Seminar on Models of Intuitionism - Läuchli realizability

Hand-out lecture 8

April 13, 2017

1 Proof assignments

During this talk, countably infinite sets Γ and Π, and a fixed element $c_0 \in \Gamma$ are given. We consider a basic language $\mathcal{L}$ without function or constant symbols. We adopt the following convention: the tuple $\langle a, b\rangle$ denotes a function f such that $f(0) = a$ and $f(1) = b$.

Definition 1.1. For an $\mathcal{L}(\Gamma)$-formula A, we define $S(A)$ by:

- (i) $S(A) = \Pi$ if A is atomic;
- (ii) $S(A \wedge B) = S(A) \times S(B)$;
- (iii) $S(A \vee B) = S(A) \sqcup S(B) = (\{0\} \times S(A)) \cup (\{1\} \times S(B))$;
- (iv) $S(A \to B) = S(B)^{S(A)}$;
- (v) $S(\forall x\, A) = S(A)^{\Gamma}$;
- (vi) $S(\exists x\, A) = \Gamma \times S(A)$;

Using induction, one may show that $S(A[c/x]) = S(A)$ for all $c \in \Gamma$.

Definition 1.2. A *proof assignment* assign to each $\mathcal{L}(\Gamma)$-sentence A a set $p(A)$ such that:

- (i) $p(\bot) \subseteq p(A) \subseteq \Pi$ for all atomic A;
- (ii) $p(A \times B) = p(A) \times p(B)$;
- (iii) $p(A \vee B) = p(A) \sqcup p(B)$;
- (iv) $p(A \to B) = \{f : S(A) \to S(B) \mid f(p(A)) \subseteq p(B)\}$;
- (v) $p(\forall x\, A) = \{f : \Gamma \to S(A) \mid \forall c \in \Gamma : f(c) \in p(A[c/x])\}$;
- (vi) $p(\exists x\, A) = \{\langle c, y\rangle \mid c \in \Gamma, y \in p(A[c/x])\}$.

Using induction, one may show that $p(A) \subseteq S(A)$ for all $\mathcal{L}(\Gamma)$-sentences A.

2 Simple functionals

Definition 2.1. Let $\mathcal{D}$ be the least set such that:

- (i) $\{0\}, \{1\}, \Gamma, \Pi \in \mathcal{D}$;
- (ii) if $D_1, D_2 \in \mathcal{D}$, then also $D_1 \times D_2, D_1 \cup D_2, D_1^{D_2} \in \mathcal{D}$.

We call $\mathcal{F} := \bigcup \mathcal{D}$ the set of *functionals.*

Proposition 2.1. *(i) Every functional is either an element of* $\{0, 1\} \cup \Gamma \cup \Pi$*, or it is a function* $D_1 \to D_2$ *for certain* $D_1, D_2 \in \mathcal{D}$.

(ii) For every $\mathcal{L}(\Gamma)$*-formula* A*, we have* $S(A) \in \mathcal{D}$ *and thus* $S(A) \subseteq \mathcal{F}$.

Definition 2.2. The set of *terms* is defined by:

- (i) 0, 1, c_0 and all variables are terms;
- (ii) if s and t are terms, then $s(t)$ and $\langle s, t\rangle$ are also terms;
- (iii) if s is a term and $D \in \mathcal{D}$, then $\lambda x^D(s)$ is also a term.

Definition 2.3. Suppose to each variable x, a functional $V(x)$ has been assigned. We extend V to all terms by:

(i) $V(0) = 0$, $V(1) = 1$ and $V(c_0) = c_0$;

(ii) $V(s(t)) = V(s)(V(t))$, which is defined as the value of $V(s)$ at $V(t)$ if $V(s)$ is a function and $V(t) \in \mathrm{dom}(V(s))$; and 0 otherwise;

(iii) $V(\langle s,t\rangle) = \langle V(s), V(t)\rangle$;

(iv) $V(\lambda x^D(t))$ is the function with domain D that assigns to $d \in D$ the value $V'(t)$, where V' is the same as V except that $V'(x) = d$.

We write $t(\theta_1,\dots,\theta_n) = V(t(x_1,\dots,x_n))$ where $V(x_i) = \theta_i$ for $1 \le i \le n$. In particular, we do not distinguish between a closed term t and the functional $V(t)$ (where V is arbitrary). We call such a functional that is given by a closed term a *simple functional*.

Example 2.1. Let $D = S((A \to C) \wedge (B \to C))$ and $E = S(A \vee B)$. Then

$$t = \lambda x^D \big(\lambda y^E (\langle x(0)(y(1)), x(1)(y(1))\rangle (y(0)))\big)$$

denotes a simple functional such that for all proof assignments p:

$$t \in p[(A \to C) \wedge (B \to C) \to (A \vee B \to C)].$$

3 Invariant functionals

We consider permutations σ of $\Gamma \cup \Pi$ such that $\sigma(\Gamma) = \Gamma$, $\sigma(\Pi) = \Pi$ and $\sigma(c_0) = c_0$. We extend it to $\mathcal{F}$ by:

(i) $\sigma(0) = 0$, $\sigma(1) = 1$;

(ii) if $g : D_1 \to D_2$, then $\sigma(g) = \sigma \circ g \circ \sigma^{-1} : \sigma(D_1) \to \sigma(D_2)$. In other words, we have $(\sigma(g))(\sigma(x)) = \sigma(g(x))$ for all $x \in D_1$. In particular, $\sigma(\langle a,b\rangle) = \langle \sigma(a), \sigma(b)\rangle$.

Definition 3.1. We say that a functional $\theta \in \mathcal{F}$ is *invariant* if for all such σ, we have $\sigma(\theta) = \theta$.

Proposition 3.1. *(i) For all $D \in \mathcal{D}$, we have that $\sigma|_D$ is a permutation of D.*

(ii) For all terms $t(x_1,\dots,x_n)$ and functionals $\theta_1,\dots,\theta_n$, we have that $\sigma(t(\theta_1,\dots,\theta_n)) = t(\sigma(\theta_1),\dots,\sigma(\theta_n))$. In particular, all simple functionals are invariant.

4 Soundness and completeness

Theorem 4.1. *Let A be an $\mathcal{L}$-sentence. Then the following are equivalent:*

(i) **IQC** $\vdash A$*;*

(ii) there is a simple functional θ such that for all proof assignments p: $\theta \in p(A)$;

(iii) for all proof assignments p, there is a simple functional θ such that: $\theta \in p(A)$;

(iv) there is an invariant functional θ such that for all proof assignments p: $\theta \in p(A)$;

(v) for all proof assignments p, there is an invariant functional θ such that: $\theta \in p(A)$.

The implication $(v) \Rightarrow (i)$ is most difficult. We need to show: if **IQC** $\nvdash A$, then there exists a proof assignment p such that $p(A)$ contains no invariant functionals.

5 Setup for the proof

We consider the set $\Sigma = \omega^{<\omega} \cup \{U\}$, where U is some object not in $\omega^{<\omega}$. For $s, s' \in \Sigma$, we say that $s \preceq s'$ if either $s' = U$, or $s, s' \in \omega^{<\omega}$ and $s \sqsubseteq s'$. To each $s \in \Sigma$, a countably infinite set $\Psi(s)$ (the *domain* at s) is assigned, such that:

(i) if $s \preceq s'$, then $\Psi(s) \subseteq \Psi(s')$;

(ii) if $s \preceq s'$ and $s \neq s'$, then $\Psi(s') - \Psi(s)$ is infinite.

Definition 5.1. A *model* will be a fallible Kripke model for $\mathcal{L}$ on the frame and domains defined above. That is, a model is a relation $\Vdash$ between elements of Σ and $\mathcal{L}(\Psi(U))$-sentences such that:

(i) if A is atomic then:

- if $s \Vdash A$, then A is an $\mathcal{L}(\Psi(s))$-sentence;
- if $s \Vdash A$ and $s \preceq s'$, then $s' \Vdash A$;
- if $s \Vdash \bot$ and A is an $\mathcal{L}(\Psi(s))$-sentence, then $s \Vdash A$.

In the following clauses, assume that A and B are $\mathcal{L}(\Psi(s))$-sentences:

(ii) $s \Vdash A \wedge B$ iff $s \Vdash A$ and $s \Vdash B$;

(iii) $s \Vdash A \vee B$ iff $s \Vdash A$ or $s \Vdash B$;

(iv) $s \Vdash A \to B$ iff for all s': if $s \preceq s'$ and $s' \Vdash A$, then $s' \Vdash B$;

(v) $s \Vdash \exists x\, A$ iff there exists a $c \in \Psi(s)$ such that $s \Vdash A[c/x]$.

(vi) $s \Vdash \forall x\, A$ iff for all s' such that $s \preceq s'$ and $c \in \Psi(s')$, we have $s' \Vdash A[c/x]$.

Theorem 5.1. *Let A be an $\mathcal{L}$-sentence. If* **IQC** $\nvdash A$*, then there exists a model $\Vdash$ such that $\langle\rangle \nVdash A$.*

6 The actual proof

Definition 6.1. Fix some one-to-one function q from Σ to the set of prime numbers. We define $\varphi : \Sigma \to \mathbb{N}$ as follows:

- $\varphi(\langle\rangle) = 1$,
- $\varphi(s * n) = \varphi(s)q(s * n)$ and
- $\varphi(U) = 0$.

Definition 6.2. Every subset in Σ has a greatest lower bound (glb), namely the initial segment that all elements in the subset share. Define $s_n = \text{glb}\{s : n \mid \varphi(s)\}$.

Impose the following structure on Γ and Π:

$$\Gamma = \Pi = \bigcup\{\Psi(s_n) \times \mathbb{Z}/n\mathbb{Z} : n \in \mathbb{N}\}.$$

Note that we identified these sets with one another, even though they may not actually be the same. We just do this as a notational convenience. In addition we require this structure to be such that $c_0 \in \Psi(s_1) \times \mathbb{Z}/1\mathbb{Z}$.

Definition 6.3. We define $\Gamma_n = \bigcup\{\Psi(s_k) \times \mathbb{Z}/k\mathbb{Z} : k \mid n\}$.

Definition 6.4. Any $c \in \Gamma$ is now actually an ordered pair. We denote $\bar{c}$ for its first component. Similarly, for each $A \in \mathcal{L}(\Gamma)$ we define $\bar{A}$ to be the formula with each constant c in A replaced by $\bar{c}$.

Definition 6.5. Define $\sigma : \Gamma \to \Gamma$ by $\sigma(\langle d, [i]_n\rangle) = \langle d, [i+1]_n\rangle$. Where $[i]_n \in \mathbb{Z}/n\mathbb{Z}$ denotes the equivalence class of the integer i modulo n.

Definition 6.6. To a model Φ we associate a proof assignment p that is given by

$$p(A) = \bigcup\{\Psi(s_n) \times \mathbb{Z}/n\mathbb{Z} : s_n \Vdash \bar{A} \text{ or } s_n \Vdash \bot\}$$

for all atomic A.

Lemma 6.1. *Let a model Φ be given, and let p be the proof assignment that is associated to it. Then for all $n \in \mathbb{N}$ and $A \in \mathcal{L}(\Gamma_n)$ we have that σ^n has a fixed element in $p(A)$ if and only if $s_n \Vdash \bar{A}$.*