Seminar on Constructible Sets

Exercises Session 4

14th March 2018

Exercises

Note: mentions to results and proofs refer back to those in Devlin.

Exercise 1. (4 points) Why is Corollary 10.4 immediate from Lemma 10.3? (Hint: notation!)

Answer: This is simply a matter of looking at the notation: we have that $f\colon M \to M$, which means that it is a total function, i.e. $\mathrm{dom}(f) = M$. This is Δ_0, thus also in particular Π_n. Then it follows from Lemma 10.3 that f is Δ_n^M

Exercise 2. (6 points) Explain how Lemma 10.6 is used in the proof of Lemma 10.7. Be precise: how does one "use the method of 10.6"?

Answer: Since R is $\Sigma_n^M(\{p_0, \dots, p_m\})$, we have that $(\forall x \in M)(R(x) \leftrightarrow \models_M \varphi(\mathring{x}))$ for some Σ_n formula $\varphi(v)$ of the M-language with constants among $\{p_1, \dots, p_m\}$. The basic idea here is that, if $p = (p_0, \dots, p_m)$, we can replace each occurrence of p_i in φ by $(p)_i$ without altering the complexity of the formula, thereby "contracting" all necessary information about the parameters inside of p and thus meaning that we can see R as being $\Sigma_n^M(\{p\})$.

More precisely: if we substitute each constant p_i in φ by a new variable v_i, then we can define a Σ_n formula $\psi(v, w_0, \dots, w_m)$ satisfying that $(\forall x \in M)(\varphi(\mathring{x}) \leftrightarrow \psi(\mathring{x}, \mathring{p_0}, \dots, \mathring{p_m}))$. Now, following the technique used in the proof of Lemma 10.6, we can define $\tilde{\psi}(v, w)$, where

$$\tilde{\psi}(v, w) \leftrightarrow \text{“}w \text{ is an } (m+1)\text{-tuple”} \wedge w_0 = (w)_0 \wedge \cdots \wedge w_m = (w)_m \wedge \psi(v, w_0, \dots, w_m)$$

which is still Σ_n by Lemma 8.4, and so we have that $(\forall x \in M)(R(x) \leftrightarrow \models_M \tilde{\psi}(\mathring{x}, \mathring{p}))$, showing that R is $\Sigma_n^M(\{p\})$.

Anton proposed an alternative 'method', different to that of Lemma 10.6: the idea is to note that given a relation $R(x, y)$ we can use twice the trick from Lemma 10.6 to obtain

$$Q(x, y) \leftrightarrow [x \text{ is an ordered pair} \wedge y \text{ is an ordered pair} \wedge R((x)_0, (y)_1)].$$

It is clear that $\exists xy Q(x, y)$ iff $\exists x Q(x, x)$, and that this construction can be extended to m-tuples. This gives another way to replace the parameters p_i in R by $(p)_i$, which is what we wanted to do.