Seminar on Set Theory
Solutions to Exercise 10

Solution to Exercise 1

In this solution we will always drop the Γ*-superscripts on Boolean evaluations.*

(a) *Induction base.* Suppose ϕ is an atomic formula. Then $\phi(x_1, \dots, x_n)$ is of the form $u = v$ or $u \in v$ with $u, v \in V^{(\Gamma)}$. That the thesis holds for ϕ now is an immediate consequence of the fact that G acts on $V^{(\Gamma)}$.

Induction step. Suppose that the thesis holds for formulas φ and ψ. Since all formulas can be written in a form that only uses connectives in $\{\wedge, \neg, \exists\}$ and, by Theorem 3.18, this does not change their truth value in $V^{(\Gamma)}$, we only consider those connectives.

Suppose ϕ can be written as $\varphi \wedge \psi$. Then we have

$$\begin{aligned} g \cdot [\![\phi(x_1, \dots, x_n)]\!] &= g \cdot [\![\varphi(x_1, \dots, x_n) \wedge \psi(x_1, \dots, x_n)]\!] \\ &= g \cdot [[\![\varphi(x_1, \dots, x_n)]\!] \wedge [\![\psi(x_1, \dots, x_n)]\!]] \\ &= g \cdot [\![\varphi(x_1, \dots, x_n)]\!] \wedge g \cdot [\![\psi(x_1, \dots, x_n)]\!] \\ &= [\![\varphi(gx_1, \dots, gx_n)]\!] \wedge [\![\psi(gx_1, \dots, gx_n)]\!] \\ &= [\![\varphi(gx_1, \dots, gx_n) \wedge \psi(gx_1, \dots, x_n)]\!] = [\![\phi(gx_1, \dots, gx_n)]\!], \end{aligned}$$

where the third equality holds due to fact that π_g as defined on p. 71 is an automorphism on B, and the fourth by the induction hypothesis.

Suppose that ϕ can be written as $\neg\varphi$. Then, again using the induction hypothesis and the fact that g induces an automorphism on B, we have

$$\begin{aligned} g \cdot [\![\phi(x_1, \dots, x_n)]\!] &= g \cdot [\![\neg\varphi(x_1, \dots, x_n)]\!] \\ &= g \cdot [[\![\varphi(x_1, \dots, x_n)]\!]^*] \\ &= [g \cdot [\![\varphi(x_1, \dots, x_n)]\!]]^* \\ &= [\![\varphi(gx_1, \dots, gx_n)]\!]^* \\ &= [\![\neg\varphi(gx_1, \dots, gx_n]\!] = [\![\psi(gx_1, \dots, gx_n)]\!]. \end{aligned}$$

Suppose that ϕ can be written as $\exists x\varphi$. Then, again using the same facts, we obtain

$$\begin{aligned} g\cdot [\![\phi(x_1,\dots,x_n)]\!] &= g\cdot [\![\exists x\varphi(x,x_1,\dots,x_n)]\!] \\ &= g\cdot [\bigvee_{u\in V^{(\Gamma)}} [\![\varphi(u,x_1,\dots,x_n)]\!]] \\ &= \bigvee_{u\in V^{(\Gamma)}} [g\cdot [\![\varphi(u,x_1,\dots,x_n)]\!]]] \\ &= \bigvee_{u\in V^{(\Gamma)}} [\![\varphi(gu,gx_1,\dots,gx_n)]\!] \\ &= \bigvee_{u\in V^{(\Gamma)}} [\![\varphi(u,gx_1,\dots,gx_n)]\!] \\ &= [\![\exists x\varphi(x,gx_1,\dots,gx_n)]\!] = [\![\phi(gx_1,\dots,gx_n)]\!], \end{aligned}$$

where the fifth equality holds due to the fact that g is a permutation of $V^{(\Gamma)}$.

Students obtain 0.25 points for the induction base and for each induction step.

(b) Let $b\in B$. Then

$$1\cdot b=\{f\in X \mid 1*f=f\in b\}=b$$

and for all $f\in X$, we have

$$\begin{aligned} f\in (gh)\cdot b &\text{ iff } (gh)^*f\in b \\ &\text{ iff } \exists u\in b\ \forall m,n\in\omega[u\langle m,n\rangle = f\langle m,ghn\rangle] \\ &\text{ iff } \exists u\in h\cdot b\ \forall m,n\in\omega[u\langle m,n\rangle = f\langle m,gn\rangle] \\ &\text{ iff } \exists u\in g\cdot(h\cdot b)\ \forall m,n\in\omega[u\langle m,n\rangle = f\langle m,n\rangle] \\ &\text{ iff } f\in g\cdot(h\cdot b), \end{aligned}$$

as required.

We still have to show that the induced π_g is, in fact, an automorphism of B. Let $x,y\in B$. Then

$$\begin{aligned} \pi_g(x\wedge y) &= \{f\in X \mid g^*f\in x\cap y\} \\ &= \{f\in X \mid g^*f\in x\}\cap\{f\in X \mid g^*f\in y\} = \pi_g(x)\wedge\pi_g(y), \end{aligned}$$

and

$$\begin{aligned} \pi_g(x^*) &= \{f\in X \mid g^*f\in x^*\} = \{f\in X \mid g^*f\in x^c\} \\ &= \{f\in X \mid g^*f\in x\}^c = \pi_g(x)^c, \end{aligned}$$

as required.

Students obtain 0.5 points for showing that G acts on B as a regular set and 0.5 points for showing that the π_g are automorphisms, thereby showing that G acts on B as a Boolean algebra.

(c) Let $H \in \Gamma$. Then there is some finite $J \subseteq \omega$ such that $G_J \subseteq H$. For arbitrary $g \in G$, we consider the set $gJ := \{gj \mid j \in J\}$. Then

$$\begin{aligned} ux = x \text{ for all } x \in G_{gJ} &\text{ so } ugj = gj \text{ for all } j \in J \\ &\text{ so } g^{-1}ugj = j \text{ for all } x \in J \\ &\text{ so } g^{-1}ug \in G_J \\ &\text{ so } g^{-1}ug \in H \\ &\text{ so } gg^{-1}ugg^{-1} = u \in gHg^{-1}. \end{aligned}$$

We see that the finite $gJ \subseteq gHg^{-1}$, so $gHg^{-1} \in H$, as required.

Students obtain 0.75 points for finding the correct subset of ω and 0.25 points for noting that it is finite.

(d) *This proof is analogous to the proof in Theorem 2.12.*

We know that $P = C(\omega \times \omega, 2)$ is a basis for B. First notice that

$$[\![\hat{m} \in u_n]\!] = \bigvee_{x \in \omega} [u_n(\hat{x}) \wedge [\![\hat{n} = \hat{x}]\!]] = u_n(\hat{m}) = \{h \in 2^{\omega \times \omega} \mid h\langle m, n\rangle = 1\}.$$

From here we see that for $p \in P$,

$$\begin{aligned} p \Vdash \hat{m} \in u_n &\text{ iff } p \leq [\![\hat{m} \in u_n]\!] \\ &\text{ iff } \{h \in 2^{\omega \times \omega} \mid p \subseteq h\} \subseteq \{h \in 2^{\omega \times \omega} \mid h\langle m, n\rangle = 1\} \\ &\text{ iff } p\langle m, n\rangle = 1, \end{aligned}$$

and

$$\begin{aligned} p \Vdash \hat{m} \notin u_n &\text{ iff } p \leq [\![\hat{m} \in u_n]\!]^* \\ &\text{ iff } \{h \in 2^{\omega \times \omega} \mid p \subseteq h\} \subseteq \{h \in 2^{\omega \times \omega} \mid h\langle m, n\rangle = 0\} \\ &\text{ iff } p\langle m, n\rangle = 0. \end{aligned}$$

Now let $n, n' \in \omega$ with $n \neq n'$ and suppose, by contradiction, that

$$V^{(\Gamma)} \not\models u_n \neq u_n'.$$

Then $[\![u_n = u_n']\!] \neq 0$, so there is a $p \in P$ such that $p \Vdash u_n = u_n'$. Choose $m \in \omega$ such that $\langle m, l\rangle \notin \mathrm{dom}(p)$ for any $l \in \omega$ (possible since $\mathrm{dom}(p)$ is finite) and put

$$p' = p \cup \{\langle\langle m, n\rangle, 1\rangle\} \cup \{\langle\langle m, n'\rangle, 0\rangle\}.$$

Then $p' \Vdash \hat{m} \in u_n \wedge \hat{m} \in u'_n$ and thus $p \Vdash u_n \neq u'_n$. However, since $p \leq p'$, we also have $p' \Vdash u_n = u'_n$, a contradiction.

Students obtain 1 point for correctly showing the conditions under which the relevant membership relation (and its negation) are forced, 0.5 points for constructing p', and 0.5 points for finishing the proof.

Solution to Exercise 2

We prove the Theorem by induction on the complexity of the formula ϕ. The atomic cases are immediate, as $x^U \in_U y^U$ and $x^U = y^U$ have been defined as $[\![x \in y]\!] \in U$ and $[\![x = y]\!] \in U$ respectively. We carry out the induction step for $\wedge, \neg$ and $\exists$, which is sufficient.
$\wedge$: Suppose that the result holds for the formulas $\phi(v_1, ..., v_n)$ and $\psi(v_1, ..., v_n)$ and for arbitrary $x_1, ..., x_n \in M^{(B)}$. We see by definition that $M^{(B)}/U \models \phi[x_1^U, ..., x_n^U] \wedge \psi[x_1^U, ..., x_n^U]$ precisely when both $M^{(B)}/U \models \phi[x_1^U, ..., x_n^U]$ and $M^{(B)}/U \models \psi[x_1^U, ..., x_n^U]$. By the induction hypothesis, the latter is equivalent to $[\![\phi(x_1, ..., x_n)]\!] \in U$ and $[\![\psi(x_1, ..., x_n)]\!] \in U$. By the properties of a filter, this is equivalent to $[\![\phi(x_1, ..., x_n)]\!] \wedge [\![\psi(x_1, ..., x_n)]\!] \in U$, that is, $[\![\phi(x_1, ..., x_n) \wedge \psi(x_1, ..., x_n)]\!] \in U$.
$\neg$: Suppose that the result holds for the formula $\phi(v_1, ..., v_n)$ and for arbitrary $x_1, ..., x_n \in M^{(B)}$. Note that $M^{(B)}/U$ deals with ordinary truth values rather than Boolean truth values, hence either $M^{(B)}/U \models \phi[x_1^U, ..., x_n^U]$ or $M^{(B)}/U \models \neg\phi[x_1^U, ..., x_n^U]$. Thus $M^{(B)}/U \models \neg\phi[x_1^U, ..., x_n^U]$ is equivalent to $M^{(B)}/U \not\models \phi[x_1^U, ..., x_n^U]$, which by the induction hypothesis is equivalent to $[\![\phi(x_1, ..., x_n)]\!] \notin U$. Since U is an ultrafilter this in turn is equivalent to $[\![\phi(x_1, ..., x_n)]\!]^* \in U$, that is, $[\![\neg\phi(x_1, ..., x_n)]\!] \in U$.
$\exists$: Suppose that the result holds for the formula $\phi(u, v_1, ..., v_n)$ and for arbitrary $y, x_1, ..., x_n \in M^{(B)}$. By definition we have $M^{(B)}/U \models \exists y \phi[y, x_1^U, ..., x_n^U]$ precisely when $M^{(B)}/U \models \phi[y^U, x_1^U, ..., x_n^U]$ for some $y \in M^{(B)}$. By the induction hypothesis this is equivalent to $[\![\phi(y, x_1, ..., x_n)]\!] \in U$ for some $y \in M^{(B)}$. As U is a filter this implies $\bigvee_{y \in M^{(B)}} [\![\phi(y, x_1, ..., x_n)]\!] \in U$, which is $[\![\exists y \phi(y, x_1, ..., x_n)]\!] \in U$. For the converse we use the Maximum Principle to find some $y \in M^{(B)}$ such that $[\![\exists y \phi(y, x_1, ..., x_n)]\!] = [\![\phi(y, x_1, ..., x_n)]\!]$.
Points awarded: $\frac{1}{2}$ for beginning the induction by noting the triviality of the atomic cases; $\frac{1}{2}$ per what amounts to a correct and sufficiently motivated proof of one of the three required cases.

Solution to Exercise 3

Consider $\{\hat{x}_i : i \in I\}$, and take x to be the mixture $\sum_{i \in I} a_i \cdot \hat{x}_i$. By the Mixing Lemma we have $a_i \leq [\![x = \hat{x}_i]\!]$ for all $i \in I$, so this leaves us to

show that $[\![x = \hat{x}_i]\!] \leq a_i$ for all $i \in I$. We see that $\mathrm{dom}(x) = \bigcup_{i\in I}\{\hat{y} : y \in x_i\}$ and $x(\hat{y}) = \bigvee_{i\in I}[a_i \wedge [\![\hat{y} \in \hat{x}_i]\!]] = \bigvee_{\substack{i\in I \\ y\in x_i}} a_i$. For arbitrary $i \in I$ we similarly find that $\bigwedge_{\hat{y}\in\mathrm{dom}(x)}[x(\hat{y}) \Rightarrow [\![\hat{y} \in \hat{x}_i]\!]] = \bigwedge_{\substack{\hat{y}\in\mathrm{dom}(x) \\ y\notin x_i}} x(\hat{y})^*$, and for any $y \in x_i$ we have $\hat{x}_i(\hat{y}) \Rightarrow [\![\hat{y} \in x]\!] = x(\hat{y})$. By combining these results we find that $[\![x = \hat{x}_i]\!] = \bigwedge_{\substack{\hat{y}\in\mathrm{dom}(x) \\ y\notin x_i}} x(\hat{y})^* \wedge \bigwedge_{y\in x_i} x(\hat{y})$. At this point we note that $\bigwedge_{y\in x_i} x(\hat{y}) = \bigwedge_{y\in x_i} \bigvee_{\substack{j\in I \\ y\in x_j}} a_j \leq \bigvee_{\substack{j\in I \\ x_i\subseteq x_j}} a_j$. On the other hand we have $\bigwedge_{\substack{\hat{y}\in\mathrm{dom}(x) \\ y\notin x_i}} x(\hat{y})^* = \bigwedge_{\substack{\hat{y}\in\mathrm{dom}(x) \\ y\notin x_i}} \bigwedge_{\substack{j\in I \\ y\in x_j}} a_j^* = \bigwedge_{\substack{j\in I \\ x_j\not\subseteq x_i}} a_j^* \leq \bigwedge_{\substack{j\in I \\ x_i\subset x_j}} a_j^*$. Thus we find that $[\![x = \hat{x}_i]\!] \leq \bigwedge_{\substack{j\in I \\ x_i\subset x_j}} a_j^* \wedge \bigvee_{\substack{j\in I \\ x_i\subseteq x_j}} a_j \leq a_i$ as x_i is the unique x_j such that $x_i \subseteq x_j$ and $x_i \not\subset x_j$ because we did not allow duplicates, and we are done.
Points awarded: 1 for taking the mixture x and unpacking its definition; 1 for intelligibly arriving at an intermediate step such as "results combined"; finally, 1 for motivating the inequalities necessary for completing the proof. Credit to those who do not use the Mixing Lemma and the estimates for the other direction and instead prove $[\![x = \hat{x}_i]\!] = a_i$ straight from the definition!