Seminar Ultracategories - Hand in exercise 5

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To be handed in 3rd April, 2019

In the lectures, we have seen that $\mathrm{Fun}^{\mathrm{LUlt}}(X, \mathsf{Set}) \simeq \mathsf{Shv}(X)$ for compact Hausdorff spaces X. We will also prove in later lectures that $\mathrm{Fun}^{\mathrm{LUlt}}(\mathrm{Mod}(\mathcal{C}), \mathsf{Set}) \simeq \mathsf{Shv}(\mathcal{C})$ for pretoposes $\mathcal{C}$. In this exercise, we will look at the situation for a different set of ultracategories.

Let P be a complete linear order, i.e. a linear order such that all subsets of P have a least upper and greatest lower bound. Note that P, as a poset category, is then also complete, hence it has a categorical ultrastructure.

1. (2pt.) Show that the category Stone_P is equivalent to the category with:

- Objects (X, f), where X is a Stone space and $f : X \to P$ is a function such that for all μ there is an $S_0 \in \mu$ and $s \in S_0$ with $f(\int_{s \in S} x_s \, \mathrm{d}\mu) \leq f(x_s)$;
- Arrows $(X, f) \xrightarrow{\phi} (Y, g)$ are continuous functions $\phi : Y \to X$ such that for all $y \in Y$, $g(y) \leq f(\phi(y))$.

2. (3pt.) Let A_P be the topology on P where a set $U \subseteq P$ is open iff it is an upset (i.e. $x \in U$ and $x \leq y$ imply $y \in U$). Show that $\mathsf{Shv}(A_P)$ is equivalent to $\mathrm{Fun}(P, \mathsf{Set})$.[1]
3. (2pt.) Show that all functors in $\mathrm{Fun}(P, \mathsf{Set})$ have a left ultrastructure.
4. (3pt.) Can we conclude that $\mathrm{Fun}^{\mathrm{LUlt}}(P, \mathsf{Set})$ is equivalent to $\mathsf{Shv}(A_P)$?

[1] Note that A_P is almost never a Stone space since it is almost never Hausdorff: if there are $x < y \in P$, then all opens containing x also contain y.