Homework exercise 1

In this exercise you will fill in a gap in the proof of theorem 3.1 (see handout). Recall that we fixed a complete Heyting algebra Ω and some Ω-set (X, δ). Let $A \subset \Omega$ and suppose that we are given an arrow $\alpha_a : 1_a \to (X, \delta)$ for each $a \in A$ such that for all $a, a' \in A$ we have

$$\alpha_a \circ e_{a \wedge a'\ a} = \alpha_a \circ e_{a \wedge a'\ a'},$$

where e_{qp} is the unique arrow $1_q \to 1_p$ for $q \leq p$. Set $p = \bigvee A$ and define $\alpha : \{*_p\} \times X \to \Omega$ by $\alpha(*_p, x) = \bigvee_{a \in A} \alpha_a(*_a, x)$. Show that α is an arrow $1_p \to (X, \delta)$.

Hint. In the previous homework we have seen that any complete lattice that satisfies the infinitary distributive law is a complete Heyting algebra. In this exercise you may use, without proof, the converse of that statement: any complete Heyting algebra satisfies the infinitary distributive law. That is, for any $p \in \Omega$ and any subset $A \subset \Omega$ one has:

$$p \wedge \bigvee A = \bigvee_{a \in A} p \wedge a.$$

Scoring. There are four properties to check, the first three are each rewarded with one point, the fourth is rewarded with two points.

Homework exercise 2

Your goal is to prove that the implication subsheaf $A \to B$ defined during the lecture is indeed a sheaf. Recall that given subsheaves $A, B \subset F$ over some fixed complete Heyting algebra Ω, we define the implication to be

$$(A \to B)_p := \{x \in F(p) \mid \forall q \leq p.\, x|_q \in A_q \Rightarrow x|_q \in B_q\}.$$

Let $p \in \Omega$ and let $Q \subset \Omega$ be such that $\bigvee Q = p$. Let $(x_q \in (A \to B)_q)_{q \in Q}$ be such that for every $q, q' \in Q$, $x_q|_{q \wedge q'} = x_{q'}|_{q \wedge q'}$. Note that this is an arbitrary compatible family.

Show that this compatible family has a unique amalgamation. It may be helpful to split the work as follows:

a. (0.5 points) Show there is some unique $x \in F_p$ such that $x|_q = x_q$ for every $q \in Q$. Conclude that if an amalgamation exists, it must be unique.

b. (0.5 points) Show that if $x \in A_p$ then for every $q \in Q$, $x_q \in A_q$.

c. (1 points) Show that if $x \in A_p$ then $x \in B_p$.

d. (2 points) Let $p' \leq p$ and suppose $x|_{p'} \in A_{p'}$. Show that $x|_{p'} \in B_{p'}$.

e. (1 points) Conclude that $(x_q \in (A \to B)_q)_{q \in Q}$ has a unique amalgamation in $(A \to B)_p$.

Solution to exercise 1

Let us denote $\alpha(x)$ for $\alpha(*_p, x)$ and likewise $\alpha_a(x)$ for $\alpha_a(*_a, x)$. We will check properties (1) to (4) as numbered on the handout.

For property (1) we can use a direct calculation where we use property (1) from the arrows $\alpha_a : 1_a \to (X, \delta)$.

$$\begin{aligned}
&\alpha(x) &&= \\
&\bigvee_{a \in A} \alpha_a(x) &&\leq \\
&\bigvee_{a \in A} \delta_a(*_a, *_a) \wedge \delta(x, x) &&= \\
&\bigvee_{a \in A} a \wedge \delta(x, x) &&= \\
&\delta(x, x) \wedge \bigvee A &&= \\
&\delta(x, x) \wedge p &&= \\
&\delta(x, x) \wedge \delta_p(*_p, *_p).
\end{aligned}$$

Property (2) is also found by direct calculation, but this time using properties (1) and (2) from the arrows $\alpha_a : 1_a \to (X, \delta)$ and using property (1) from α as well.

$$\delta_p(*_p, *_p) \wedge \alpha(x) \wedge \delta(x, x') = \delta(x, x') \wedge \bigvee_{a \in A} \alpha_a(x) = \bigvee_{a \in A} \delta(x, x') \wedge \alpha_a(x) =$$
$$\bigvee_{a \in A} \delta_a(*_a, *_a) \wedge \alpha_a(x) \wedge \delta(x, x') \leq \bigvee_{a \in A} \alpha_a(x') = \alpha(x').$$

Like the previous two properties, we again find property (3) by direct calculation.

$$\delta_p(*_p, *_p) = p = \bigvee A \leq \bigvee_{a \in A} \bigvee_{x \in X} \alpha_a(x) = \bigvee_{x \in X} \bigvee_{a \in A} \alpha_a(x) = \bigvee_{x \in X} \alpha(x).$$

Finally, property (4) is a little bit more work. First we note that

$$\alpha(x) \wedge \alpha(x') = \left(\bigvee_{a \in A} \alpha_a(x) \right) \wedge \left(\bigvee_{a \in A} \alpha_a(x') \right) = \bigvee_{a, a' \in A} \alpha_a(x) \wedge \alpha_{a'}(x').$$

It is now sufficient to show that $\alpha_a(x) \wedge \alpha_{a'}(x') \leq \delta(x, x')$ for any $a, a' \in A$. By property (1) for α_a we have $\alpha_a(x) \leq a$, and likewise $\alpha_{a'}(x') \leq a'$. So we see that $\alpha_a(x) \wedge \alpha_{a'}(x') \leq a \wedge a'$, hence

$$\alpha_a(x) \wedge \alpha_{a'}(x') = (\alpha_a(x) \wedge (a \wedge a')) \wedge (\alpha_{a'}(x') \wedge (a \wedge a')).$$

Now note that $\alpha_a(y) \wedge (a \wedge a') = (\alpha_a \circ e_{a \wedge a'\ a})(y)$ and $\alpha_{a'}(y) \wedge (a \wedge a') = (\alpha_a \circ e_{a \wedge a'\ a'})(y)$, for all $y \in X$. As α_a and $\alpha_{a'}$ are part of a compatible family we can define $\alpha_{a \wedge a'} := \alpha_a \circ e_{a \wedge a'\ a} = \alpha_{a'} \circ e_{a \wedge a'\ a'}$, which is an arrow $1_{a \wedge a'} \to (X, \delta)$. So by property (4) of that arrow and the above equality we find indeed

$$\alpha_a(x) \wedge \alpha_{a'}(x') = \alpha_{a \wedge a'}(x) \wedge \alpha_{a \wedge a'}(x') \leq \delta(x, x').$$

Solution to exercise 2

Clarification: The exercise should have explicitly stated that we assume $A \to B$ defined this way to be a presheaf, my apologies for the ambiguity.

Fix Ω, F, A, B, p, Q, and $(x_q)_{q\in Q}$ as in the exercise.

(a). Since F is a sheaf and $(x_q)_{q\in Q}$ satisfies the conditions of a compatible family, there is a unique amalgamation $x \in F_p$. Any amalgamation x' of $(x_q)_{q\in Q}$ in $(A \to B)_p$ would also be an amalgamation in F_p, and hence $x = x'$ by uniqueness of x.

(b). Suppose $x \in A_p$. Since $x_q = x|_q$ by the definition of an amalgamation, and $x|_q = A(q \leq p)(x) \in A_q$, we have $x_q \in A_q$.

(c). Suppose $x \in A_p$. For every $q \leq p$, $x_q \in A_q$, and since $x_q \in (A \to B)_q$, it follows that $x_q \in B_q$. By assumption, B is a sheaf and thus the compatible family $(x_q)_{q\in Q}$ has an amalgamation in B_p. By the logic in (a), this amalgamation is x, hence $x \in B_p$.

(d). Let $p' \leq p$ and suppose $x|_{p'} \in A_{p'}$. Define $Q' = \{q \wedge p' \mid q \in Q\}$ and define a new compatible family $(x'_{q'})_{q'\in Q'}$ by $x'_{q'} = x|_{q'}$. Note that $x'_{q'} \in A_{q'}$ since $x|_{q'} = x|_{p'}|_{q'}$ and $x|_{p'} \in A_{p'}$. This is a compatible family, since for any $q'_1, q'_2 \in Q'$ we have

$$x|_{q'_1}|_{q'_1 \wedge q'_2} = x|_{q'_1 \wedge q'_2} = x|_{q'_2}|_{q'_1 \wedge q'_2}.$$

Since Ω is a complete Heyting algebra, $\bigvee Q' = \bigvee_{q\in Q} q \wedge p' = p' \wedge \bigvee Q = p' \wedge p = p'$, since $p' \leq p$. By the same logic as in (a), this compatible family has a unique amalgamation $x' \in F_{p'}$. However, since $x|_{p'}$ is also an amalgamation of the compatible family, $x' = x|_{p'}$ and thus $x' \in A_{p'}$, and thus by the same logic as in (c) we in fact have $x' \in B_{p'}$.

Grading: 1 point for the choice of compatible family, 1 point for the rest of the argument.

(e). By (d), for any $p' \leq p$, if $x|_{p'} \in A_{p'}$ then $x|_{p'} \in B_{p'}$, and thus $x \in (A \to B)_p$, and hence $(x \in q)_{q\in Q}$ has a unique amalgamation in $(A \to B)_p$.