Seminar on Set Theory

Part 1 of Model Solution of Hand-In 11
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As always, let M be a transitive $\in$-model of ZFC and let $B \in M$ be a complete Boolean algebra in the sense of M.

Atoms in B

An *atom* in B is an element $a \in B$ such that $a \neq 0$ and for all $x \in B$, $x \leq a$ implies $x = 0$ or $x = a$.

Part a (1 point): Show that the set $U_a = \{x \in B \,|\, a \leq x\}$ is an ultrafilter in B.

Solution: Check that the properties of a filter hold:

1. $1 \in U_a$: $1 \geq a$.
2. $0 \notin U_a$: $0 \ngeq a$.
3. Upwards closed: Let $x \in U_a$, let $y \geq x$. Then $y \geq x \geq a$ so $y \in U_a$.
4. Closed under joins: Let $x, y \in U_a$. Since $x \geq a$ and $y \geq a$, $x \wedge y \geq a$, since $x \wedge y$ is the greatest upper bound of x and y. Hence $x \wedge y \in U_a$.

Now check that U_a is an ultrafilter. Suppose $x \notin U_a$. Then $x \ngeq a$, hence $x \wedge a \neq a$. Since $x \wedge a \leq a$, it follows that $x \wedge a = 0$, since a is an atom. Hence $a \leq x^*$, and so $x^* \in U_a$.

Part b (1 point) Show that U_a is M-generic.

Solution: Let $\{b_i \,|\, i \in I\}$ be an M-partition of unity. It suffices to show that there is an $i \in I$ such that $b_i \geq a$ (Lemma 4.3). Suppose $b_i \ngeq a$ for all $i \in I$, then by the same argument as above, $b_i \wedge a = 0$. But then

$$a = a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i) = 0$$

which contradicts the assumption that a is an atom.

Part c (1 point) Show that $U_a \in M$ and deduce that $M[U_a] = M$.

Solution: By assumption, $B \in M$. Furthermore, by transitivity we see that $B \subset M$, and in particular, $a \in M$. It follows by the axiom of separation that we can define $U_a^M \in M$ such that $M \models \forall x. (x \in U_a^M \leftrightarrow x \in B \wedge a \leq x)$. Since both U_a and U_a^M are subsets of B, it suffices to show that $x \in U_a$ iff $x \in U_a^M$ for elements of B. But since for any $x \in B$, $M \models x \in B \wedge x \geq a$ iff $x \in B \wedge x \geq a$, this is indeed the case.

Part d (2 points) Let U be an ultrafilter in B such that $U \in M$ and put $a = \bigwedge U$. Show that the following are equivalent: (a) $a \neq 0$, (b) a is an atom, (c) $U = U_a$.

Solution: We prove (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

(a) $\Rightarrow$ (c): Suppose $a \neq 0$ and let $x \in U$. Since $a = \bigwedge U$, $a \leq x$, hence $x \in U_a$. Thus $U \subseteq U_a$. By maximality of U, $U = U_a$.

(c) $\Rightarrow$ (b): Suppose $U = U_a$. If $a = 0$ then $U_a = B$, contradicting the assumption that U is a filter; hence $a \neq 0$. Let $b \leq a$ and assume $b \neq 0$. Then, by the same argument as in part (a), U_b is a filter. For any $x \in U$, $x \geq a \geq b$, so $x \in U_b$. Thus, by maximality of U, $U = U_b$, so $b \in U_a$. Thus $b \geq a$, and since we already know $b \leq a$, we have $b = a$. Thus $b \leq a$ implies $b = a$ or $b = 0$.

(b) $\Rightarrow$ (a): By definition.