Seminar on Models of Intuitionism

Hand-in exercise 9

20 April (due 4 May)

Exercise 1. For $f, g\colon \mathbb{N} \to \mathbb{N}$ we define $f \oplus g\colon \mathbb{N} \to \mathbb{N}$ by

$$x \mapsto \begin{cases} f(y) & \text{if } x = 2y; \\ g(y) & \text{if } x = 2y+1. \end{cases}$$

We write $\mathfrak{M}_w^{\mathrm{op}}$ for the poset of Muchnik degrees ordered by $[\mathcal{A}] \leq^{\mathrm{op}} [\mathcal{B}]$ iff $\mathcal{B} \leq_w \mathcal{A}$. In this exercise you will show that $\mathfrak{M}_w^{\mathrm{op}}$ is also a Heyting algebra with $[\mathcal{A}] \to [\mathcal{B}] = [\mathcal{A} \to \mathcal{B}]$ where

$$\mathcal{A} \to \mathcal{B} = \{f\colon \mathbb{N} \to \mathbb{N} \mid \forall g \in A \exists h \in B[h \leq_T f \oplus g]\}$$

for mass problems $\mathcal{A}$ and $\mathcal{B}$.

(a) Prove that for $f, g\colon \mathbb{N} \to \mathbb{N}$ the function $f \oplus g$ is the least upper bound with respect to $\leq_T$. *(2 points)*

(b) Verify that $[\mathcal{A}] \to [\mathcal{B}]$ is well-defined, i.e. that $[\mathcal{A} \to \mathcal{B}] = [\mathcal{A}' \to \mathcal{B}']$ if $\mathcal{A} \equiv_w \mathcal{A}'$ and $\mathcal{B} \equiv_w \mathcal{B}'$. *(1 point)*

(c) Show that $\mathfrak{M}_w^{\mathrm{op}}$ is a Heyting algebra, i.e. that for mass problems $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ we have $[\mathcal{A}] \vee [\mathcal{C}] \geq [\mathcal{B}]$ iff $[\mathcal{C}] \geq [\mathcal{A} \to \mathcal{B}]$. *(2 points)*

Exercise 2. The goal of this exercise is to show that $\mathfrak{M}_w$ is not dd-like. Recall that an element a of a lattice L is *join-reducible* if there exist $b, c \in L$ both different from a such that $a = b \vee c$.

(a) Suppose $[\mathcal{A}] \in \mathfrak{M}_w$. Give a neccesary and sufficient condition on $C(\mathcal{A})$ such that $[\mathcal{A}]$ is join-reducible. (Remark: At the cost of 1 point you can ask us what this desired condition is.) *(3 points)*

(b) Show that $\mathfrak{M}_w$ is not dd-like. *(2 points)*