Seminar on Intuitionism

Hand-out lecture 4

March 9, 2017

1 First talk

1.1 The model

Definition. Given a topological space T, we create a model for intuitionistic logic by associating to each formula A an open set in T such that:

$$\begin{aligned}
[\![A \wedge B]\!] &= [\![A]\!] \cap [\![B]\!] \\
[\![A \vee B]\!] &= [\![A]\!] \cup [\![B]\!] \\
[\![\neg A]\!] &= Int(T - [\![A]\!]) \\
[\![A \to B]\!] &= Int((T - [\![A]\!]) \cup [\![B]\!]) \\
[\![\exists x A(x)]\!] &= \bigcup_{\xi \in \mathcal{R}} [\![A(\xi)]\!] \\
[\![\forall x A(x)]\!] &= Int \bigcap_{\xi \in \mathcal{R}} [\![A(\xi)]\!]
\end{aligned}$$

We say that a formula A is *valid* in this model if $[\![A]\!] = T$.

Theorem. (Rasiowa-Sikorski) If A is provable in IQC, then $[\![A]\!] = T$.

Remarks. We will assume that $\mathbb{Q}$ is contained in the domain $\mathcal{R}$, and that the following formulas are always valid, where q, r denote variables in the rationals:

1. $\forall x, y \neg(x < y \wedge x < y)$
2. $\forall x, y, z(x < y \to (x < z \vee z < y))$
3. $\forall x \exists q, r(q < x \wedge x < r)$
4. $\forall x, y(x < y \to \exists q(x < q \wedge q < y))$

Theorem. For $\xi \in \mathcal{R}$, define the function $\xi : T \to \mathbb{R}$ by $\xi(t) = \inf\{r \in \mathbb{Q} \mid t \in [\![\xi < r]\!]\}$. Then this function is continuous.

Remarks. From the above, it follows that $[\![\xi < \eta]\!] = \{t \in T \mid \xi(t) < \eta(t)\}$. From now on we take $\mathcal{R}$ to be the collection of all continuous functions $T \to \mathbb{R}$. We also fix our topological space T to be the *Baire space*. This is the space of all infinite sequences of natural numbers. A basic open in this space is a set of infinite sequences containing exactly those sequences starting with a given finite sequence.

1.2 Decision method

Theorem. $\forall x, y A(x < y, y < x)$ is intuitionistically provable iff $\neg(P \wedge Q) \to A(P, Q)$ is provable in intuitionistic propositional logic.

Theorem. (Kreisel) A universal sentence is a consequence of a universal axiom in IQC iff it's matrix is a propositional consequence of a finite number of substitution instances of the axiom, using the variables in the conclusion.

Remark. From the above theorem by Kreisel, a generalization to an arbitrary number of variables of the decision method given above easily follows.

1.3 Completeness

Theorem. (Completeness) If a universal sentence is not intuitionistically provable, then it also fails in the model.

2 Second talk

2.1 Maximality of IPC

Definition. Let $\tau : T \to T$ be a homeomorphism. We define $\tau : \mathcal{R} \to \mathcal{R}$ by $\tau(\xi) = \xi \circ \tau^{-1}$ for $\xi \in \mathcal{R}$.

Properties.

(i) $\tau[\![\xi < \eta]\!] = [\![\tau(\xi) < \tau(\eta)]\!]$ for $\xi, \eta \in \mathcal{R}$.

(ii) $\tau[\![A(\xi_1, \cdots, \xi_k)]\!] = [\![A(\tau(\xi_1), \cdots, \tau(\xi_k))]\!]$ for all formulae $A(x_1, \ldots, x_k)$ containing no parameters in $\mathcal{R}$, and all $\xi_1, \ldots, \xi_k \in \mathcal{R}$.

Proposition. Let $A(p_1, \ldots, p_n)$ be a sentence in the language of the propositional calculus containing the propositional letters $p_1, \ldots, p_n$. Suppose that IPC $\nvdash A(p_1, \ldots, p_n)$. Then $[\![\neg\forall y_1, \ldots, y_k A(y_1 > 0, \ldots, y_k > 0)]\!] = T$.

2.2 Adding functions

We introduce variables $f, g, \dots$ that range over functions. We interpret them as elements of

$$\mathcal{R}^{\mathcal{R}} := \{\varphi : \mathcal{R} \to \mathcal{R} \mid \forall \xi, \eta \in \mathcal{R}\,([\![\xi = \eta]\!] \subseteq [\![\varphi(\xi) = \varphi(\eta)]\!])\}.$$

Definition. For $\varphi, \psi \in \mathcal{R}^{\mathcal{R}}$, we set:

$$[\![\varphi \neq \psi]\!] = \bigcup_{\xi \in \mathcal{R}} [\![\varphi(\xi) \neq \psi(\xi)]\!];$$
$$[\![\varphi = \psi]\!] = \mathrm{Int}(T \backslash [\![\varphi \neq \psi]\!]).$$

2.3 Strict extensionality

Theorem. $[\![\forall f \forall xy\,(f(x) \neq f(y) \to x \neq y)]\!] = T$.

Definition. Given $\varphi \in \mathcal{R}$, define $\Phi : T \times \mathbb{R} \to \mathbb{R}$ such that:

$$\text{For all } \xi \in \mathcal{R} : \quad \text{if } \xi(t) = a, \text{ then } \Phi(t, a) = \varphi(\xi)(t).$$

Properties.

(i) $\Phi(t, \xi(t)) = \varphi(\xi)(t)$ for all $t \in T$, $\xi \in \mathcal{R}$ and $\varphi \in \mathcal{R}^{\mathcal{R}}$.

(ii) For all $\varphi \in \mathcal{R}^{\mathcal{R}}$, the function Φ is continuous.

2.4 Unique choice

Consider formulae $A(x, y)$ such that $\forall x x' y y'\,(A(x, y) \wedge x = x' \wedge y = y' \to A(x', y'))]\!] = T$.

Theorem. $[\![\forall x \exists! y\, A(x, y) \to \exists f \forall x\, A(x, f(x))]\!] = T$

Definition. For a homeomorphism $\tau : T \to T$ and its corresponding $\tau : \mathcal{R} \to \mathcal{R}$, we define $\tau : \mathcal{R}^{\mathcal{R}} \to \mathcal{R}^{\mathcal{R}}$ by $\tau(\varphi) = \tau \circ \varphi \circ \tau^{-1}$, for $\varphi \in \mathcal{R}^{\mathcal{R}}$.

Properties.

(i) $\tau(\varphi) \in \mathcal{R}^{\mathcal{R}}$ for all $\varphi \in \mathcal{R}^{\mathcal{R}}$.

(ii) $\tau[\![\varphi \neq \psi]\!] = [\![\tau(\varphi) \neq \tau(\psi)]\!]$ for all $\varphi, \psi \in \mathcal{R}^{\mathcal{R}}$.

(iii) $\tau[\![A(\xi_1, \cdots, \xi_k, \varphi_1, \dots, \varphi_\ell)]\!] = [\![A(\tau(\xi_1), \cdots, \tau(\xi_k), \tau(\varphi_1), \dots, \tau(\varphi_\ell))]\!]$ for all formulae $A(x_1, \dots, x_k, f_1, \dots, f_\ell)$ containing no parameters in $\mathcal{R}$ or $\mathcal{R}^{\mathcal{R}}$, all $\xi_1, \dots, \xi_k \in \mathcal{R}$, and all $\varphi_1, \dots, \varphi_\ell \in \mathcal{R}^{\mathcal{R}}$.

Theorem. Suppose $A(x,y)$ contains no parameters in $\mathcal{R}$ or $\mathcal{R}^{\mathcal{R}}$, and let $\varphi \in \mathcal{R}$. If $[\![\forall x \exists! y\, A(x,y)]\!] = T$ and $[\![\forall x\, A(x,\varphi(x))]\!] = T$, then there exists a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $\varphi(\xi) = F \circ \xi$ for all $\xi \in \mathbb{R}$.