Axiom of Choice holds in **Set**, we can represent $\mathcal{P}_{\mathrm{wf}}(X)$ as the closed subobject of $\mathcal{P}(X)$ on those functions $\alpha : W \to \mathcal{P}(A)$ for which there are no infinite sequences $(x_0, x_1, \ldots)$ in W, such that for each i, both $\alpha(x_i)$ and $[x_{i+1} < x_i]$ are nonempty.

2 A generalized form of Markov's Principle

Let us formulate a consequence of the phenomenon observed in section 1. Suppose $\mathcal{E}$ is a topos and j is an internal topology in $\mathcal{E}$. We say that $\mathcal{E}$ *has property* Q_j if the statement of Corollary 1.2 holds in $\mathcal{E}$:

$$Q_j \qquad \text{for all } (X,<),\ \mathcal{P}_{\mathrm{wf}}(X) = \mathcal{P}_{\mathrm{jwf}}(X)$$

If furthermore $\mathcal{E}$ has a natural numbers object, we say that $\mathcal{E}$ satisfies *Generalized Markov's Principle* (GMP) with respect to j, if for each object X the following internal statement is true:

$$\mathrm{GMP}_j \qquad \begin{array}{c} \forall P, T : \mathcal{P}(X) \forall f : X \to X \\ \forall x : X\, ((\neg T(x) \to P(fx)) \to P(x)) \to \\ \forall x : X\, (j(\exists n : N.T(f^n x)) \to P(x)) \end{array}$$

Markov's Principle (MP) with respect to j is the axiom:

$$\mathrm{MP}_j \quad \forall R : \mathcal{P}(N)\, (\forall n : N.R(n) \vee \neg R(n)) \to (j(\exists n : N.R(n)) \to \exists n : N.R(n))$$

Theorem 2.1 *Let E be a topos with natural numbers object and internal topology j.*

1. GMP_j *follows from property* Q_j.
2. MP_j *follows from* GMP_j.

Proof. For 1), given X, P, T, f, define $<$ on X by: $y < x$ iff $y = f(x) \wedge \neg T(x)$. Then for any subobject P of X, $\mathrm{Prog}(P)$ is equivalent to $\forall x : X\, ((\neg T(x) \to P(fx)) \to P(x))$. So $\mathrm{Prog}(P)$ implies $\forall x : X(T(x) \to P(x))$ and $\forall x : X\, (P(fx) \to P(x))$, and hence $\forall x : X\, (j(\exists n : N.T(f^n x)) \to j(P(x)))$. We see then, that if $X' = \{x : X \,|\, j(\exists n : N.T(f^n x))\}$, $X' \in \mathcal{P}_{\mathrm{jwf}}(X)$ holds: suppose $A \subset X'$ j-closed, progressive. First we prove by induction on n that $T(f^n x)$ implies $x \in A$ (for $n = 0$ this is trivial since $\forall y < x(y \in A)$ holds vacuously, and the induction step is also easy), so we see that $\exists n : N.T(f^n x) \to x \in A$. By the axioms for j we have $\forall x : X\, (j(\exists n : N.T(f^n x)) \to j(x \in A))$. By definition of X' and the assumption that A is j-closed and progressive, $A = X'$ follows.

By Property Q_j, $X' \in \mathcal{P}_{\mathrm{wf}}(X)$, which gives the desired conclusion.

For 2), take N for X, R for T and $\exists k : N.R(n+k)$ for $P(n)$. Let $f : N \to N$ be the successor function. The premiss of GMP_j in this case,

$$\forall n : N\,((\neg R(n) \to \exists k\, R(n+k+1)) \to \exists k\, R(n+k))$$

is easily seen to follow from the decidability of $R(n)$. By GMP_j we conclude

$$\forall n : N\,(j(\exists k\, R(n+k)) \to \exists k\, R(n+k))$$

which, by instantiating 0 for n, gives the conclusion of MP_j. $\square$

Remark. In the context of first-order arithmetic **HA** one can consider the following form of GMP: the axiom scheme

GMP_0 $\quad [\forall n((\neg T(n) \to P(n+1)) \to P(n))] \to \forall n(\neg\neg\exists k T(n+k) \to P(n))$

where T and P are arbitrary formulas. One can also consider the axiom GMP_1 in second-order arithmetic **HAS**, which is the universally quantified form (quantifiers over T and P) of GMP_0. One has the following theorem, the proof of which we leave as an exercise (see also [15]). Here, MP stands for $\mathrm{MP}_{\neg\neg}$ and ShP is *Shanin's Principle* for second-order arithmetic:

ShP $\quad \forall X \exists Y(\forall z(\neg\neg z \in Y \to z \in Y) \wedge \forall x(x \in X \leftrightarrow \exists y\langle x, y\rangle \in Y))$

Theorem 2.2

a) *In* $\mathbf{HA} + \mathrm{MP}$*,* GMP_0 *is realizable.*

b) $\mathbf{HA} + \mathrm{GMP}_0 \vdash \mathrm{MP}$

c) *In* $\mathbf{HAS} + \mathrm{MP} + \mathrm{ShP}$*,* GMP_1 *is realizable.*

3 Applications

In categorical logic, well-foundedness often manifests itself in the form of *initiality* of algebras of a particular type. The fact that an algebra is initial (has a unique algebra homomorphism to any other algebra) is usually proved by recursion along the well-founded relation, but in a number of cases also the converse holds: initiality implies well-foundedness.

In this section we shall briefly discuss three examples of this: the categorical treatment of W-types, the initial Lift-algebras of Synthetic Domain Theory (these are, actually, special W-types), and initial ZF-algebras in Algebraic Set Theory.

3.1 Well-founded trees and W-types

Let us work in the following context: $(\Gamma \dashv \nabla) : \mathcal{F} \to \mathcal{E}$ is a geometric inclusion of toposes; we denote the local operator in $\mathcal{E}$ induced by this inclusion by j. We assume that both $\mathcal{E}$ and $\mathcal{F}$ have natural numbers objects; in the discussion below, the variable n is assumed to run over the natural numbers object.

Let $f : B \to A$ be a morphism in $\mathcal{E}$. A P_f*-algebra* is an object X together with, for each $a \in A$, a function from the set of all $f^{-1}(a)$-tuples $\{x_b \mid f(b) = a\}$ to X; categorically this is expressed as a map

$$\left(\begin{array}{c} X \times A \\ \downarrow \\ A \end{array}\right)^{\left(\begin{array}{c} B \\ \downarrow f \\ A \end{array}\right)} \to \left(\begin{array}{c} X \times A \\ \downarrow \\ A \end{array}\right)$$

in the slice $\mathcal{E}/A$. The W-type of f, $W(f)$, is the initial such P_f-algebra, if it exists.

In [11], it is shown that in a topos, $W(f)$ always exists and may be constructed in the following way.

For any object X, let X^* be the free monoid on X; that is the object of finite sequences of elements of X. An X*-labelled tree* t is a subset of X^* that contains exactly one sequence of length 1, and is such that if $(x_1, \ldots, x_{n+1}) \in t$, then $(x_1, \ldots, x_n) \in t$, for all $n \geq 1$.

Given $f : B \to A$ in $\mathcal{E}$, an f*-tree* is an $(A + B)$-labelled tree t with the following properties:

i) If $(x_1, \ldots, x_n) \in t$ then $x_i \in A$ if i is odd, and $x_i \in B$ if i is even;

ii) if $(x_1, \ldots, x_{2n}) \in t$ then there is a unique $x_{2n+1} \in A$ such that $(x_1, \ldots, x_{2n+1}) \in t$;

iii) if $(x_1, \ldots, x_{2n+1}) \in t$ then $\{x \mid (x_1, \ldots, x_{2n+1}, x) \in t\} = \{b \in B \mid f(b) = x_{2n+1}\}$

Clearly, one can form the object $T(X)$ of X-labelled trees and the object $T(f)$ of f-trees. $W(f)$, the W-type associated to f, is now constructed as the set of *well-founded* f-trees.

Our objective is to calculate W-types in $\mathcal{E}$ from knowledge about $\mathcal{F}$.

Lemma 3.1 *Suppose that the local operator j is dense (that is, $j(\bot) = \bot$) and that A is j-separated. Then $T(f)$ embeds into $\mathcal{P}_j((A + B)^*)$, the object of j-closed subobjects of $(A + B)^*$.*

Proof. We have to show that for $t \in T(f)$ and $(x_1, \ldots, x_n) \in (A + B)^*$, that $j((x_1, \ldots, x_n) \in t)$ implies $(x_1, \ldots, x_n) \in t$. This is done by induction on n. For $n = 1$, let a be the unique element of A such that $(a) \in t$. $j((x_1) \in t)$ is equivalent to $j(x_1 = a)$. This implies $x_1 = a$ since A is j-separated; hence, $(x_1) \in t$ follows.

Suppose $j((x_1,\ldots,x_{n+1})\in t)$. Then $j((x_1,\ldots,x_n)\in t)$ since $t\in T(f)$, so $(x_1,\ldots,x_n)\in t$ by induction hypothesis.

If n is even, there is a unique $a\in A$ such that $(x_1,\ldots,x_n,a)\in t$. Also, we have $j(x_{n+1}\in A)$ because $t\in T(f)$. It follows that $x_{n+1}\in A$ because A is complemented in $A+B$ and j is dense. Therefore $j(x_{n+1}=a)$ and again by separatedness of A, $x_{n+1}=a$ so $(x_1,\ldots,x_{n+1})\in t$.

If n is odd, we have $j(x_{n+1}\in B)$ from which, just as in the previous case, we conclude $x_{n+1}\in B$, and $j(f(x_{n+1})=x_n)$ by $t\in T(f)$. Separatedness of A gives $f(x_{n+1})=x_n$, and $(x_1,\ldots,x_{n+1})\in t$ now follows from $t\in T(f)$. $\square$

The adjunction $\Gamma\dashv\nabla$ carries over to an adjunction between the categories of monoids in $\mathcal{F}$ and $\mathcal{E}$ (this is true for arbitrary geometric morphisms; see e.g. [5]), and Γ preserves free monoids as well as sums, so $\Gamma((A+B)^*)\cong(\Gamma(A)+\Gamma(B))^*$, the free monoid on $\Gamma(A)+\Gamma(B)$ in $\mathcal{F}$. Moreover, we have an isomorphism $\mathcal{P}_j((A+B)^*)\cong\nabla(\mathcal{P}(\Gamma((A+B)^*)))$ (on the RHS, $\mathcal{P}$ denotes the power-object in $\mathcal{F}$). Hence Lemma 3.1 gives an inclusion from $T(f)$ into $\nabla(\mathcal{P}((\Gamma A+\Gamma B)^*))$. It is easy to see that this inclusion factors through $\nabla(T(\Gamma A+\Gamma B))$, the (image of the) object of $\Gamma A+\Gamma B$-labelled trees in $\mathcal{F}$.

Let us specialize to the case where $\mathcal{F}$ is **Set** and $\mathcal{E}$ is $\mathbf{RT}(A)$. The local operator j is dense in this case. We calculate $W(f)$ in the case that A is j-separated, using Corollary 1.2 and Lemma 3.1.

Since $T(f)$ is a subobject of $\nabla(T(\Gamma A+\Gamma B))$ the underlying set of $T(f)$ may be taken as the set of $\Gamma A+\Gamma B$-labelled trees in **Set**. There is an element relation $\in\hookrightarrow(A+B)^*\times T(f)$, realized by:

$$e\Vdash(x_1,\ldots,x_n)\in t \text{ iff } e\Vdash E(x_1,\ldots,x_n) \text{ and } ([x_1],\ldots,[x_n])\in t$$

Here, $E(x_1,\ldots,x_n)$ is short for $(x_1,\ldots,x_n)=(x_1,\ldots,x_n)$ (the equality relation on $(A+B)^*$), and for the second condition t is identified with the corresponding $(\Gamma A+\Gamma B)$-labelled tree.

Then the predicate that singles out $T(f)$ from $\nabla(T(\Gamma A+\Gamma B))$ is the realizability interpretation of the conditions for t being an f-tree:

e realizes $t\in T(f)$ if and only if for all natural numbers $i>0$, $e{\cdot}i$ is defined and:

- there is an $a\in A$ such that $(e{\cdot}1)_0\Vdash a=a$ and

$$(e{\cdot}1)_1\Vdash\forall x:(A+B).(x)\in t\leftrightarrow x=a$$

-

$$e{\cdot}2i\Vdash\left\{\begin{array}{c}\forall(x_1,\ldots,x_{2i}):(A+B)^*.(x_1,\ldots,x_{2i})\in t\to\exists y:(A+B)\\ (y\in A\wedge\forall z:(A+B)((x_1,\ldots,x_{2i},z)\in t\leftrightarrow y=z))\end{array}\right.$$

- $$e{\cdot}(2i+1) \Vdash \left\{ \begin{array}{l} \forall (x_1,\ldots,x_{2i+1}) : (A+B)^*.(x_1,\ldots,x_{2i+1}) \in t \to \\ \forall y : (A+B)((x_1,\ldots,x_{2i+1},y) \in t \leftrightarrow F(y,x_{2i+1})) \end{array} \right.$$

  (where F is the predicate on $A+B$ defining the morphism $f : B \to A$)

$W(f)$, being (by Corollary 1.2) a j-closed subobject of $T(f)$, has the same realizers but the underlying set consists of those $t \in T(\Gamma A + \Gamma B)$ which are well-founded in **Set**.

This simplifies (and generalizes) somewhat the computation of W-types for separated objects, presented in [14].

3.2 The initial Lift algebra

Let us look at a simple application in Synthetic Domain Theory. A *dominance* in a topos $\mathcal{E}$ with natural numbers object ([12]) is a subobject Σ of Ω such that

1. $\top \in \Sigma$, and
2. $\forall p, q : \Omega.(p \in \Sigma \wedge p \to (q \in \Sigma)) \to ((p \wedge q) \in \Sigma)$

both hold in E. Given a dominance Σ we have a *lift functor* L: internally,

$$LX \;=\; \begin{array}{c} \{\alpha \in \mathcal{P}(X) \,|\, \forall xy : X\,(x \in \alpha \wedge y \in \alpha \to x = y) \wedge \\ [\exists x : X.x \in \alpha] \in \Sigma\} \end{array}$$

Note that LX is isomorphic to the set of pairs

$$\{(\sigma, f) \,|\, \sigma \in \Sigma, f : \{* \,|\, \sigma\} \to X\}$$

That is, to the domain of the exponential

$$\left(\begin{array}{c} X \times \Sigma \\ \downarrow \\ \Sigma \end{array} \right)^{\left(\begin{array}{c} 1 \\ \downarrow \top \\ \Sigma \end{array} \right)}$$

in E/Σ.

Hence the *initial* L-algebra, usually denoted I, is the W-type associated to $(1 \stackrel{\top}{\to} \Sigma)$, that is: the set of well-founded trees with nodes labelled by elements of Σ, such that the set of branches out of a node labelled p is in bijective correspondence with the set $\{* \,|\, p\}$.

One can show that the set of all (not just well-founded) such trees is isomorphic to the set F of functions $p : N \to \Sigma$ satisfying $\forall n : N.p_{n+1} \to p_n$.

From property Q_j it follows that I is a j-closed subobject of F. Since it is easy to see that we always have for $p \in F$,

$$\exists n.\neg p_n \to p \in I \to \neg\neg\exists n.\neg p_n$$

we see at once that if j is the local operator $\neg\neg$,

$$I = \{p \in F \mid \neg\neg\exists n.\neg p_n\}$$

This reproves Theorem 3.2 from [16], where this was shown for the Effective Topos. The proof there generalizes to arbitrary $\mathbf{RT}(A)$, but here we see that the result follows axiomatically from property $Q_{\neg\neg}$.

We can refine this a bit and show that it is also a consequence of $\mathrm{GMP}_{\neg\neg}$.

Theorem 3.2 *Let $\mathcal{E}$ be a topos with dominance Σ and natural numbers object. If $\mathcal{E}$ satisfies $\mathrm{GMP}_{\neg\neg}$, then the initial lift algebra I is*

$$\{p \in F \mid \neg\neg\exists n : N.\neg p_n\}$$

In particular if Σ is $\neg\neg$-separated, $I = \{p \in F \mid \neg\forall n : N.p_n\}$.

Proof. M. Jibladze ([4]) has shown that the well-foundedness condition for $p \in F$ can be simplified to:

$$p \in I \text{ iff } \forall\phi : \Omega.(\forall n : N.((p_n \to \phi) \to \phi)) \to \phi$$

For an application of $\mathrm{GMP}_{\neg\neg}$, let $X = F$, $P = I$, $T = \{p \in F \mid \neg p_0\}$, $f(p) = \lambda n.p_{n+1}$.

Since $I \subseteq \{p \in F \mid \neg\neg\exists n.\neg p_n\} = \{p \in F \mid \neg\neg\exists n.T(f^n p)\}$, by $\mathrm{GMP}_{\neg\neg}$ we are done if we can show

$$\forall p\,((\neg\neg p_0 \to \lambda n.p_{n+1} \in I) \to p \in I)$$

We use Jibladze's formula. Suppose $\neg\neg p_0 \to \lambda n.p_{n+1} \in I$; let $\phi : \Omega$ and assume $\forall n.(p_n \to \phi) \to \phi$. We have to conclude that ϕ holds.

The second assumption gives

$$\begin{array}{ll} \text{(i)} & (p_0 \to \phi) \to \phi \\ \text{(ii)} & \forall n.(p_{n+1} \to \phi) \to \phi \end{array}$$

Suppose p_0. Then $\neg\neg p_0$. By the assumption $\neg\neg p_0 \to \lambda n.p_{n+1} \in I$, $\lambda n.p_{n+1} \in I$. By ii) and Jibladze's formula, ϕ follows. Summarizing: $p_0 \to \phi$ holds. Now using i) we have ϕ, as desired. $\square$

3.3 Algebraic Set Theory

A. Joyal and I. Moerdijk, in their monograph [6], develop a way to do set theory algebraically. Cornerstone of their theory is a set of axioms for a so-called "class of small maps". Given such a class $\mathcal{S}$, one defines the notion of a *ZF-algebra*. It is shown that if V is an initial ZF-algebra, then V is a model of intuitionistic set theory IZF (with Collection). They give some examples of classes of small maps;

one example is for the Effective Topos. In [7], this is compared with McCarty's realizability model for IZF, [10]. It is shown that McCarty's universe can be viewed as an object in the Effective Topos and that it is actually the initial ZF-algebra for the class of small maps that Joyal and Moerdijk define in their book. This is done by first proving a theorem which characterizes initial ZF-algebras by a few conditions. One of the conditions is that the algebra is well-founded w.r.t. the $\in$-relation. Checking this for McCarty's model is an easy application of Theorem 1.1, since it is easy to see that its Γ-image is well-founded in **Set**.

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