Seminar on Constructible Sets

Answers to the Exercises Session 1

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7th of March 2018

Exercise 1. Lemma 6.2 states that if κ is *regular* and *uncountable*, $\lambda < \kappa$, and $\mathcal{A} = \{A_i \mid i < \lambda\}$ is a family of club subsets of κ, then $\bigcap \mathcal{A}$ is club in κ. Explain, for each case below, whether this also holds:

(i) (20pts) if κ is *singular*;

(ii) (10pts) if κ is *countable*;

(iii) (10pts) if $\lambda = \kappa$.

Answer.

(i) If κ is singular, take $\lambda = \mathrm{cf}(\kappa) < \kappa$, let $\langle \xi_\alpha \rangle_{\alpha<\lambda}$ be a cofinal sequence in κ, and define for each ordinal β the interval $X_\beta := \{\alpha \mid \beta < \alpha \leq \kappa\}$. Clearly X_β is club for every $\beta < \kappa$, so in particular $\{X_{\xi_\alpha} \mid \alpha < \lambda\}$ is a set of club sets. The intersection of this set is empty, since for any $\gamma < \kappa$, we can find $\alpha < \lambda$ such that $\xi_\alpha > \gamma$ (since $\langle \xi_\alpha \rangle_{\alpha<\lambda}$ was cofinal), and then $\gamma \notin X_{\xi_\alpha}$.

(ii) If $\kappa = \omega_0$ is countable, then we can take a strictly increasing sequence $\langle a_n \rangle_{n\in\omega}$, which is unbounded. It is easy to see that $\langle a_{2n} \rangle_{n\in\omega}$ and $\langle a_{2n+1} \rangle_{n\in\omega}$ are both unbounded as well, but their intersection is empty.

(iii) If $\lambda = \kappa$, we can use the same argument as for κ singular, where we let the sequence $\langle \xi_\alpha \rangle_{\alpha<\kappa}$ be such that $\xi_\alpha = \alpha$ for all $\alpha < \kappa$.

Exercise 2. (Devlin I.6.4) Let κ be an uncountable regular cardinal.

(i) (15pts) If $A \subseteq \kappa$ is club, then the enumeration of A in increasing order (as ordinals) is a normal function from κ to κ.

(ii) (15pts) If $f\colon \kappa \to \kappa$ is a normal function, then $\mathrm{ran}(f)$ is a club subset of κ.

Answer.

(i) Let f be the enumeration function in increasing order. Then f is strictly increasing by construction. Since κ is regular and A is unbounded, we have $\kappa = \mathrm{cf}(\kappa) \leqslant |A| \leqslant \kappa$, so $\kappa = |A|$ and f indeed goes from κ to κ.

To see continuity, let $\gamma < \kappa$ be a limit ordinal, $\gamma = \bigcup_{\beta<\gamma} \beta$. We have $f(\gamma) \in A$ because A is closed, and by construction it must be $f(\gamma) = \bigcup_{\beta<\gamma} f(\beta)$, so f is continuous.

(ii) Clearly $\mathrm{ran}(f)$ is unbounded, since for all $\alpha < \kappa$ we have $\alpha \leqslant f(\alpha) < f(\alpha+1)$ because f is strictly increasing.

To see that $\mathrm{ran}(f)$ is closed, take $\gamma < \kappa$ a limit point of $\mathrm{ran}(f)$, that is, $\gamma = \sup(\mathrm{ran}(f) \cap \gamma)$. Let $\alpha < \kappa$ be the smallest ordinal such that $\gamma \leqslant f(\alpha)$. Then $\gamma = \bigcup_{\beta<\alpha} f(\beta)$ and α is a limit ordinal, so by continuity we have $f(\alpha) = \bigcup_{\beta<\alpha} f(\beta) = \gamma$, thus $\gamma \in \mathrm{ran}(f)$.

Exercise 3. (30pts) Investigate whether $\langle \omega, \in \rangle$ and $\langle V_\omega, \in \rangle$ are models for the Axiom of Extensionality.

Answer. For ω, it is enough to show that for all $n, m \in \omega$ we have $n \neq m$ implies $n \in m$ or $m \in n$, while of course $n \notin n$ and $m \notin m$, therefore $\langle \omega, \in \rangle$ is extensional.
For V_ω, let $u, v \in V_\omega$ such that $u \neq v$ and let $\text{rank}(u) = n$, $\text{rank}(v) = m$. W.l.o.g. assume $n \leq m$. If $n < m$ then v must contain an element x of rank $m - 1$. Clearly $x \notin u$. On the other hand if $n = m$, then we see that $n, m \subset V_n$, and then $u \neq v$ implies there is an $x \in V_n$ such that $x \in u \setminus v$ or $x \in v \setminus u$.

An alternative route is to see that transitive sets are models for the Axiom of Extensionality, since both ω and V_ω are transitive. Indeed, let X be a transitive set, and let $x, y \in X$. If $x \neq y$, then by the Axiom of Extensionality there exists a set z such that either $z \in x$ and $z \notin y$, or $z \notin x$ and $z \in y$. Suppose without loss of generality that $z \in x$. Then $z \in X$ by transitivity, and therefore X is extensional.