Seminar ultracategories: hand-in chapter 4
To be handed in on March 27.

Throughout this exercise, $\mathcal{M}$ is an arbitrary ultracategory, T is a set, and $\{M_t\}_{t\in T}$ is a collection of objects of $\mathcal{M}$. For $t \in T$, we define the arrow $\underline{M_t} \to (\beta T, \mathcal{O}_{\beta T})$ of $\mathrm{Comp}_{\mathcal{M}}$ to be the map corresponding (via Remark 4.2.6) to $\delta_t \in \beta T$ and

$$\mathcal{O}_{\beta T,\delta_t} = \int_T M_{t'} d\delta_t \xrightarrow{\varepsilon_{T,t}} M_t.$$

(a) Show that these maps exhibit $(\beta T, \mathcal{O}_{\beta T})$ as the coproduct of the collection of objects $\{\underline{M_t}\}_{t\in T}$ in the category $\mathrm{Comp}_{\mathcal{M}}$. [3pt]

Now let T_0 be a subset of T, and write $u\colon T_0 \hookrightarrow T$ for the inclusion map. We write $u_*\colon \beta T_0 \hookrightarrow \beta T$ for the corresponding continuous map given by $u_*\nu = \int_{T_0} \delta_{t_0} d\nu$. Observe that this is just the pushforward map along u as provided by Definition 1.1.4. For each $\nu \in \beta T_0$, we have the ultraproduct diagonal map $\Delta_{\nu,u}$, defined as the composition:

$$\int_T M_t d(u_*\nu) = \int_T M_t d\left(\int_{T_0} \delta_{t_0} d\nu\right) \xrightarrow{\Delta_{\mu,\delta_\bullet}} \int_{T_0} \left(\int_T M_t d\delta_{t_0}\right) d\nu \xrightarrow{\int_{T_0} \varepsilon_{T,t_0} d\nu} \int_{T_0} M_{t_0} d\nu.$$

We define the natural transformation $\alpha\colon \mathcal{O}_{\beta T} \circ u_* \to \mathcal{O}_{\beta T_0}$ by $\alpha_\nu = \Delta_{\nu,u}$ for $\nu \in \beta T_0$.

(b) Show that $(u_*, \alpha)\colon (\beta T_0, \mathcal{O}_{\beta T_0}) \to (\beta T, \mathcal{O}_{\beta T})$ is a cartesian morphism of $\mathrm{Comp}_{\mathcal{M}}$. [3pt. Beware: showing that this morphism is cartesian is *not* the difficult part of this exercise.]

For $t_0 \in T_0$, we have that $u_*\delta_{t_0} = \delta_{t_0}$.

(c) Show that the diagram

$$\begin{array}{ccccc} \int_T M_t d\delta_{t_0} & = & \int_T M_t d(u_*\delta_{t_0}) & \xrightarrow{\alpha_{\delta_{t_0}}} & \int_{T_0} M_{t'_0} d\delta_{t_0} \\ & \searrow^{\varepsilon_{T,t_0}} & & \swarrow_{\varepsilon_{T_0,t_0}} & \\ & & M_{t_0} & & \end{array}$$

commutes for every $t_0 \in T_0$. [2pt]

(d) Use the previous exercises to conclude that the map (u_*, α) from exercise (b) is the canonical map $\bigsqcup_{t_0\in T_0} \underline{M_{t_0}} \to \bigsqcup_{t\in T} \underline{M_t}$ between coproducts. [2pt]