Homework 3 model solutions

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Rob Schellingerhout

Since this material is about classifying categories and not about the basic logical calculus I am not too fussy about basic logical rules. I allow most of the rules from pages 19 and 20 of Butz to be used implicitly for example.

Exercise 1. (4 points)

We have the following chain of entailments

$$\begin{aligned}
\gamma(\bar{x},\bar{y}_1) \wedge \chi(\bar{y}_1,\bar{z}_1) \wedge \gamma(\bar{x},\bar{y}_2) \wedge \chi(\bar{y}_2,\bar{z}_2) &\vdash^T_\Gamma y_1 = y_2 \wedge \chi(\bar{y}_1,\bar{z}_1) \wedge \chi(\bar{y}_2,\bar{z}_2) && (\gamma \text{ functional})\\
&\vdash^T_\Gamma \chi(\bar{y}_2,\bar{z}_1) \wedge \chi(\bar{y}_2,\bar{z}_2) && (\text{Lemma 4.2(ii)})\\
&\vdash^T_\Gamma \bar{z}_1 = \bar{z}_2, && (\chi \text{ functional})
\end{aligned}$$

where $\Gamma = \bar{x}, \bar{z}_1, \bar{z}_2, \bar{y}_1, \bar{y}_2$. Hence we conclude by 2.3 (adjusted version or Lemma 4.2(i)) that

$$\exists \bar{y}_1(\gamma(\bar{x},\bar{y}_1) \wedge \chi(\bar{y}_1,\bar{z}_1)) \wedge \exists \bar{y}_2(\gamma(\bar{x},\bar{y}_2) \wedge \chi(\bar{y}_2,\bar{z}_2)) \vdash^T_{\bar{x},\bar{z}_1,\bar{z}_2} \bar{z}_1 = \bar{z}_2$$

i.e

$$T \vdash_{\bar{x},\bar{z}_1,\bar{z}_2} \chi\gamma(\bar{x},\bar{z}_1) \wedge \chi\gamma(\bar{x},\bar{z}_2) \Rightarrow \bar{z}_1 = \bar{z}_2.$$

Grading. These are all worth one point each:

- Eliminating quantifiers using 2.3 and 4.2(i) or something similar.
- Correctly applying functionality.
- Using Lemma 4.2(ii) for substitution.
- The remaining details.

Exercise 2. (7 points)

($\Rightarrow$) Choose $q = \exists\bar{x}(p(\bar{x}))$ and define maps

$$\{\cdot \mid q\} \underset{\{\psi\}}{\overset{\{\phi\}}{\rightleftarrows}} \{\bar{x}|\ p(\bar{x})\}$$

given by $\phi(\bar{x}) = p(\bar{x})$ and $\psi(\bar{x}) = p(\bar{x})$. We must check that these are indeed arrows in $\mathcal{R}(T)$. Since there is an injection $\{\gamma\} : \{\bar{x} \mid p(\bar{(x)})\} \to \{\cdot \mid \top\}$ we know by totality of γ and Lemma 6.2 that

$$p(\bar{x}_1) \wedge p(\bar{x}_2) \vdash_{\bar{x}_1,\bar{x}_2} \gamma(\bar{x}_1) \wedge \gamma(\bar{x}_2) \vdash_{\bar{x}_1,\bar{x}_2} \bar{x}_1 = \bar{x}_2. \tag{1}$$

Additionally we know that

$$p(\bar{x}) \vdash_{\bar{x}} \exists\bar{x}(p(\bar{x})) \tag{2}$$

by rule (2.3).

For ϕ we have

1. $\phi(\bar{x}) \vdash_{\bar{x}} q \wedge p(\bar{x})$ by (2) and the definitions of ϕ and q,
2. $q \vdash_{\bar{x}} \exists\bar{x}(\phi(\bar{x}))$ by rule (1.1),
3. and $\phi(\bar{x}_1) \wedge \phi(\bar{x}_2) \vdash_{\bar{x}_1,\bar{x}_2} \bar{x}_1 = \bar{x}_2$ by (1).

For ψ we obtain

1. $\psi(\bar{x}) \vdash_{\bar{x}} p(\bar{x}) \wedge q$ (same argument as ϕ),
2. $p(\bar{x}) \vdash_{\bar{x}} \psi(\bar{x}) = \exists_{\emptyset}\psi(\bar{x})$ by rule (1.1),
3. and $\psi(\bar{x}) \wedge \psi(\bar{x}) \vdash_{\bar{x}} \top$ by rule (2.1).

We check explicitly that they define an isomoprhism. Computations yield

$$\begin{aligned}\psi\phi &= \exists\bar{x}(\phi(\bar{x}) \wedge \psi(\bar{x}))\\ &= \exists\bar{x}(p(\bar{x}) \wedge p(\bar{x}))\\ &\Leftrightarrow \exists\bar{x}(p(\bar{x}))\\ &\Leftrightarrow q \wedge \top\\ &= \mathrm{id}_{\{\cdot|q\}},\end{aligned}$$

and

$$\begin{aligned}\phi\psi(\bar{x}_1,\bar{x}_2) &= \exists_{\emptyset}\psi(\bar{x}_1) \wedge \phi(\bar{x}_2)\\ &= p(\bar{x}_1) \wedge p(\bar{x}_2)\\ &\Leftrightarrow p(\bar{x}_1) \wedge p(\bar{x}_2) \wedge \bar{x}_1 = \bar{x}_2 && \text{by (1)}\\ &\Leftrightarrow p(\bar{x}_1) \wedge \bar{x}_1 = \bar{x}_2 && \text{by Lemma 4.2(ii)}\\ &= \mathrm{id}_{\{\bar{x}|p(x)\}}(\bar{x}_1,\bar{x}_2).\end{aligned}$$

We conclude that $\{\bar{x} \mid p(\bar{x}\}$ and $\{\cdot \mid q\}$ are isomorphic.

($\Leftarrow$) The context of $\{\cdot \mid q\}$ being empty implies that the unique arrow $\{\cdot \mid q\} \to \{\cdot \mid \top\}$ is vacuously injective. This makes $\{\cdot \mid q\}$ (and hence $\{\bar{x} \mid p(\bar{x})\}$) a subobject of the terminal object $\{\cdot \mid \top\}$.

Grading. These are all worth one point each:

- Find $q = \exists\bar{x}(p(\bar{x}))$.
- Use lemma 6.2 to conclude (1).
- Define the maps ϕ and ψ.
- Check that ϕ defines an arrow.
- Check that ψ defines an arrow.
- Check that they define an isomorphism.
- The other direction.

Exercise 3. (3 + 2 points)

(a) Let

$$\Gamma \xrightarrow{f} [y:\sigma] \underset{[F(y)]}{\overset{[y]}{\rightrightarrows}} [y:\sigma]$$

be an equalizer diagram. In this case $f = [M]$ for some term M such that $M : \sigma\ [\Gamma]$. Note that in this language all terms are of the form $F^n(x)$ for some variable x and some $n \in \mathbb{N}$. Hence $M = F^n(x)$ for some variable x. Composition is given by substitution hence

$$[F^n(x)] = [y] \circ [M] = [F(y)] \circ [M] = [F^{n+1}(x)],$$

and since equality is defined to be provable equality we have that

$$F^n(x) = F^{n+1}(x)\ [x:\sigma]$$

is provable.

Grading. These are all worth one point each:

- Use the equalizer $[y], [F(y)]$.
- Observe that all terms are of the form $F^n(x)$.
- Use the definitions of equality and composition to complete the argument.

(b) Suppose for the sake of a contradiction that $\mathcal{R}(T)$ and $C\ell(T)$ are equivalent. Since $\mathcal{R}(T)$ is regular it has equalizers. This would then imply that $C\ell(T)$ had equalizers and hence by the previous part

$$F^n(x) = F^{n+1}(x)\ [x:\sigma]$$

would be a theorem of T. However, this cannot be true as this statement fails in the following model N of T given in Set by

$$N[\![\sigma]\!] = \mathbb{N} \quad \text{and} \quad N[\![F]\!] = n \mapsto n+1.$$

Grading. These are all worth one point each:

- Use the equivalence and (a) to conclude that $F^n(x) = F^{n+1}(x)\ [x:\sigma]$ is provable.
- Prove that this entails a contradiction by constructing a model.