contain the σ-invariant element c_0). Then $x = x_1 \cup x_2$ belongs to $p[B \to C]$ and is invariant under σ^n.

4. Let A be $\forall vB$. '*only if*': Let $x \in p[\forall vB]$, $\sigma^n x = x$. Let $s_n R s$, $d \in \Psi(s)$. As before, we have $n|m$ and $s = s_m$ for $m = \varphi(s)$. By (8) there is $c \in \Gamma_m$ such that $c^- = d$. By (10) $\sigma^m c = c$. Therefore $\sigma^m(xc) = xc$. Also $xc \in p[B_c^v]$ and $B_c^v \in F(\Gamma_m)$. Induction hypothesis gives $\Phi((B_c^v)^-, s) = T$. $(B_c^v)^-$ is $(B^-)_d^v$. Hence $\Phi(\forall vB^-, s_n) = T$.

'*if*': Let $\Phi(\forall vB^-, s_n) = T$. Let $c \in \Gamma$. If H is the group generated by σ^n, then $H(c)$ is generated by σ^m for some m with $n|m$. By (10) $c \in \Gamma_m$. By (7) $c^- \in \Psi(s_m)$. By (4) $s_n R s_m$. Therefore $\Phi((B_c^v)^-, s_m) = T$. Also $B_c^v \in F(\Gamma_m)$. By induction hypothesis $H(c)$ has fixed elements in $p[B_c^v]$. As before we get an invariant function. There is no trouble with the range, since for all $h \in H$, $h(p[B_c^v]) = p[B_c^v] = p[B_{hc}^v]$ in virtue of (12) and (11). Therefore σ^n has fixed elements in $p[\forall vB]$.

5. Let A be $\exists vB$. The proof is straightforward, using (7), (8) and (10).

This concludes the proof of the lemma.

Proof of part (2) of the theorem:

Let A be a closed formula containing no individual constants other than c_0. Then $A \in F(\Gamma_1)$ and $A^- \in F(\Psi(s_1)) = F(\Psi(A))$. Assume not $\vdash A$. Lemma 1 gives a model Φ such that $\Phi(A^-, A) = F$. Let p be the proof assignment associated to Φ. Then by lemma 2, σ has no fixed element in $p[A]$. Therefore $p[A]$ contains no invariant functional.

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EXTENDING THE TOPOLOGICAL INTERPRETATION TO INTUITIONISTIC ANALYSIS, II

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This paper is a sequel to the paper [12] written for Professor Heyting under the same title. Nearly all of the questions left open in [12] have been answered. In particular the results of section 3 in [12] having to do with universal formulae of the theory of < in *three* variables have been extended to *arbitrary* universal formulae in section 5. (Our numbering of sections continues that of [12].) We then discuss in section 6 the general metamathematical implications of the method of section 5 for the theory of the topological model of intuitionistic analysis. In section 7 the important step is taken of enlarging the model to encompass arbitrary (extensional) real functions. The main result is the verification in the model of Brouwer's theorem on continuity: *all functions are uniformly continuous on closed intervals.* The proof is given in detail along with several related results. (The reader will have to refer to [12] for notation and the definition of the model.)

The author was thus able to conclude this paper feeling that he had a rather good grasp of the basic properties of the real numbers of the model. Several further projects remain to be carried out, however. The next important step is to discuss the corresponding topological interpretation of *second-order arithmetic* and the theory of free-choice sequences of integers. This will make possible an exact comparison of the theory of the model and the usual axiomatic theories of intuitionistic analysis (which will no doubt be one of the main topics of part III of this series of papers.) Following such work it is obvious that attention must be given to obtaining a constructive version of the model. Kreisel has suggested that the theory of constructive and lawless sequences (the system of [8]) may provide the proper framework

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for this discussion in view of the connection between the topological interpretation and lawless sequences [5] (part IV?) In this connection it is appropriate to take this opportunity to retract a remark of [12], p. 194. Contrary to what was said, Kreisel has all along felt that the topological interpretation has relevance. Indeed in [6] he mentions it along with realizability and his own elimination of free choice sequences. Moreover in [5], p. 370, paragraph 2, he had already presented the clearest possible statement of his view of the interpretation.

In another direction the theory of species and higher-order functions will require consideration. That study will of course bring in connections with the Boolean models for set theory ([11] and [13].) To date no particular technical results about the Boolean models have played a role in the discussion of these topological models. In fact, the main effort was given to the consideration of those properties of the models and topological spaces most appropriate to foundations of intuitionism. However, it was the success of the Boolean models in making clear certain aspects of Cohen's remarkable independence results (such as the type of construction of generic and random reals) that encouraged the author to take up again the study of the topological interpretation which he had put down some ten years before. And he is most happy to see now just how well things fit together. It does not seem altogether impossible that work in the intuitionistic theories may throw additional light on the properties of the Boolean models (cf. [2] and other references given there).

5. Further independence results. In section 3 we discussed universally quantified three-variable consequences of the axioms of order (1.1)–(1.2) and noted that such questions could be thrown back to propositional calculus as well as being determined by the model. At the time of writing the earlier section the results were rather special, and the author suggested that they should generalize. They do. Kreisel has pointed out the relevance of the following well-known fact about HPC (Heyting's predicate calculus):

A universal sentence is a consequence in HPC *of a given universal axiom if and only if its matrix is a propositional consequence of a finite number of substitution instances of the axiom using the variables mentioned in the conclusion.*

Note that our language has only relation symbols, in fact only $<$, which makes the result so simple; in particular, we have a decision method for

universal consequences of a universal axiom. The proof of this general result requires no special knowledge of models.

Let us consider the theory of $<$. We can schematically indicate the general universal sentence as

$$\forall x_0, \ldots, x_{n-1}\, A(x_i < x_j : i, j < n),$$

where we mean to suggest that the expression involves the various atomic formulae $x_i < x_j$. Let P_{ij} be propositional letters and let

$$A(P_{ij} : i, j < n)$$

indicate the result of replacing $x_i < x_j$ by P_{ij} in the matrix of our universal sentence. Further let

$$B(P_{ij} : i, j < n)$$

be the conjunction of all formulae of the two forms

$$\neg [P_{ij} \wedge P_{ji}]$$

and

$$[P_{ij} \rightarrow P_{ik} \vee P_{kj}]$$

for $i, j, k < n$. In our case the general result mentioned above shows that the given universal sentence is a consequence of (1.1)–(1.2) if and only if

$$[B(P_{ij} : i, j < n) \rightarrow A(P_{ij} : i, j < n)]$$

is a *propositional* theorem in HPC; hence the decision method. This approach is definitely only suited to universal prenex formulae, however. The author is rather inclined to believe that the full first-order intuitionistic theory based on (1.1)–(1.2) is *undecidable*.

Having noted the above purely formal result, we may now ask: *Are the axioms* (1.1)–(1.2) *complete for the universal sentences valid in our model?* We shall prove that, under suitable conditions on the topological space T, if the universal sentence is *not* provable in the theory of $<$, then it *fails* in the model. Indeed we shall find that the contradictory sentence

$$\forall x_0 \neg \forall x_1, \ldots, x_{n-1}\, A(x_i < x_j : i, j < n)$$

is actually *valid* in the model. For this purpose in view of [10], pp. 130–131 we find it convenient to assume that T is a non-empty, dense-in-itself, totally disconnected, metric space; the two best known examples being the Cantor space 2^N and the Baire space N^N. We prefer the latter because of its naturalness in interpreting second-order number theory. Thus for the remainder of the paper we may as well fix $T = N^N$. In the future there may be reason for greater generality, but this choice of T gives a complicated enough model with sufficiently many interesting properties.

Suppose, then, that the universal sentence is *not* provable from (1.1)–(1.2). Thus the corresponding propositional formula is not provable either. By what we know about $T = N^N$, we can assign open sets $[\![P_{ij}]\!]$ to the P_{ij} in such a way that

$$[\![B(P_{ij}: i, j < n)]\!] = T;$$

while

$$[\![A(P_{ij}: i, j < n)]\!] \neq T.$$

We wish to find continuous functions $\xi_i \in \mathscr{R}$ such that

$$[\![\xi_i < \xi_j]\!] = [\![P_{ij}]\!],$$

and this will give us the counterexample in the model.

It seems convenient to construct certain auxiliary functions before obtaining the ξ_i. In the first place, given open sets $[\![P_{ij}]\!]$ and given a subset $I \subseteq \{0, 1, \ldots, n\}$, we can define continuous functions $\sigma_I \in \mathscr{R}$ that are *non-negative* and such that

$$[\![\sigma_I > 0]\!] = \bigcap_{k \in I} \bigcap_{l \notin I} [\![P_{kl}]\!].$$

Next, for given $i, j < n$, we define $\pi_{ij} \in \mathscr{R}$ such that

$$\pi_{ij} = \sum_{\substack{i \in I \\ j \notin I}} \sigma_I - \sum_{\substack{j \in J \\ i \notin J}} \sigma_J,$$

where the summations run over all the subsets satisfying the indicated restrictions. Finally we set for $i < n$

$$\xi_i = \pi_{0i},$$

but it takes several steps to see that these functions so defined have the desired properties. (These formulae, by the way, do indeed generalize those of section 3.)

We note first that, for $i, j < n$,

$$\pi_{ii} = 0$$

and

$$\pi_{ji} = -\pi_{ij}$$

hold by definition. The π_{ij} are by no means non-negative, but we shall see that

$$[\![\pi_{ij} > 0]\!] = [\![P_{ij}]\!].$$

Assume that $\pi_{ij}(t) > 0$, then by the definition of the π_{ij} and the non-negative character of the σ_I, we find $\sigma_I(t) > 0$ for at least one subset I such that $i \in I, j \notin I$. But then by the construction of σ_I, we have $t \in [\![P_{ij}]\!]$. Now for the converse, assume that $t \in [\![P_{ij}]\!]$. We must obtain a particular $I \subseteq \{0, 1, \ldots, n-1\}$ such that $i \in I$, $j \notin I$, and

$$t \in \bigcap_{k \in I} \bigcap_{l \notin I} [\![P_{kl}]\!],$$

where of course k, l range over all indices less than n satisfying the restrictions. To construct I we proceed by induction. Let $I_0 = \{i\}$ and $I_0' = \{j\}$. Since

$$[\![P_{ij}]\!] \cap [\![P_{ji}]\!] = \emptyset$$

we know that $i \neq j$, because $t \in [\![P_{ij}]\!]$ but $t \notin [\![P_{ji}]\!]$. Hence $I_0 \cap I_0' = \emptyset$ and

$$t \in \bigcap_{k \in I_0} \bigcap_{l \in I_0'} [\![P_{kl}]\!].$$

Suppose now $q < n$, and $I_q \cap I_q' = \emptyset$, and

$$t \in \bigcap_{k \in I_q} \bigcap_{l \in I_q'} [\![P_{kl}]\!].$$

Let m be the least integer where $m < n$, $m \notin I_q \cap I_q'$. We will show how to adjoin m to I_q or to I_q' preserving the above properties. Because, from the assumptions about the P_{ij}, we have

$$[\![P_{kl}]\!] \subseteq [\![P_{km}]\!] \cup [\![P_{ml}]\!],$$

so that

$$\bigcap_{k \in I_q} \bigcap_{l \in I_q'} [\![P_{kl}]\!] \subseteq \Big(\bigcap_{k \in I_q} [\![P_{km}]\!] \cup \bigcap_{l \in I_q'} [\![P_{ml}]\!]\Big).$$

Now if $t \in \bigcap_{k \in I_q} [\![P_{km}]\!]$, set $I_{q+1} = I_q$ and $I_{q+1}' = I_q' \cup \{m\}$. While if $t \in \bigcap_{l \in I_q'} [\![P_{ml}]\!]$, set $I_{q+1} = I_q \cup \{m\}$ and $I_{q+1}' = I_q'$. The required properties are obvious in either case. Proceeding in this way to $q = n$ (or is it $n-2$ since we started with i, j as given?) we obtain I and its complement as desired. Having found this I we see $\sigma_I(t) > 0$. Now if J is any subset with $j \in J$ and $i \notin J$, then since $t \notin [\![P_{ji}]\!]$, we have $\sigma_J(t) = 0$. We have thus proved that $\pi_{ij}(t) > 0$.

To complete the proof we must prove in addition

$$\pi_{ij}+\pi_{jk}+\pi_{ki} = 0.$$

Note that in view of our above remarks about the π_{ij}, it is sufficient to assume the $i, j, k < n$ are all *distinct*. We must, unfortunately, write out the sum on the left-hand side of the above equation, namely:

$$\sum_{\substack{i\in I\\ j\notin I}} \sigma_I - \sum_{\substack{j\in J\\ i\notin J}} \sigma_J + \sum_{\substack{j\in I\\ k\notin I}} \sigma_I - \sum_{\substack{k\in J\\ j\notin J}} \sigma_J + \sum_{\substack{k\in I\\ i\notin I}} \sigma_I - \sum_{\substack{i\in J\\ k\notin J}} \sigma_J .$$

There are two kind of terms: positive and negative, and there are *equal* numbers of each. Thus if we can show that each positive term can be cancelled by a *unique* negative term that is formally equal to it, then the whole expression cancels to zero. In view of the symmetry of the situation, consider a typical term σ_I with $i \in I$ and $j \notin I$. By inspection, of the *six* kinds of restrictions on the capital letter subscripts in the expression only *two* could contain an occurrence of the *same* subscript I. And actually only *one* can arise: the cases $k \in I$ and $k \notin I$ are disjoint. Thus indeed a positive occurrence of σ_I cancels with a unique negative occurrence with the same subscript. Our argument is complete.

From the equation just proved we see that

$$\xi_j - \xi_i = \pi_{ij}$$

which implies that

$$[\![\xi_i < \xi_j]\!] = [\![P_{ij}]\!].$$

It follows that

$$[\![A(\xi_i < \xi_j : i, j < n)]\!] \neq T.$$

Since the only interesting situation is where $n \geq 2$, we can derive exactly as in section 3 the validity in the model of

$$\forall x_0 \neg \forall x_1, \ldots, x_{n-1} A(x_i < x_j : i, j < n), \tag{5.1}$$

which gives us a large number of independence results. What we would like to discuss next is whether there are reasonable *axioms* for intuitionistic analysis from which these formulae (5.1) become *provable*.

6. Maximality of propositional calculus. In classical logic we have a very easy time of it: an unprovable propositional schema leads at once to contradiction after a suitable substitution of quite trivial formulae for the propositional letters. Thus we know in a most direct way that in our formaliza-

tion no valid principles have been overlooked. The predicate calculus is not that simple, otherwise it would be decidable. Nevertheless, if we consider a theory such as that of all (classically) true sentences of first-order arithmetic, we conclude from Gödel's theorem that each unprovable predicate calculus schema has an instance in the language of arithmetic that is indeed false. Let us formulate this idea as a general metamathematical notion.

On the one hand we have a *logical calculus* (classical or intuitionistic propositional or predicate calculus, to be definite) which consists of various schemata and some rules (maybe, only modus ponens and generalization.) On the other hand we have a *theory* which consists of a set (species) of valid formulae in a certain (applied) language. We assume that we know what it means to form an *instance* of a logical schema in the applied language. Obviously we want to assume that the logic is *sound* for the theory; that is, all instances of schemata are valid and all instances of rules lead from valid formulae to valid formulae. What is more interesting is the question of whether the logic is *maximal* for the theory. This means that if we take a schema *not* provable in the logical calculus, then the adjunction of all instances of this new schema render the given theory *inconsistent* in the sense of the original logical calculus. (We can take the inconsistency to be the resultant provability of arbitrary formulae.) We use the word 'maximal' here as being more descriptive than the overworked 'complete'. As pointed out, classical propositional calculus is maximal for *any* (classical) theory. Classical predicate calculus is maximal for certain theories, though not for any recursively enumerable theory. What about the intuitionistic logical calculi?

An analysis of the argument of section 5 shows that atomic formulae of the form $y > 0$ have instances which, in the model, take on any value (any open set). Thus with our choice of $T = N^N$ we can easily argue that if $C(P_i : i < m)$ is a propositional formula not provable in HPC, then

$$\neg \forall y_0, \ldots, y_{m-1} C(y_i > 0 : i < m) \tag{6.1}$$

is valid in the model. Note that the negation falls outside the scope of the universal quantifiers. Thus no instance of the unprovable $C(P_i : i < m)$ is likely to lead to a *propositional* contradiction (unless it is classically invalid.) However, an obvious instance combined with *universal generalization* does lead to a contradiction. Therefore we can say that the propositional part of HPC is maximal for the theory of valid sentences of the model in the context of *all* the rules of HPC. Whether the full HPC is maximal for this theory the author cannot see at the moment; though this may in fact follow from the known topological completeness proofs for HPC.

A question that should be answered concerns those axiomatic theories of analysis for which we have the same HPC maximality. Kreisel has verified in [5], p. 378, propositional maximality for a theory of lawless sequences. The theories of Kleene–Vesley [4] and Kreisel–Troelstra [14] need investigation on this point. The many fragmentary results of [4] point in this direction; and one should consider in this connection whether the full $\forall\alpha\exists\beta$-continuity is really needed or just $\forall\alpha\exists!\beta$. Another question: could there be any intuitionistically acceptable extension of Heyting's arithmetic (HA) for which we would have propositional maximality? It seems unlikely.

From the axiomatic point of view, propositional maximality has certain useful consequences. We shall now show that (6.1) together with some very elementary *algebra* allow us to formally derive the results of section 5. Hence, if in [4] it would have been possible to obtain (6.1), many of the results of [4], ch. IV would have followed directly.

Let us recall the notation from the previous section: $B(P_{ij}: i, j < n)$ stood for a certain conjunction related to the axioms of order; $A(P_{ij}: i, j < n)$ was an arbitrary formula such that

$$[B(P_{ij}: i, j < n) \rightarrow A(P_{ij}: i, j < n)]$$

was *unprovable* in HPC. Let p_{ij} be distinct real variables for $i, j < n$. Our construction of the ξ_i in section 5 could be turned into a formal proof of

$$B(p_{ij} > 0: i, j < n) \leftrightarrow \exists x_0, \ldots, x_{n-1} \bigwedge_{i,j<n} [x_i < x_j \leftrightarrow p_{ij} > 0], \quad (6.2)$$

where $\bigwedge$ is the sign of conjunction of several formulae. The proof would be rather long: Assuming the left-hand side, we would have to first introduce

$$p_{ij}^+ = \max(p_{ij}, 0),$$

to get non-negative variables, and then invoke the elementary theorems to have

$$p_{ij}^+ > 0 \leftrightarrow p_{ij} > 0.$$

But that is easy. Next we would have to introduce for $I \subseteq \{0, 1, \ldots, n-1\}$

$$s_I = \prod_{k \in I} \prod_{l \notin I} p_{kl}^+,$$

where $\prod$ is the sign of arithmetic product. Elementary algebra tells us that a product of non-negative terms is strictly positive if and only if *all* the terms

are strictly positive. Thus the formal s_I correspond to our use of σ_I in the informal proof of section 5. Then we could introduce $\bar{p}_{ij}$ corresponding to the π_{ij} and then the x_i corresponding to the ξ_i. Again by elementary algebra we could transcribe our proof to finally establish (6.2).

Now we invoke (6.1) and the unprovability of the implication combined with (6.2) to prove

$$\neg \forall x_0, \ldots, x_{n-1} A(x_i < x_j: i, j < n), \quad (6.3)$$

which is not quite as strong as what we validated in section 5. But note that $[x_i < x_j \leftrightarrow x_i + y < x_j + y]$ by elementary algebra. This translation invariance makes it easy to prove, as in section 3, the following

$$\forall x_0, \ldots, x_{n-1} A(x_i < x_j: i, j < n) \leftrightarrow \forall x_1, \ldots, x_{n-1} A(x_i < x_j: i, j < n), \quad (6.4)$$

where on the right-hand side we have left x_0 *free*. Obviously the implication goes from left to right; for the converse one just translates the *fixed* x_0 to an *arbitrary* x_0. The reader can surely do this for himself. That is how we get the formal proof of (5.1).

The deduction just outlined illustrates a familiar phenomenon: a result purely in the theory of one predicate (in this case it was $<$) was obtained only with the aid of several auxiliary notions (viz. $+, \cdot, -$, max, min) together with their elementary properties. In classical theories we have examples in various fragments of integer arithmetic but not in the theory of order in the continuum. This is another indication that the intuitionistic theory of $<$ is more complicated than the classical theory.

To further illustrate the usefulness of the auxiliary notions for understanding the basic theory of $<$ we may mention the definitions of $z \in [x, y]$ and $z \in (x, y)$ given in (4.2)–(4.3). These are equivalent to the definitions in [3] (3.3.2.1, p. 40), and we find in [3], p. 41 a proof that

$$z \in [x, y] \leftrightarrow \min(x, y) \leqslant z \leqslant \max(x, y). \quad (6.5)$$

That obviously makes very good sense and clears up the doubts the author had when he wrote section 4. However, in the case of the open interval the best we can do is

$$z \in (x, y) \leftrightarrow \neg\neg [\min(x, y) < z < \max(x, y)]. \quad (6.6)$$

The author is only very slowly beginning to understand the role of $\neg\neg$ in intuitionistic mathematics, and hopes to discuss it again at another time.

A different kind of consequence of the method used to verify (6.1) shows us that for arbitrary formulae A

$$\exists y[y > 0 \leftrightarrow A] \tag{6.7}$$

is valid in the model. This is a strong version of *Kripke's schema* first mentioned in print by Myhill [7], p. 174. Myhill's weaker statement of the principle would read in our notation:

$$\exists y[[y \leqslant 0 \rightarrow \neg A] \wedge [y > 0 \rightarrow A]].$$

Clearly this statement follows from (6.7) but one would guess not conversely. Assuming that the system of analysis supplied by our model is at all reasonable intuitionistically, we thus obtain a consistency proof for this form of Kripke's schema. As Myhill has already pointed out, (6.7) is incompatible with $\forall\alpha\exists\beta$-continuity ([7], pp. 173–174). Thus to justify the reasonableness of our model, we must now turn to a discussion of what continuity properties actually are valid since such principles are desirable intuitionistically.

7. Functions and continuity. We are now going to enrich our language of analysis by the adjunction of *function variables*. We reserve f, g, h (with or without subscripts) for such variables and employ the usual function-value notation $f(x)$ in the formal language. We are assuming our functions to be everywhere defined, real-valued functions of *one* argument. The extension of the language and the results to functions of several arguments seems to present no difficulties and is left to the reader.

The interpretation of these functions in the model will be given in a very straightforward way and will lead to several obviously basic principles being valid. In the first place our functions will be extensional:

$$\forall f \forall x, y[x = y \rightarrow f(x) = f(y)], \tag{7.1}$$

and *apartness* and *equality* of functions will be given by the following definitions:

$$\forall f, g[f \neq g \leftrightarrow \exists x\, f(x) \neq g(x)], \tag{7.2}$$

$$\forall f, g[f = g \leftrightarrow \neg f \neq g]. \tag{7.3}$$

Next we will have the principle of *function existence*:

$$\forall x \exists! y\; A(x, y) \rightarrow \exists! f \forall x\; A(x, f(x)), \tag{7.4}$$

where $A(x, y)$ is an arbitrary formula which is assumed to be *extensional* (as are all of our formulae for the time being) in the sense that

$$\forall x, x', y, y'\; [A(x, y) \wedge x = x' \wedge y = y' \rightarrow A(x', y')]$$

must be valid.

There are no surprises here, since (7.1)–(7.4) are clearly rather fundamental for any reasonable theory of functions. What is surprising is that, without seeming to have built the fact into our interpretation, we do validate the principle of *continuity*:

$$\forall f \forall z, w \forall e > 0 \exists d > 0 \forall x, y \in [z, w][|x-y| < d \rightarrow |f(x)-f(y)| < e], \tag{7.5}$$

where we have used e and d as rational variables in place of the conventional ε and δ. Note that (7.5) is not just *point-wise* continuity of every function but *uniform* continuity on every closed interval.

A simple consequence of uniform continuity is: *every rational-valued function is constant.* A moment's reflection shows it is enough to prove that every *integer-valued* function is constant; and, even simpler, it is sufficient to prove that every *two-valued* function is constant. This last principle has a simple schematic statement:

$$\forall x[A(x) \vee \neg A(x)] \rightarrow [\forall x A(x) \vee \forall x \neg A(x)], \tag{7.6}$$

a principle which states, in Brouwer's terminology, that the continuum is 'unzerlegbar' (cf. [4], *R10.4, p. 155). The proof of (7.6) begins with the introduction of a function f such that

$$\forall x[[A(x) \wedge f(x) = 0] \vee [\neg A(x) \wedge f(x) = 1]],$$

which is justified by (7.4). Then one applies (7.5) with $e = 1$ for an arbitrary pair of numbers z, w. The interval $[z, w]$ can easily be covered by intervals of length d, where d is supplied by (7.5), and the inequalities together with an inductive argument imply that f is constant on $[z, w]$.

It is tempting on rather formal grounds to assume a version of the *axiom of choice*:

$$\forall x \exists y A(x, y) \rightarrow \exists f \forall x A(x, f(x)), \tag{AC-RR}$$

where we have followed Kreisel [8], pp. 233f, in style of terminology with RR meaning that the choices are from *reals* to *reals*. Without thought, (AC-RR) would seem to be a reasonable strengthening of (7.4); however, in view of the remarks in connection with (7.6) we see that it is actually *invalid* for the *extensional* notion of function intended here. Indeed the formula $[x < y \wedge y \in Q]$ gives the obvious counter-example.

It is quite pleasant that to be able to achieve all the above results the interpretation in the model can be as simple as possible: we take as functions those mappings $\varphi : \mathscr{R} \to \mathscr{R}$ such that for $\xi, \eta \in \mathscr{R}$ we have

$$\llbracket \xi = \eta \rrbracket \subseteq \llbracket \varphi(\xi) = \varphi(\eta) \rrbracket.$$

This is the extensionality principle: we make (7.1) valid *by definition*. We let $\mathscr{R}^{\mathscr{R}}$ denote the class of all these extensional functions $\varphi : \mathscr{R} \to \mathscr{R}$. Further we define for $\varphi, \psi \in \mathscr{R}^{\mathscr{R}}$

$$\llbracket \varphi \neq \psi \rrbracket = \bigcup_{\xi \in \mathscr{R}} \llbracket \varphi(\xi) \neq \psi(\xi) \rrbracket.$$

This convention makes (7.2) valid also *by definition*; similarly for (7.3). But this is all that is trivial by definition; the remaining facts require proof.

To prove the validity of (7.4) it is clearly enough to prove *existence* of f, because the uniqueness will follow in view of the hypothesis and (7.2–7.3). Thus we must show that

$$\llbracket \forall x \exists ! y A(x, y) \rrbracket \subseteq \llbracket \exists f \forall x A(x, f(x)) \rrbracket.$$

We will take advantage here of the fact that $T = N^N$ is *totally disconnected* (though no doubt all we really need to know is that T is metric.) In such a space every open set is the union of its *clopen* subsets. Hence it is sufficient to prove for every clopen set $K \subseteq T$ that if

$$K \subseteq \llbracket \forall x \exists ! y A(x, y) \rrbracket,$$

then

$$K \subseteq \llbracket \exists f \forall x A(x, f(x)) \rrbracket.$$

Indeed, to prove this last inclusion, we need only find a function $\varphi_K \in \mathscr{R}^{\mathscr{R}}$ corresponding to K such that

$$K \subseteq \llbracket \forall x A(x, \varphi_K(x)) \rrbracket.$$

The advantage of working with *clopen* K is that for $t \in K$ and $\xi \in \mathscr{R}$ we will have a special way of defining $\varphi_K(\xi)(t)$, but for $t \notin K$ we can simply set $\varphi_K(\xi)(t) = 0$. The clopenness of K will assure that $\varphi_K(\xi) \in \mathscr{R}$ so defined is *continuous*.

Thus let $\xi \in \mathscr{R}$ be fixed. Our assumption implies that

$$K \subseteq \llbracket \exists ! y A(\xi, y) \rrbracket.$$

To define $\varphi_K(\xi)$ we use the same method as in section 2 and set

$$\varphi_K(\xi)(t) = \inf \{ r \in Q : t \in \llbracket \exists y [A(\xi, y) \wedge y < r] \rrbracket \}$$

for $t \in K$; otherwise the value is 0, as mentioned above. As in the proof of (2.7) it is elementary to check that

$$K \cap \varphi_K(\xi)^{-1}(q, q') = K \cap \llbracket \exists y [A(\xi, y) \wedge q < y < q'] \rrbracket,$$

which shows that $\varphi_K(\xi)$ is continuous on K. Therefore $\varphi_K(\xi)$ is continuous everywhere in T. This does not yet prove that $\varphi_K \in \mathscr{R}$. We must still verify

$$K \cap \llbracket \xi = \eta \rrbracket \subseteq \llbracket \varphi_K(\xi) = \varphi_K(\eta) \rrbracket;$$

but this easily follows from the extensionality of A. Since $\varphi_K(\xi)$ is 0 outside of K, we see that indeed $\varphi_K \in \mathscr{R}^{\mathscr{R}}$. Finally it is clear by construction that

$$K \subseteq \llbracket A(\xi, \varphi_K(\xi)) \rrbracket,$$

and the desired conclusion follows.

The proof that (7.5) is valid is by no means as straightforward. Note first that (7.5) implies

$$\forall f \forall x, y [f(x) \neq f(y) \to x \neq y], \qquad (7.7)$$

which is a stronger version of (7.1). As it turned out, it is very useful to validate (7.7) first as a lemma, and then establish (7.5). (This point was a stumbling block for the author, who formulated the problem in a lecture at the University of Texas at Austin. Shortly thereafter the idea of the following proof was communicated to the author by Professor Jack Hardy, whose assistance it is a pleasure to acknowledge.)

To prove (7,6) we note first that what we have to establish is this inclusion for $\xi, \eta \in \mathscr{R}$ and $\varphi \in \mathscr{R}^{\mathscr{R}}$:

$$\llbracket \varphi(\xi) \neq \varphi(\eta) \rrbracket \subseteq \llbracket \xi \neq \eta \rrbracket.$$

Now after translation back to subsets of T, we can complement both sides of the inclusion to obtain:

$$\{ t \in T : \xi(t) = \eta(t) \} \subseteq \{ t \in T : \varphi(\xi)(t) = \varphi(\eta)(t) \}. \qquad (*)$$

That is what we must prove; what we are assuming (because $\varphi \in \mathscr{R}^{\mathscr{R}}$) is equivalent to:

$$\mathrm{In}\, \{ t \in T : \xi(t) = \eta(t) \} \subseteq \{ t \in T : \varphi(\xi)(t) = \varphi(\eta)(t) \}. \qquad (**)$$

We can state the problem in words: (*) means that φ is an operator on continuous functions (to continuous functions) such that if two arguments ξ, η are equal at a point, then their images $\varphi(\xi)$, $\varphi(\eta)$ are equal at the *same* point. Thus $\varphi(\xi)(t)$ only depends on t and $\xi(t)$, not on the whole of ξ. It would seem that (**) is much weaker: $\varphi(\xi)(t)$ depends on knowing t and knowing ξ *in some neighborhood of* t, that is, on the local behaviorof ξ around t. (The author imagined at first that one could have a function φ satisfying (**) without its satisfying (*). It seems odd that a problem like this has not been considered before, but inquiry failed to uncover any relevant information. It would fair to say that most of the operators considered in mathematics usually depend on the *whole* of ξ (like integration) or do *not* give continuous functions $\varphi(\xi)$ as values (like right-hand derivatives).)

The first main step in deducing (*) (for *all* $\xi, \eta \in \mathscr{R}$) from (**) (for *all* $\xi, \eta \in \mathscr{R}$) is to remark that the set on the right-hand side of (**) is *closed* because $\varphi(\xi)$ and $\varphi(\eta)$ are each *continuous*. This means that we can strenghten (**) to read:

$$\mathrm{Cl\,In}\{t \in T : \xi(t) = \eta(t)\} \subseteq \{t \in T : \varphi(\xi)(t) = \varphi(\eta)(t)\}, \qquad (***)$$

which we are assuming for *all* $\xi, \eta \in \mathscr{R}$. Now we must prove (*) for a particular pair $\xi_0, \eta_0 \in \mathscr{R}$. Suppose $t_0 \in T$ is a point such that $\xi_0(t_0) = \eta_0(t_0)$. We will show below that we can find *open* subsets U, V of T such that

$$\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \{t_0\},$$

and such that ξ_0 is *bounded* on U and η_0 is *bounded* on V. Now the function

$$\zeta_0 = \xi_0 \restriction \mathrm{Cl}(U) \cup \eta_0 \restriction \mathrm{Cl}(V)$$

is obviously *continuous* on $\mathrm{Cl}(U) \cup \mathrm{Cl}(V)$ because $\xi_0(t_0) = \eta_0(t_0)$, and it is *bounded*. Thus by the well-known Tietze extension theorem (T is a metric space) we can extend the function to the whole of T. We call the extension ζ_0 also. (This construction of ζ_0, which is due to Prof. Hardy, is the main idea of the proof.) Now we make use of (***) for ξ_0, ζ_0 and for ζ_0, η_0. In the first instance we see by the construction of ζ_0 that:

$$t_0 \in \mathrm{Cl\,In}\{t \in T : \xi_0(t) = \zeta_0(t)\},$$

and so by (***), $\varphi(\xi_0)(t_0) = \varphi(\zeta_0)(t_0)$. In the second instance we have

$$t_0 \in \mathrm{Cl\,In}\{t \in T : \zeta_0(t) = \eta_0(t)\},$$

and so again by (***), $\varphi(\zeta_0)(t_0) = \varphi(\eta_0)(t_0)$. Now it follows at once that $\varphi(\xi_0)(t_0) = \varphi(\eta_0)(t_0)$, which is what we needed to establish (*).

Filling in the part of the argument about U and V that we had postponed, we note first that if t_0 is *isolated* we can take $U = V = \{t_0\}$. Assume then that t_0 is not isolated and let $t_1, t_2, t_3, \ldots, t_n, \ldots$ be an infinite sequence of *distinct* points converging to t_0. Inasmuch as ξ_0 and η_0 are continuous, they both must be *bounded* on $\{t_1, t_2, \ldots, t_n, \ldots\}$. Around each of these points we can put small *open* spheres with $t_n \in S_n$ such that the *closures* of the spheres are pairwise disjoint (T is a metric space!) and on which ξ_0 and η_0 are uniformly bounded. It follows that

$$U = \bigcup_{n=1}^{\infty} S_{2n}$$

and

$$V = \bigcup_{n=1}^{\infty} S_{2n-1}$$

are the required open sets.

We shall now investigate the nature of the functions $\varphi \in \mathscr{R}^{\mathscr{R}}$ more closely. The reader will appreciate that it is necessary to keep his wits about him in thinking about the model. We already use *functions* $\xi : T \to R$ to play the role of *real* numbers in the model, so that when we come to *functions* in the model we have to use *operators* $\varphi : \mathscr{R} \to \mathscr{R}$. Thus $\varphi(\xi)$ is a function for each $\xi \in \mathscr{R}$, as we had to consider in the above proof of (7.7). Looking from the outside, the type of the objects used in the model is always *higher* than the formal type of the variable which ranges over them. This process of going up in type for the definition of the model is very convenient for making the construction simple, but we shall now show that a reduction in type is possible helping in the analysis of the model.

Since $\varphi \in \mathscr{R}^{\mathscr{R}}$ satisfies (*) we define $\Phi : T \times R \to R$ by the condition:

$$\Phi(t, a) = b \textit{ iff for some } \xi \in \mathscr{R},\ \xi(t) = a \text{ and } \varphi(\xi)(t) = b.$$

We thus have Φ well-dertermined and such that for $\xi \in \mathscr{R}$, $t \in T$:

$$\varphi(\xi)(t) = \Phi(t, \xi(t)).$$

The point of bringing in Φ is that, as we shall now show, the function Φ is *continuous* on the *product space* $T \times R$. We argue by contradiction: suppose that Φ is *not* continuous at some point (t, a). Then there must exist an $\varepsilon > 0$ and a sequence of neighborhoods U_n of t and a sequence of reals δ_n, such that the U_n monotonically decrease to $\{t\}$ and the δ_n to 0. Further we can find a sequence of points (t_n, a_n) where $t_n \in U_n$, and $|a_n - a| < \delta_n$, and

$$|\Phi(t_n, a_n) - \Phi(t, a)| > \varepsilon$$

for all n. Since for fixed a_n, a the two expressions $\Phi(t, a_n)$ and $\Phi(t, a)$ represet *continuous* functions of t, we can clearly assume that all $t_n \neq t$. (Otherwise we could 'adjust' $t_n \in U_n$ and still preserve the above strict inequality). Furthermore, because the diameters of the U_n are converging to 0, we can assume that the t_n are all distinct. (If not, we can choose a suitable subsequence.) After that fuss, we can now remark that we are able to construct a *continuous* function $\xi \in \mathscr{R}$ such that $\xi(t_n) = a_n$ and $\xi(t) = a$, because the a_n *do* converge to a. But then since $\varphi(\xi)$ is continuous, $\varphi(\xi)(t_n)$ would have to converge to $\varphi(\xi)(t)$, which contradicts the above inequality.

Now that we see why Φ is continuous, we can quickly establish the validity of (7.5) for φ. Note first that it is sufficient to consider $z = q$ and $w = r$, where $q, r \in Q$ and $q < r$. Let $\varepsilon > 0$, $\varepsilon \in Q$. We must show:

$$T = \bigcup_{\delta < 0} \text{In} \bigcap_{\xi, \eta \in \mathscr{R}} \text{In}\ \{t \in T: [\xi(t), \eta(t) \in [q, r] \wedge |\xi(t) - \eta(t)| < \delta \rightarrow |\Phi(t, \xi(t)) - \Phi(t, \eta(t))| < \varepsilon]\}.$$

Introduce an auxiliary function $\varepsilon: T \times (0, \infty) \rightarrow R$ by:

$$\varepsilon(t, \delta) = \sup \{|\Phi(t, a) - \Phi(t, b)| : a, b \in [q, r],\ |a - b| \leqslant \delta\}.$$

The function inside the sup is a continuous function of three variables t, a, b, and the sup is being taken over a *compact* subset of $R \times R$. It follows for a fixed δ that $\varepsilon(t, \delta)$ is a continuous (and well-defined!) function of t. Since for fixed t the real function $\Phi(t, a)$ is *uniformily continuous* for $a \in [q, r]$, we conclude that $\varepsilon(t, \delta)$ decreases to 0 as δ decreases to 0. Let, then, $t_0 \in T$ and choose δ so that $\varepsilon(t_0, \delta) < \varepsilon$. Let $U \subseteq T$ be a neighborhood of t_0 such that $\varepsilon(t, \delta) < \varepsilon$ for all $t \in U$. We must verify that U is included in the right-hand side of the above topological equation. It is immediate.

The method of proof just employed for (7.5) is also very helpful in other problems: for example we shall now validate in the model a principle which states that every positive function on a closed interval is bounded away from zero:

$$\forall f \forall z, w[\forall x \in [z, w][f(x) > 0] \rightarrow \exists y > 0 \forall x \in [z, w][f(x) \geqslant y]]. \quad (7.8)$$

In view of continuity, and the denseness of the rationals, it is sufficient to

treat only the case where $z = q$, $w = r$, $q < r$. For simplicity we shall in addition assume for our $\varphi \in \mathscr{R}^{\mathscr{R}}$ that

$$T = [\![\forall x \in [q, r][\varphi(x) > 0]]\!].$$

The more general case can be treated as in the proof of (7.4). In terms of Φ we are thus assuming that

$$\Phi(t, a) > 0$$

for all $t \in T$ and all $a \in [q, r]$. Define $\eta \in \mathscr{R}$ by $\eta(t) = \inf\{\Phi(t, a) : a \in [q, r]\}$ for $t \in T$. Again by virtue of the compactness of $[q, r]$, we can check that η is continuous. But for fixed t we have $\Phi(t, a)$ as a positive continuous function of $a \in [q, r]$, therefore $\eta(t) > 0$ for all $t \in T$. We can thus easily verify that

$$T = [\![\forall x \in [q, r][\varphi(x) \geqslant \eta]]\!],$$

and (7.8) is established. We may remark that this construction of η shows also that every function has a greatest lower bound on every closed interval – a fact that can fairly easily be deduced from continuity (cf [1], p. 35 for example.)

Our reduction of the $\varphi: \mathscr{R} \rightarrow \mathscr{R}$ to the $\Phi: T \times \mathrm{R} \rightarrow R$ is very useful in discussing other questions about the model. For example, every ordinary continuous function $F: R \rightarrow R$ can be extended to $\mathscr{R}$, as we have often remarked, by the formula

$$\varphi(\xi)(t) = F(\xi(t)).$$

The question is whether *every* $\varphi \in \mathscr{R}^{\mathscr{R}}$ is *ordinary* in this sense. Well, obviously not: let $\alpha \in \mathscr{R}$ be fixed and define

$$\varphi(\xi)(t) = \xi(t) + \alpha(t).$$

Could there be an $F: R \rightarrow R$ such that

$$F(\xi(t)) = \xi(t) + \alpha(t)$$

holds in general? If so, we could take ξ to be constant, say 0, and deduce

$$F(0) = \alpha(t),$$

thus making α constant. Since α need not be constant, the desired counterexample is produced.

The foregoing simple argument leads naturally to a modification of our question: given $\varphi \in \mathscr{R}^{\mathscr{R}}$ could we find a continuous function $G : R \times R \to R$ and a particular $\alpha \in \mathscr{R}$ such that

$$\varphi(\xi)(t) = G(\alpha(t), \xi(t))$$

for all $\xi \in \mathscr{R}$ and $t \in T$? The answer is again *no*, but the argument is more difficult. (The author is indebted to Kenneth Kunen for the suggestion of the counter-example.) In terms of the correspondng Φ the problem is to find G and α such that

$$\Phi(t, \xi(t)) = G(\alpha(t), \xi(t))$$

for all $\xi \in \mathscr{R}$ and $t \in T$. This simplifies to

$$\Phi(t, a) = G(\alpha(t), a)$$

for all $a \in R$ and $t \in T$. We have only to choose a very bad Φ to get into trouble. In view of the universal properties of N^N, we can choose Φ so that the functions $\psi_t : R \to R$ where

$$\psi_t(a) = \Phi(t, a)$$

give us *all* continuous functions from reals into reals. That is, Φ would be a *universal* function for all real continuous functions with the parameter in $T = N^N$. That is possible; but, by the equation for G above, *it* would also be universal with parameter in R. That is *not* possible.

We have just shown that in the model not every function need be ordinary or even quasi-ordinary (i.e. ordinary in a parameter α.) There is a common special case where we can, however, conclude that the function is indeed ordinary. Suppose φ has been introduced to validate

$$\forall x\, A(x, \varphi(x))$$

where $\forall x \exists! y A(x, y)$ is valid *and* $A(x, y)$ has *no* additional parameters. (The formula can involve $<, +, \cdot$, other ordinary functions and predicates, the predicates Q and D, and so on.) We note first that φ must be somehow invariantly determined.

Now we ask: in what sense is φ invariant? Well, let $\tau : T \to T$ be any autohomeomorphism of T. This extends to the model. For $\xi \in \mathscr{R}$ we define $\tau(\xi)$ so that

$$\tau(\xi)(\tau(t)) = \xi(t)$$

for all $t \in T$. (Thus $\tau(\xi) = \xi \circ \tau^{-1}$) This makes it possible to regard $\tau : \mathscr{R} \to \mathscr{R}$. However, it is *not* the case in general that $\tau \in \mathscr{R}^{\mathscr{R}}$; for we have

$$[\![\tau(\xi) = \tau(\eta)]\!] = \tau([\![\xi = \eta]\!]).$$

(Recall that τ is naturally defined on open subsets of T also!) We can regard τ as acting on $\mathscr{R}^{\mathscr{R}}$, nevertheless. For any $\psi \in \mathscr{R}^{\mathscr{R}}$ we define $\tau(\psi)$ so that

$$\tau(\psi)(\tau(\xi)) = \tau(\psi(\xi))$$

for all $\xi \in \mathscr{R}$. It is easy to check that $\tau(\psi) \in \mathscr{R}$. Having done this, we remark that by the usual kind or argument, if B is any formula involving ordinary functions and predicates bound real and function variables, *and* parameters from $\mathscr{R}$ and $\mathscr{R}^{\mathscr{R}}$, then

$$\tau([\![B(\xi, \ldots, \psi, \ldots)]\!]) = [\![B(\tau(\xi), \ldots, \tau(\psi), \ldots)]\!].$$

Applying this automorphism principle to A we have

$$\tau([\![A(\xi, \eta)]\!]) = [\![A(\tau(\xi), \tau(\eta))]\!]$$

because there are no other parameters. But

$$[\![A(\xi, \eta)]\!] = [\![\varphi(\xi) = \eta]\!],$$

hence

$$[\![\tau(\varphi)(\tau(\xi)) = \tau(\eta)]\!] = [\![\varphi(\tau(\xi)) = \tau(\eta)]\!].$$

It follows at once that $\tau(\varphi) = \varphi$.

We have thus proved that if $\varphi \in \mathscr{R}^{\mathscr{R}}$ is defined by a parameterless formula, then it is invariant under all automorphisms of the model determined by autohomeomorphisms of T. Let us see what this means about the corresponding continuous function $\Phi : T \times R \to R$. Now

$$\varphi(\xi)(t) = \Phi(t, \xi(t)),$$

so by substitution

$$\begin{aligned}\varphi(\tau(\xi))(\tau(t)) &= \Phi(\tau(t), \tau(\xi)(\tau(t))) \\ &= \Phi(\tau(t), \xi(t)).\end{aligned}$$

But $\tau(\varphi) = \varphi$, so we derive

$$\begin{aligned}\varphi(\tau(\xi))(\tau(t)) &= \tau(\varphi)(\tau(\xi))(\tau(t)) \\ &= \tau(\varphi(\xi))(\tau(t)) \\ &= \varphi(\xi)(t).\end{aligned}$$

Putting two and two together we find:

$$\Phi(t, \xi(t)) = \Phi(\tau(t), \xi(t)).$$

Since ξ is arbitrary we can say

$$\Phi(t, a) = \Phi(\tau(t), a)$$

for all $t \in T$ and all $a \in R$. But hold, the autohomeomorphism τ is arbitrary, and $T = N^N$ has a transitive autohomeomorphism group; thus

$$\Phi(t, a) = \Phi(t', a)$$

for all $t, t' \in T$. This means that Φ does not depend on t at all and we can write

$$\Phi(t, a) = F(a)$$

where $F: R \to R$ so obtained is continuous. Thus

$$\varphi(\xi)(t) = F(\xi(t)),$$

and we have finally proved that φ is just an ordinary function.

In words we could say that the above argument suggests that in intuitionistic analysis it is impossible to give an outright *extensional* definition of any function not already known from classical analysis. (Of course, the simple-minded identification of intuitionistic analysis with the theory of this model is not justified because the model is defined classically.)

We close this section with a problem about continuity. In (7.6) the hypothesis means that the property $A(x)$ decomposes the continuum into two disjoint parts. Let us weaken this by having *two* properties $A(x)$ and $B(x)$ where $\forall x[A(x) \vee B(x)]$, but where we do not make any disjointness assumption. (For example $\forall x[x < 1 \vee x > 0]$ is valid.) We can obviously not hope to have any such strong conclusion as $[\forall x A(x) \vee \forall x B(x)]$ (by the example!), but what about this principle:

$$\forall x[A(x) \vee B(x)] \to \exists q, r[q < r \wedge [\forall x \in [q, r]\, A(x) \vee \forall x \in [q, r]\, B(x)]]?$$

The author has not yet been able to see an answer to the question of the validity of this formula. The situation may become clearer after a study of second-order arithmetic in the model.

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