Seminar on models of Intuitionism

Hand-in exercise 12

May 17, 2017

Exercise 1. Let H be an aH algebra.

(a) *(2 points)* Prove the prime filter existence theorem: for every filter F on H and $x \in H$ such that $x \notin F$, there is a prime filter P such that $F \subseteq P$ and $x \notin P$.

(b) *(3 points)* Consider the evaluation map $e : H \to 2^{[H,2]}$, given by $e(x)(i) = i(x)$. Use part (a) to prove that the map e is injective.

Exercise 2. The completeness theorem we have this week is for minimal logic, with the possibility to extend that with additional axioms. To obtain **IPC** one would have to add a notion of falsehood. By adding an initial object[1] $\mathbf{f}$ to $\mathcal{C}$ we obtain the category of proofs of intuitionistic logic. We interpreted the functor $F : \mathcal{C} \to \mathbf{Set}^{\mathbb{Z}}$ to be the proof assignment of a certain set of proofs (a $\mathbb{Z}$-set) to every formula. It would be natural to require the set of proofs of falsehood to be empty, that is $F(\mathbf{f}) = \emptyset$. In this exercise we will see that if we require this, our completeness theorem becomes false. So we let $\mathcal{C}$ be connectially closed with initial object $\mathbf{f}$ (i.e. bicartesian closed) and let $F : \mathcal{C} \to \mathbf{Set}^{\mathbb{Z}}$ be a connectially closed functor such that $F(\mathbf{f}) = \emptyset$ (i.e. bicartesian closed). See below for a few facts you may use.

(a) *(3 points)* We interpret $\neg A$ to be the formula $A \to \mathbf{f}$, as usual. It is well-known that $\neg A \vee \neg\neg A$ is not provable in **IPC**, so $\mathcal{C}(\mathbf{t}, \neg A \vee \neg\neg A) = \emptyset$ (where $\mathbf{t}$ is the terminal object we have seen last week). Show that $\mathbf{Set}^{\mathbb{Z}}(F(\mathbf{t}), F(\neg A \vee \neg\neg A)) \neq \emptyset$. This invalidates our completeness theorem because the emptiness of $\mathcal{C}(\mathbf{t}, \neg A \vee \neg\neg A)$ is then never preserved.

(b) *(2 points)* Why not take $A \vee \neg A$ as a counterexample? Well, simply because it is not a counterexample. Show that there is a proof assignment $F(A)$ such that $\mathbf{Set}^{\mathbb{Z}}(F(\mathbf{t}), F(A \vee \neg A)) = \emptyset$. You do not have to construct the entire functor F, just show that there is a $\mathbb{Z}$-set X such that if $F(A) = X$ there is no arrow $F(\mathbf{t}) \to F(A \vee \neg A)$.

You may of course use the homework from last week (in particular exercise 1), and may regard the hint given there as a fact (i.e. the forgetful functor $U : \mathbf{Set}^{\mathbb{Z}} \to \mathbf{Set}$ is a connectially closed functor). Additionally, you may assume that for $\mathbb{Z}$-sets X and Y a coproduct $X + Y$ is given by the set $X \sqcup Y = \{0\} \times X \cup \{1\} \times Y$ with as group action

$$n \cdot (i, z) = \begin{cases} (i, n \cdot_X z) & \text{if } i = 0 \\ (i, n \cdot_Y z) & \text{if } i = 1 \end{cases}$$

Finally, you may assume that the singleton $\{*\}$ is a terminal object in $\mathbf{Set}^{\mathbb{Z}}$.

[1] An initial object $\mathbf{f}$ in a category $\mathcal{C}$ is an object such that there is exactly one arrow from $\mathbf{f}$ to A for every object A in $\mathcal{C}$.