Seminar on Models of Intuitionism - Läuchli realizability

Model solution 8

April 13, 2017

Exercise 1.
Each part is worth two points: one point for a giving a correct term and one point for the explanation as to why that term is in all proof assignments.
(a) Write $D = S(\forall x\,(A(x) \wedge B(x)))$ and let θ be given by the closed term

$$\lambda x^D \left(\left\langle \lambda y^\Gamma (x(y)(0)), \lambda y^\Gamma (x(y)(1))\right\rangle\right).$$

Let p be an arbitrary proof assignment. Suppose that $\alpha \in p(\forall x\,(A(x) \wedge B(x)))$. Then, for any $c \in \Gamma$, we have that $\alpha(c) \in p(A(c) \wedge B(c))$, so we get $\alpha(c)(0) \in p(A(c))$. This means that $\lambda y^\Gamma(\alpha(y)(0)) \in p(\forall x\, A(x))$. Similarly, $\lambda y^\Gamma(\alpha(y)(1)) \in p(\forall x\, B(x))$. We conclude that

$$\theta(\alpha) = \left\langle \lambda y^\Gamma(\alpha(y)(0)), \lambda y^\Gamma(\alpha(y)(1))\right\rangle \in p(\forall x\, A(x) \wedge \forall x\, B(x)).$$

Since $\alpha \in p(\forall x\,(A(x) \wedge B(x)))$ was arbitrary, we can conclude that $\theta \in p(\varphi)$. □
(b) Write $D = S((A \to B) \vee (A \to C))$, $E = S(A)$ and let θ be given by the closed term:

$$\lambda x^D \left(\lambda y^E (\langle x(0), x(1)(y)\rangle)\right).$$

Let p be an arbitrary proof assignment. Suppose that $\alpha \in p((A \to B) \vee (A \to C))$ and $\beta \in p(A)$. Then $\theta(\alpha)(\beta) = \langle \alpha(0), \alpha(1)(\beta)\rangle$. Notice that $\alpha(0) \in \{0,1\}$. If $\alpha(0) = 0$, then $\alpha(1) \in p(A \to B)$, so $\alpha(1)(\beta) \in p(B)$, which in turn means that $\theta(\alpha)(\beta) = \langle 0, \alpha(1)(\beta)\rangle \in p(B \vee C)$. If $\alpha(0) = 1$, then we show in a similar fashion that $\theta(\alpha)(\beta) \in p(B \vee C)$. We conclude that $\theta(\alpha) \in p(A \to B \vee C)$ and thus that $\theta \in p(\varphi)$. □
(c) Write $D = S(\neg(A \vee \neg A))$, $E = S(A)$ and let θ be given by the closed term

$$\lambda x^D \left(x\left(\left\langle 1, \lambda y^E (x(\langle 0, y\rangle))\right\rangle\right)\right).$$

Let p be an arbitrary proof assignment. Suppose that $\alpha \in S(\neg(A \vee \neg A))$ and $\beta \in p(A)$. Then $\langle 0, \beta\rangle \in p(A \vee \neg A)$, so we get $\alpha(\langle 0, \beta\rangle) \in p(\bot)$. Since $\beta \in p(A)$ was arbitrary, this means that $\lambda y^E(\alpha(\langle 0, y\rangle)) \in p(\neg A)$. We get $\left\langle 1, \lambda y^E(\alpha(\langle 0, y\rangle))\right\rangle \in p(A \vee \neg A)$, and thus

$$\theta(\alpha) = \alpha\left(\left\langle 1, \lambda y^E(\alpha(\langle 0, y\rangle))\right\rangle\right) \in p(\bot).$$

Since $\alpha \in p(\neg(A \vee \neg A))$ was arbitrary, we can conclude that $\theta \in p(\varphi)$. □

Exercise 2.
(a) *Constructing a suitable θ was worth two points, and showing that it works was worth the remaining point.* Since Γ is countably infinite, we can pick an enumeration of Γ. We define the function

$$\theta : S(\forall x\,(P(x) \vee Q)) = (\Pi \sqcup \Pi)^\Gamma \to \Pi^\Gamma \sqcup \Pi = S(\forall x\, P(x) \vee Q)$$

as follows. Let $\alpha : \Gamma \to \Pi \sqcup \Pi$ be given. For $c \in \Gamma$, we have $\alpha(c)(0) \in \{0,1\}$. Suppose that for all $c \in \Gamma$, we have $\alpha(c)(0) = 0$. Then define the function $\tilde{\alpha} : \Gamma \to \Pi$ by $\tilde{\alpha}(c) = \alpha(c)(1)$ for all $c \in \Gamma$, and set $\theta(\alpha) = \langle 0, \tilde{\alpha}\rangle$. Now suppose that there exists a $c \in \Gamma$ such that $\alpha(c)(0) = 1$. Then let $\tilde{c}$ be the least (in the enumeration of Γ we picked) such c, and define $\theta(\alpha)$ as $\langle 1, \alpha(\tilde{c})(1)\rangle$. This completes the definition of θ.

Now let a proof assignment p be given, and suppose that $\alpha \in p(\forall x\,(P(x) \vee Q))$. We have to show that $\theta(\alpha) \in p(\forall x\,P(x) \vee Q) = p(\forall x\,P(x)) \sqcup p(Q)$. For all $c \in \Gamma$, we have $\alpha(c) \in p(P(c) \vee Q) = p(P(c)) \sqcup p(Q)$. If for all $c \in \Gamma$, we have $\alpha(c)(0) = 0$, then $\alpha(c)(1) \in p(P(c))$ for all $c \in \Gamma$. This means that $\tilde{\alpha}$, as defined above, is an element of $p(\forall x\,P(x))$, and we conclude that $\theta(\alpha) = \langle 0, \tilde{\alpha}\rangle \in p(\forall x\,P(x)) \sqcup p(Q)$. Now suppose that there is a $c \in \Gamma$ such that $\alpha(c)(0) = 1$, and let $\tilde{c}$ be the least such c. Then $\alpha(\tilde{c})(1) \in p(Q)$ and therefore $\theta(\alpha) = \langle 1, \alpha(\tilde{c})(1)\rangle \in p(\forall x\,P(x)) \sqcup p(Q)$, which completes the proof. $\square$

(b) *Half a point was awarded for the idea of giving two proof assignments p_1 and p_2 such that $p_1(Q \vee \neg Q) \cap p_2(Q \vee \neg Q) = \emptyset$, and the other half point was awarded for carrying this out.* Suppose there exists a θ such that $\theta \in p(Q \vee \neg Q) = p(Q) \sqcup p(\neg Q)$ for all proof assignments p. Consider a proof assignment p such that $p(Q) = \Pi$ and $p(\bot) = \emptyset$. Since there are no functions $\Pi \to \emptyset$, we see that $p(\neg Q) = p(\bot)^{p(Q)} = \emptyset^{\Pi} = \emptyset$, so $\theta \in p(Q) \sqcup p(\neg Q) = \Pi \sqcup \emptyset$. In particular, $\theta(0) = 0$. Now consider a proof assignment p such that $p(\bot) = p(Q) = \emptyset$. Then $\theta \in p(Q) \sqcup p(\neg Q) = \emptyset \sqcup p(\neg Q)$, so $\theta(0) = 1$. We have arrived at a contradiction, so we conclude that such θ cannot exist. $\square$