Solutions for the Homework of the 18th of April

Bart Keller & Tristan van der Vlugt

$\boxed{\mathbf{1}}$ **In the proof of theorem 1.5 of Devlin, we constructed an isomorphism π. Prove the following properties:**

1) $\pi(\omega_1) = \alpha$,
2) $\pi(\mathbf{T}) = \mathbf{T} \restriction \alpha$,
3) $\pi(A) = A \cap (T \restriction \alpha)$.

Proof. If for some $X \in M$ we have that $X \cap M \subseteq L_\alpha$, then $\pi(X) = X \cap M$. We can see this as follows. For $x \in X \cap M$, we know that $\pi^{-1}(x) = x$, since we know that $\pi \restriction L_\alpha = \mathrm{id}_{L_\alpha}$, so $x \in \pi(X)$. This gives that $X \cap M \subseteq \pi(X)$. For $x \in \pi(X)$, we immediately have that $x = \pi^{-1}(x) \in X \cap M$, so this gives us that $\pi(X) \subseteq X \cap M$. We may conclude that $\pi(X) = X \cap M$.
We can now prove the three asked properties of π:
We can use the claim for $X = \omega_1$, since $\omega_1 \cap M = \mathrm{On} \cap L_{\omega_1} \cap M = \mathrm{On} \cap L_\alpha = \alpha$. We know that $\alpha \subseteq L_\alpha$, so we may use the claim to conclude that $\pi(\omega_1) = \omega_1 \cap M = \alpha$
We can also use the claim for $X = T$. We have been given in the prove of Theorem 1.5 in [1] that $T \cap M = T \restriction \alpha$. Since we can easily see that $T \restriction \alpha \subseteq L_{\omega_1}$, since element of $T \restriction \alpha$ is an α-sequence of 0s and 1s, and we can see that $T \restriction \alpha \subseteq M$, we have that $T \restriction \alpha \subseteq M \cap L_{\omega_1} = L_\alpha$. So we may use the claim to conclude that $\pi(T) = T \cap M = T \restriction \alpha$. Since we also have that the structure of $\mathbf{T}$ is preserved under the bijection π, we also have that $\pi(\mathbf{T}) = \mathbf{T} \restriction \alpha$.
Finally, we may use the claim for $X = A$. By the previous, we have that $A \subseteq L_\alpha$, since $A \subseteq T$. Since in the proof of Theorem 1.5 of [1] it is also given that $A \cap M = A \cap (T \restriction \alpha)$, we may use the claim to conclude that $\pi(A) = A \cap M = A \cap (T \restriction \alpha)$.
Another version of the proof uses that the bijection π distributes over unions, i.e. $\pi(A \cup B) = \pi(A) \cup \pi(B)$. Using this, one can also prove the above properties. □

$\boxed{\mathbf{2}}$ **Show that, at the bottom of page 116 in the proof of theorem 1.5 of Devlin, the sets A_α and b_x, as defined in the definition of T_α, exist for every $\alpha < \omega_1$ and $x \in T \restriction \alpha$**

Proof. For the existence of A_α, we work with contradiction. Suppose that no such maximal antichain as A_α exists. Then we have that every maximal antichain is contained in $T \restriction \beta + 1$ for some $\beta < \alpha$, where β is chosen minimally. This is however impossible, since we can easily construct a maximal antichain in $T \restriction \beta + 2$ by replacing one of the elements at level T_β in our

maximal antichain with one of its successors at level $T_{\beta+1}$, which would still be a maximal antichain. This is a contradiction with the minimality of β, so we must have that such a β cannot exist. So therefore A_α must exist.
For the existence of b_x, we also work with contradiction. So assume that no such branch exists. So then we have that every α-branch containing x is disjoint form A_α. Then we have that x is incomparable to any element of A_α, and this is a contradiction with the fact that A_α is a maximal antichain. So we must have that a branch like b_x exists. $\square$

3 **Let $\mathbb{S} = \langle S, \leq \rangle$ be a Souslin tree. Define the product $\mathbb{S} \times \mathbb{S}$ as the poset $\langle X, \preceq \rangle$, where $X = \{(s,t) \mid s,t \in S \wedge \mathrm{ht}(s) = \mathrm{ht}(t)\}$ and $(a,b) \preceq (a',b')$ if and only if $a \leq a'$ and $b \leq b'$. Show that $\mathbb{S} \times \mathbb{S}$ is not a Souslin tree.**

Proof. Every level S_α is nonempty, so by the axiom of choice let $\langle x_\alpha \rangle_{\alpha<\omega_1}$ be a sequence such that $x_\alpha \in S_\alpha$ for every $\alpha < \omega_1$. Souslin trees are normal, so there are $x_1^\alpha, x_2^\alpha \in S_{\alpha+1}$, with $x_1^\alpha \neq x_2^\alpha$ and $x_\alpha < x_i^\alpha$ for both i. Clearly $\mathrm{ht}(x_1^\alpha) = \mathrm{ht}(x_2^\alpha)$, and therefore $(x_1^\alpha, x_2^\alpha) \in \mathbb{S} \times \mathbb{S}$. For each $\alpha < \omega_1$ fix two of these direct successors of x_α to get a sequence $\langle (x_1^\alpha, x_2^\alpha) \rangle_{\alpha<\omega_1}$. This sequence has cardinality ω_1 and is an antichain:
Suppose $(x_1^\alpha, x_2^\alpha) \preceq (x_1^\beta, x_2^\beta)$. Assume without loss of generality that $\alpha < \beta$ (equality of α and β implies $(x_1^\alpha, x_2^\alpha) = (x_1^\beta, x_2^\beta)$ and is therefore not of interest). Then $x_i^\alpha \leq x_i^\beta$, and since $\mathrm{ht}(x_i^\alpha) < \mathrm{ht}(x_i^\beta)$ we even have $x_i^\alpha < x_i^\beta$. $\mathbb{S}$ is a tree, so for both i the set $B_i = \{y \in S \mid y < x_i^\beta\}$ is a well-order, with x_β as maximal element. But $x_i^\alpha \in B_i$, and thus $x_i^\alpha \leq x_\beta$, and this shows that the branch $\{y \in S \mid y < x_\beta\}$ of $\mathbb{S}$ is not a well-order, as it contains two incomparable elements. This is a contradiction, therefore all elements in $\langle (x_1^\alpha, x_2^\alpha) \rangle_{\alpha<\omega_1}$ are mutually incomparable. $\square$

References

[1] Keith J. Devlin, *Constructibility*, Springer-Verlag Berlin, ISBN 0-387-13258-9, 1984.