Seminar on Set Theory

Hand-out lecture 15

January 15, 2016

IZF and the Heyting valued universe

I. IZF and ordinals

IZF is IZ extended with the Axiom Schemes of Collection and $\in$-induction.

Definition.

i) A set x is *transitive* if $\forall y \in x\ y \subseteq x$.

ii) An *ordinal* is a transitive set of transitive sets. The class of ordinals is denoted by ORD.

iii) An ordinal α is *simple* if $\forall \beta, \gamma \in \alpha\ (\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta)$.

Examples. 0, $\{0\}$, $\{0 \mid \phi\}$, $\{0, \{0 \mid \phi\}\}$ are ordinals. We write $1 = \{0\}$.

Proposition. If every ordinal is simple, then the Law of the Excluded Middle holds.

Definition.

i) If $\alpha \in$ ORD, then its *successor* α^+ is defined as $\alpha \cup \{\alpha\}$.

ii) If $A \subseteq$ ORD is a set, then its *supremum* is defined as $\bigcup A$.

iii) For $\alpha, \beta \in$ ORD, we write $\alpha < \beta$ for $\alpha \in \beta$, and $\alpha \leq \beta$ for $\alpha \subseteq \beta$.

Properties.

- $\alpha^+ \leq \beta \leftrightarrow \alpha < \beta$.
- $\bigcup A \leq \beta \leftrightarrow \forall \alpha \in A\ \alpha \leq \beta$.
- $\alpha < \beta \leq \gamma \rightarrow \alpha < \gamma$.

Definition. An $\alpha \in$ ORD is

i) a *weak limit* if $\forall \beta \in \alpha\ \exists \gamma \in \alpha\ \beta \in \gamma$;

ii) a *strong limit* if $\forall \beta \in \alpha\ \beta^+ \in \alpha$.

Proposition. Is every weak limit is also a strong limit, then the Law of the Excluded Middle holds.

II. Heyting valued universe

For a Heyting algebra H, we can define $V^{(H)}$, $\mathcal{L}^{(H)}$ and $[\![\cdot]\!]^H$. We have:

- first order intuitionistic logic holds in $V^{(H)}$;
- $V^{(H)} \models$ IZF;
- a hat operator $\hat{\cdot} : V \rightarrow V^{(H)}$;
- $V^{(H)} \models$ Zorn's Lemma.

III. Algebra of opens

Let $(P, \leq)$ be a poset. For $p \in P$, we write $O_p = \{q \in P \mid q \leq p\}$. We give P the *left order topology*, generated by the base $\{O_p \mid p \in P\}$. We let

$$H = O(P) = \{X \subseteq P \mid \forall p \in X \forall q \in P \ (q \leq p \to q \in X)\}.$$

Definition. Let $p \in P$ and $\sigma \in \mathcal{L}^{(H)}$. We say that p *forces* σ, written $p \Vdash \sigma$, if $O_p \leq [\![\sigma]\!]$.

Property. For $p \in P$ and $\mathcal{L}^{(H)}$-sentences σ and τ, we have

$$p \Vdash \sigma \vee \tau \quad \text{iff} \quad p \Vdash \sigma \text{ or } p \Vdash \tau.$$

Theorem. For $P = \mathbb{N}^{\mathrm{op}}$ and $K = \{\langle \hat{n}, O_n \rangle \mid n \in \mathbb{N}\} \in V^{(H)}$, we have

$$V^{(H)} \models K \text{ is infinite, but not Dedekind infinite.}$$