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CT_0 IS STRONGER THAN $\mathrm{CT}_0!$

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ABSTRACT. $\mathrm{CT}_0!$ is the result of adding the uniqueness condition to the antecedent of CT_0. $\mathbf{HA} + \mathrm{CT}_0$ is shown to be essentially stronger than $\mathbf{HA} + \mathrm{CT}_0!$.

According to Church's thesis, every intuitively computable arithmetical function is recursive. In constructive mathematics, Church's thesis can be expressed by the following formula of second order arithmetic:

$$\forall f \exists z \forall x \exists v \big(T(z, x, v) \,\&\, f(x) = U(v) \big) \tag{CT}$$

(T, U are Kleene's T-predicate and result-extracting function).[1]

The most straightforward counterpart of CT in first order language is the following schema, proposed by Dr. Dragalin:

$$\forall x \exists ! y A(x, y) \to \exists z \forall x \exists v \big(T(z, x, v) \,\&\, A(x, U(v)) \big). \tag{$\mathrm{CT}_0!$}$$

The antecedent here expresses that $[\langle x, y \rangle \colon A(x, y)]$ is the graph of a total function; this function corresponds to f in the second order version.

However, in the current literature on formal systems of constructive mathematics, another schema is usually considered to be the first order counterpart of CT. This other schema is less restrictive; it is obtained from $\mathrm{CT}_0!$ by dropping the uniqueness condition:

$$\forall x \exists y A(x, y) \to \exists z \forall x \exists v \big(T(z, x, v) \,\&\, A(x, U(v)) \big). \tag{CT_0}$$

This schema is motivated as follows. Assume $\forall x \exists y A(x, y)$. This means in constructive mathematics that a computable function f exists such that $\forall x A(x, f(x))$. By Church's thesis, f is recursive; take z to be a gödelnumber of f.

Thus, CT_0 is CT combined with a choice principle [**1**, 1.11.7].

We show that CT_0 is essentially stronger than the more straightforward formulation of Dragalin's: there exists a closed instance of CT_0 underivable in intuitionistic arithmetic **HA** from $\mathrm{CT}_0!$.

Familiarity with Kleene's realizability is assumed [**1**, 3.2.2]. By j, j_1, j_2 we denote a pairing function and its inverses:

$$j_1 j(x, y) = x, \quad j_2 j(x, y) = y, \quad j(j_1(z), j_2(z)) = z;$$

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[1]It is assumed that f is a variable for *intuitively* computable functions.

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{ } denotes partial recursive function application. We consider disjunction a defined connective [**1**, 1.3.7].

1. The proof is based on a modification of Kleene's realizability in which the clause for existential quantifier only is changed. We define a sequence V_0, $V_1, \ldots$ of sets of natural numbers and define e to be a number realizing $\exists x A(x)$ if $V_e \neq \varnothing$ and every element of V_e realizes $\exists x A(x)$ in the sense of Kleene. If the family (V_e) has sufficiently good closure properties then **HA** is correct w.r.t. this modification of realizability. Assume, on the other hand, that (V_e) is chosen in such a way that the following hold:

(a) there is no effective procedure for finding an element in any given nonempty member of (V_e).

(b) there is such a procedure for the one-element members of (V_e).

One can expect then that $\mathrm{CT}_0!$ is correct under this realizability interpretation, and CT_0 is not.

Conditions (a), (b) suggest the following choice of V_e:

DEFINITION 1. For any natural e, $V_e = [n \colon n < j_2(e) \;\&\; \neg\,!\{j_1(e)\}(n)]$.

DEFINITION 2. A natural number e *realizes* a sentence A if

A is atomic and true, or

A is $B \;\&\; C, j_1(e)$ *realizes* $B, j_2(e)$ *realizes* C, or

A is $B \rightarrow C$ and for every n which *realizes* B $\{e\}(n)$ is defined and *realizes* C, or

A is $\forall x B(x)$ and for every $n \{e\}(n)$ is defined and *realizes* $A(\bar{n})$, or

A is $\exists x B(x)$, $V_e \neq \varnothing$, and for every $n \in V_e, j_2(n)$ *realizes* $A(j_1(\bar{n}))$.

2. The following two properties of (V_e) are obvious:

LEMMA 1. *There exists a unary partial recursive function α such that for every e $|V_e| = 1$ implies $!\alpha(e)$, $\alpha(e) \in V_e$.*

LEMMA 2. *There exists a unary total recursive function β such that for every n $V_{\beta(n)} = \{n\}$.*

LEMMA 3. *There exists a unary total recursive function γ such that for every e*

$$V_{\gamma(e)} = \bigcup_{n \in V_e} V_n.$$

PROOF. For any m, n, e,

$$m \in \bigcup_{n \in V_e} V_n \Rightarrow m \in \bigcup_{n < j_2(e)} V_n \Rightarrow m < \max_{n < j_2(e)} (j_2(n)),$$

hence $\bigcup_{n \in V_e} V_n$ is bounded uniformly effectively w.r.t. e. Furthermore,

$$\begin{aligned} m \notin \bigcup_{n \in V_e} V_n &\Leftrightarrow \neg\, \exists n (m \in V_n \;\&\; n \in V_e) \\ &\Leftrightarrow \neg\, \exists n_{n < j_2(e)} (m \in V_n \;\&\; n \in V_e) \\ &\Leftrightarrow \forall n_{n < j_2(e)} (m \notin V_n \vee n \notin V_e), \end{aligned}$$

hence the complement of $\bigcup_{n \in V_e} V_n$ is r.e. uniformly effectively w.r.t. e.

LEMMA 4. *For every unary partial recursive function θ there exists a unary partial recursive function θ^* such that for every e $V_e \subset \operatorname{dom}\theta$ implies $!\theta^*(e)$, $V_{\theta^*(e)} = \theta(V_e)$.*

PROOF. Take an e with $V_e \subset \operatorname{dom}\theta$. $\operatorname{dom}\theta$ and the complement of V_e are both r.e. and cover the set of natural numbers. It follows that there exists a set P_e recursive uniformly effectively w.r.t. e which is contained in $\operatorname{dom}\theta$ and disjoint with the complement of V_e, so that $V_e \subset P_e \subset \operatorname{dom}\theta$. For any $m \in \theta(V_e)$

$$m < \max_{n \in V_e} \theta(n) \leqslant \max_{n \in P_e} \theta(n);$$

hence $\theta(V_e)$ is bounded uniformly effectively w.r.t. e. Moreover, for any m

$$\begin{aligned} m \notin \theta(V_e) &\Leftrightarrow \neg\, \exists n(n \in V_e \;\&\; m \simeq \theta(n)) \\ &\Leftrightarrow \neg\, \exists n_{n \in P_e}(n \in V_e \;\&\; m \simeq \theta(n)) \\ &\Leftrightarrow \neg\, \exists n_{n \in P_e}\big(n \in V_e \;\&\; (n \in \operatorname{dom}\theta \;\&\; m \simeq \theta(n))\big) \\ &\Leftrightarrow \forall n_{n \in P_e}\big(n \notin V_e \vee \neg\,(n \in \operatorname{dom}\theta \;\&\; m \simeq \theta(n))\big). \end{aligned}$$

The first disjunctive member is r.e., and the second is recursive, uniformly effectively w.r.t. e; hence the disjunction is uniformly r.e. too. On the other hand, the condition restricting the universal quantifier is uniformly recursive; hence the quantifier can be replaced by one with a recursive bound. It follows that the complement of $\theta(V_e)$ is r.e. uniformly effectively w.r.t. e.

LEMMA 5. *For every formula A there exists a unary partial recursive function φ_A with the following property: for any nonempty V_e, if every element of V_e realizes a closed instance $\bar{A}$ of A then $\varphi_A(e)$ is defined and realizes $\bar{A}$.*

PROOF. For atomic E define $\varphi_E(e) = 0$. Let A be $B \;\&\; C$, and φ_B, φ_C are already defined. Consider the set

$$V_{j_1^*(e)} = \{j_1(n) \colon n \in V_e\}$$

(notation from Lemma 4). If V_e is nonempty then so is $V_{j_1^*(e)}$; if every element of V_e realizes $\bar{A}$ then every element of $V_{j_1^*(e)}$ realizes $\bar{B}$. Hence under these assumptions $\varphi_B(j_1^*(e))$ realizes $\bar{B}$. Similarly, $\varphi_C(j_2^*(e))$ realizes $\bar{C}$. Hence we can define

$$\varphi_A(e) = j\big(\varphi_B(j_1^*(e)), \varphi_C(j_2^*(e))\big).$$

Let A be $B \to C$. Define $\theta_m(n) = \{n\}(m)$, and consider

$$V_{\theta_m^*(e)} = \big[\{n\}(m) \colon n \in V_e\big].$$

Assume $V_e \neq \varnothing$ and every element of V_e realizes $\bar{A}$. Take $n \in V_e$ and any m which realizes $\bar{B}$; then $\{n\}(m)$ is defined and realizes $\bar{C}$. Hence $V_{\theta_m^*(e)}$ is nonempty, and its elements realize $\overline{C}$. Hence we can define

$$\varphi_A(e) = \Lambda m.\varphi_C \theta^*_m(e).$$

Let A be $\forall x C(x)$. Define θ_m and φ_A as in the preceding case.

Let A be $\exists x C(x)$. Take φ_A to be γ from Lemma 3.

3. LEMMA 6. *Every theorem of* **HA** + $\mathrm{CT}_0!$ *is realizable.*

PROOF. The correctness of all postulates which do not contain existential quantifiers explicitly is verified in exactly the same way as for Kleene's realizability. Hence the only postulates to be considered are the postulates of predicate calculus for existential quantifier and $\mathrm{CT}_0!$.

Consider an axiom of the form $A(t) \to \exists x A(x)$; assume for simplicity that x is the only parameter in $A(x)$. Assume that m realizes $A(t)$. Then $\beta(j(m, t))$ realizes $\exists x A(x)$ for β from Lemma 2; hence $\Lambda m.\beta(j(m; t))$ realizes $A(t) \to \exists x A(x)$.

Consider an inference of the form

$$\frac{A(b) \to C}{\exists x A(x) \to C},$$

and assume for simplicity that b is the only parameter in $A(b)$, C is closed. Let φ be a unary partial recursive function such that for every m $\varphi(m)$ realizes $A(\overline{m}) \to C$, and assume that e realizes $\exists x A(x)$. Then $V_e \neq \varnothing$, and for every $n \in V_e$ $j_2(n)$ realizes $A(j_1(\overline{n}))$. It follows that for every such n $\{\varphi(j_1(n))\}(j_2(n))$ realizes C.

Define $\theta(n)$ to be $\{\varphi(j_1(n))\}(j_2(n))$ and consider

$$V_{\theta^*(n)} = \left[\{\varphi(j_1(n))\}(j_2(n)) \colon n \in V_e\right].$$

For φ_C from Lemma 5, $\varphi_C(\theta^*(n))$ realizes C. Hence $\Lambda e.\ \varphi_C(\theta^*(e))$ realizes $\exists x A(x) \to C$.

Consider an instance (for simplicity, closed) of $\mathrm{CT}_0!$. Assume that e realizes $\forall x \exists ! y A(x, y)$. Then for every n $\{e\}(n)$ realizes $\exists y A(\overline{n}, y)$ & $\forall y_1 y_2[(A(\overline{n}, y) \,\&\, A(\overline{n}, y_2)) \to y_1 = y_2]$. It follows, on the one hand, that $j_1(\{e\}(n))$ realizes $\exists y A(\overline{n}, y)$, i.e. $V_{j_1(\{e\}(n))} \neq \varnothing$ and for every $m \in V_{j_1(\{e\}(n))}$ $j_2(m)$ realizes $A(\overline{n}, j_1(\overline{m}))$.

Define $\psi(n) = j_1^*(j_1(\{e\}(n)))$; then $V_{\psi(n)} = [j_1(m) \colon m \in V_{j_1(\{e\}(n))}] \neq \varnothing$ and for every $q \in V_{\psi(n)}$ $A(\overline{n}, \overline{q})$ is realizable.

On the other hand, $j_2(\{e\}(n))$ realizes $\forall y_1 y_2[(A(\overline{n}, y_1) \,\&\, A(\overline{n}, y_2)) \to y_1 = y_2]$. It follows that there exists at most one q such that $A(\overline{n}, \overline{q})$ is realizable. Hence $|V_{\psi(n)}| = 1$. Then for α from Lemma 1 $V_{\psi(n)} = \{\alpha(\psi(n))\}$. Hence, by the definition of ψ, for every $m \in V_{j_1(\{e\}(n))}$,

$$j_1(m) = \alpha(\psi(n)),$$

which implies that $j_2(m)$ realizes $A(\overline{n}, \overline{\alpha(\psi(n))})$.

Define $\nu(e, n) = j_2^*(j_1(\{e\}(n)))$; then $V_{\nu(e,n)} = [j_2(m) \colon m \in V_{j_1(\{e\}(n))}] \neq \varnothing$ and every $s \in V_{\nu(e,n)}$ realizes $A(\overline{n}, \overline{\alpha(\psi(n))})$. Hence this formula is realized also by $\varphi_A(\nu(e, n))$. Let b_e be $\Lambda n.\alpha(\psi(e, n))$, and take l_e such that $T(b_e, n, l_e)$. Then $\varphi_A(\nu(e, n))$ realizes $A(\bar{n}, U(\bar{l}_e))$, 0 realizes $T(\bar{b}_e, \bar{n}, \bar{l}_e)$, $j(0, \varphi_A(\nu(e, n)))$ realizes $(T(\bar{b}_e, \bar{n}, \bar{l}_e) \,\&\, A(\bar{n}, U(\bar{l}_e)))$; hence $\sigma(e)$ defined by

$$\sigma(e) = \Lambda n.\beta\Big[j\big[1, j\big(0, \varphi_A(\nu(e, n))\big)\big]\Big]$$

realizes $\forall x \exists v(T(\bar{b}_e, x, v) \,\&\, A(x, U(v)))$. It follows that $\beta(j(b_e, \sigma(e)))$ realizes the consequent of the instance of $\mathrm{CT}_0!$ in question, and $\Lambda e.\beta(j(b_e, \sigma(e)))$ realizes the instance itself.

4. Consider now the following "binary" version of CT_0:

$$\forall x\big(A(x) \vee B(x)\big) \to \exists z \forall x \exists v\big(T(z, x, v) \\ \& \,\big(U(v) = 0 \to A(x)\big) \,\&\, \big(U(v) \neq 0 \to B(x)\big)\big). \quad (\mathrm{CT}_0^b)$$

Every instance of CT_0^b clearly follows from an instance of CT_0: take $A(x, y)$ to be $(A(x) \,\&\, y = 0) \vee (B(x) \,\&\, y \neq 0)$. We shall find a closed instance of CT_0^b which is not realizable.

Let $[m\colon \exists n T(a, m, n)]$, $[m\colon \exists n T(b, m, n)]$ be disjoint recursively inseparable r.e. sets. Take $A(x)$, $B(x)$ to be respectively $\forall z \neg T(\bar{a}, x, z)$, $\forall z \neg T(\bar{b}, x, z)$. Assume the instance of CT_0^b is realizable. Consider its antecedent $\forall x(\forall z \neg T(\bar{a}, x, z) \vee \forall z \neg T(\bar{b}, x, z))$ and show that it is realizable too. According to the definition of disjunction, this formula is an abbreviation for

$$\forall x \exists y\Big[\big(y = 0 \to \forall z \neg T(\bar{a}, x, z)\big) \,\&\, \big(y \neq 0 \to \forall z \neg T(\bar{b}, x, z)\big)\Big].$$

Every true closed instance of the subformula in the brackets is realized by $j(\Lambda l.0, \Lambda l.0)$; denote this number by d. For any k define sets W_k', W_k'', W_k as follows:

$$W_k' = \begin{cases} \{j(0, d)\} & \text{if } \forall n \neg T(a, m, n), \\ \varnothing & \text{otherwise;} \end{cases}$$

$$W_k'' = \begin{cases} \{j(1, d)\} & \text{if } \forall n \neg T(b, m, n), \\ \varnothing, & \text{otherwise,} \end{cases}$$

$W_k = W_k' \cup W_k''$. The sets W_k', W_k'' are uniformly bounded, and their complements are uniformly effectively r.e.; hence the sets W_k have the same properties, and there exists a total recursive function π such that for every k $W_k = V_{\pi(k)}$. $W_k \neq \varnothing$ because a, b are gödelnumbers of disjoint r.e. sets. Consider an element of W_k. If it is $j(0, d)$ then $\forall n \neg T(a, m, n)$; hence

$$\big(0 = 0 \to \forall z \neg T(\bar{a}, x, z)\big) \,\&\, \big(0 \neq 0 \to \forall z \neg T(\bar{b}, x, z)\big)$$

is true and consequently realized by d.

Similarly, if it is $j(1, d)$ then

$$\big(1 = 0 \to \forall z \neg T(\bar{a}, x, z)\big) \,\&\, \big(1 \neq 0 \to \forall z \neg T(\bar{b}, x, z)\big)$$

is realized by d. It follows that $\pi(k)$ realizes

$$\exists y\Big[\big(y = 0 \to \forall z \neg T(\bar{a}, x, z)\big) \,\&\, \big(y \neq 0 \to \forall z \neg T(\bar{b}, x, z)\big)\Big],$$

and the antecedent of the instance of CT_0^b in question is realized by $\Lambda k.\pi(k)$. Hence its consequent is realizable too.

Furthermore, the consequent has the following property: every implication occurring in it has the atomic conclusion. For such formulas, the realizability implies the truth. Hence the consequent is true, i.e. there exists a total recursive function ρ such that for every m $\rho(m) = 0$ implies $\forall n \neg T(a, m, n)$, $\rho(m) \neq 0$ implies $\forall n \neg T(b, m, n)$. But this is impossible, because the r.e. sets with the gödelnumbers a, b are recursively inseparable. Thus we proved

THEOREM. *There exists a closed instance of* CT_0^b *underivable in* $\mathbf{HA} + \mathrm{CT}_0!$.

COROLLARY. *There exists a closed instance of* CT_0 *underivable in* $\mathbf{HA} + \mathrm{CT}_0!$.

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