1 Category theory

Definition 1.1. A *category* $\mathcal{C}$ is a collection $\mathcal{C}_0$ of *objects* and a collection $\mathcal{C}_1$ of *arrows* or *morphisms* such that the following holds.

- Each arrow has an object X in $\mathcal{C}_0$ as *domain* and an object Y as *codomain*, this is referred to as an arrow from X to Y.
- Given two arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, there is a composition $X \xrightarrow{g \circ f} Z$. This composition must be associative.
- Every object X has an identity arrow $Id_X : X \to X$ such that $f \circ Id_X = f$ and $Id_X \circ g = g$ for all $f : X \to Y$ and all $g : Y \to X$.

Example 1.1. There is a category Set with all possible sets as objects and the usual functions as arrows.

Example 1.2. Given a poset P, we can see this as a category with as objects the elements of P and an arrow $p \to q$ iff $p \leq q$.

Definition 1.2. A *functor* F between categories $\mathcal{C}$ and $\mathcal{D}$ consists of operations $F_0 : \mathcal{C}_0 \to \mathcal{C}_1$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ such that for each arrow $f : X \to Y$ in $\mathcal{C}$ we have $F_1(f) : F_0(X) \to F_1(Y)$. Furthermore, F should respect identities and composition:

- for $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $F_1(g \circ f) = F_1(g) \circ F_1(f)$,
- for every X in $\mathcal{C}$ we have $F_1(Id_X) = Id_{F_0(X)}$.

Example 1.3. Let Top be the category of topological spaces with continuous functions as arrows. Then there exists a functor, the *forgetful functor*, that assigns to each topological space its underlying set.

Definition 1.3. For any category $\mathcal{C}$ we can define $\mathcal{C}^{op}$ which consists of the same objects and the same arrows, only we reverse the direction of all arrows.

Example 1.4. Given a topological space X we can define a category $\mathcal{O}(X)$ with as objects all opens in X and an arrow $U \to V$ iff $U \subset V$. This also gives rise to the category $\mathcal{O}(X)^{op}$, consisting of the opens of X and an arrow $V \to U$ iff $V \supset U$.

Another category we can construct is $\mathcal{C}(X)$, with for every open U of X an object $C(U)$, the set of continuous functions on U. For each $V \supset U$ we make an arrow $C(V) \to C(U)$, which is just the restriction of the functions in $C(V)$ to U.

Now we can define a functor $F : \mathcal{O}(X)^{op} \to \mathcal{C}(X)$ by sending each open U to the set $C(U)$ of continuous functions on U, and each inclusion $V \supset U$ to the restriction of $C(V)$ to $C(U)$.

2 Sheaves

Definition 2.1. A *presheaf* on a category $\mathcal{C}$ is a functor $F : \mathcal{C}^{op} \to \text{Set}$.

Example 2.1. The functor we have seen in example 1.4 is a presheaf on $\mathcal{O}(X)$. Even though it may not be a functor to Set, we can compose it with the forgetful functor (see example 1.3) to obtain a functor to Set.

Remember from example 1.2 that every poset can also be seen as a category. Hence we can see every Heyting algebra as a category.

Definition 2.2. For a complete Heyting algebra $\mathcal{C}$, we say that F is a *sheaf* on Ω if it is a presheaf on $\mathcal{C}$ that satisfies the sheaf condition. This means that for each $A \subset \mathcal{C}$ (with $p = \bigvee A$) we have that given a family $\{x_a \in F_0(a)\}_{a \in A}$ such that for all $a, a' \in A$ we have

$$F_1(a \wedge a' \leq a)(x_a) = F_1(a \wedge a' \leq a')(x_{a'}),$$

there is a unique $x \in F_0(p)$ such that for all $a \in A$ we have $F_1(a \leq p)(x) = x_a$.

Such a family $\{x_a\}_{a \in A}$ is called a *compatible family*, and the corresponding x is called the *amalgamation*. So in other words: a sheaf should have a unique amalgamation for each compatible family.

Example 2.2. The presheaf from example 2.1 is a sheaf.

Definition 2.3. A *subsheaf* $H \subset F$ is a functor H such that:

- $H(X) \subset F(X)$ for each object X,
- $H(f) = F(f)|_{H(Y)}$ for each arrow $f : X \to Y$ and
- H itself is a sheaf.

3 Equivalence of Ω-sets and the sheaves on Ω

Definition 3.1. Let Ω be a complete Heyting algebra. We define the category of Ω-sets as follows. An object is a pair (X, δ) with $\delta : X \times X \to \Omega$ such that for all $x, y, z \in X$:

- $\delta(x, y) \wedge \delta(y, z) \leq \delta(x, z)$ and
- $\delta(x, y) = \delta(y, x)$.

An arrow $(X, \delta) \to (Y, \varepsilon)$ is then given by a function $f : X \times Y \to \Omega$, such that for all $x, x' \in X$ and $y, y' \in Y$:

(1) $f(x, y) \leq \delta(x, x) \wedge \varepsilon(y, y)$,

(2) $\delta(x, x') \wedge f(x, y) \wedge \varepsilon(y, y') \leq f(x', y')$,

(3) $\delta(x, x) \leq \bigvee_{y \in Y} f(x, y)$ and

(4) $f(x, y) \wedge f(x, y') \leq \varepsilon(y, y')$.

Composition of $(X, \delta) \xrightarrow{f} (Y, \varepsilon) \xrightarrow{g} (Z, \eta)$ is then given by $(g \circ f)(x, z) = \bigvee_{y \in Y} f(x, y) \wedge g(y, z)$.

Definition 3.2. For each $p \in \Omega$ we define the Ω-set $1_p = (\{*_p\}, \delta_p)$ with $\delta_p(*_p, *_p) = p$.

Lemma 3.1. *For any $q \leq p$ in Ω we have a unique arrow $e_{qp} : 1_q \to 1_p$ given by $e_{qp}(*_q, *_p) = q$.*

Theorem 3.1. *The category of* Ω-sets *is equivalent to the category* $\mathrm{Sh}(\Omega)$ *of sheaves on* Ω.

4 Lattice of Subsheaves

The subsheaf relation $\subset$ defines a poset structure on the subsheaves of a fixed sheaf F. We show that this structure is in fact a complete Heyting algebra, and relate this to the Heyting algebra we are taking sheaves over.

For the rest of this section, fix a complete Heyting algebra Ω. This work is based on chapters nine and ten of Jaap van Oosten's *Basic Category Theory and Topos Theory* lecture notes (BCTTT).

Lemma 4.1. *For any sheaf F over Ω, $F_\perp = \{*\}$.*

Fix now a sheaf F over Ω. Let $\mathrm{Sub}(F)$ be the poset of subsheaves of F ordered by inclusion.

Lemma 4.2. $\mathrm{Sub}(F)$ *has a greatest element, namely F itself.*

Lemma 4.3. $\mathrm{Sub}(F)$ *has a least element, which is $\emptyset$ everywhere except at $\perp \in \Omega$.*

Lemma 4.4 (c.f. BCTTT Lemma 10.13)**.** *For every $U \subset \mathrm{Sub}(F)$, the meet $\bigwedge U$ is given by the pointwise intersection*

$$(\bigwedge U)_p := \{x \in F_p \mid \forall u \in U. \, x \in u_p\}.$$

Lemma 4.5 (c.f. BCTTT Theorem 9.8)**.** *For all $A, B \subset F$, the implication $A \to B$ is given by*

$$(A \to B)_p := \{x \in F_p \mid \forall q \leq p. \, x|_q \in A_q \Rightarrow x|_q \in B_q\}.$$

Lemma 4.6 (c.f. BCTTT p. 112)**.** *For every $U \subset \mathrm{Sub}(F)$, the join $\bigvee U$ is given by*

$$(\bigvee U)_p := \{x \in F_p \mid p = \bigvee \{q \leq p \mid \exists u \in U. \, x|_q \in u_q\}\}.$$

Theorem 4.1. $\mathrm{Sub}(F)$ *is a complete Heyting algebra.*

Theorem 4.2. $\mathrm{Sub}(1) \cong \Omega$.