Seminar Set Theory: Model Solution Exercise 3.1

It seems best to work with the *transitive closure* $TC(R)$ of R: $xTC(R)y$ holds if there is a finite sequence

$$x = x_0 R x_1 R \cdots R x_n = y$$

Let u be a set. Using that Rv is a set for all v, and induction on $n \in \omega$, we find that for all n,

$$\exists z \forall f (f \in z \ \leftrightarrow \ \begin{array}{c} \mathrm{fun}(f) \wedge \mathrm{dom}(f) = n+1 \wedge \\ \forall k < n f(k) R f(k+1) \wedge f(n) R u \end{array})$$

Moreover the set z is unique by Extensionality; call it $z_{u,n}$. Using Union and Replacement we find that the collection

$$TC(R)(u) \ = \ \{x \mid \exists n \exists f \in z_{u,n} f(0) = x\}$$

is a set.

Now define the function G by

$$G \ = \ \{(u,v) \mid \exists! g \left(\begin{array}{r} \mathrm{fun}(g) \wedge \mathrm{dom}(g) = TC(R)(u) \wedge \\ \forall v \in TC(R)(u) g(v) = F(\langle v, g|Rv\rangle) \wedge F(\langle u, g|Ru\rangle) = v \end{array} \right)\}$$

We need to see that G is a function; i.e. that for every u there exists a unique v with $(u,v) \in G$. Consider the set of those $x \in TC(R)(u)$ for which there is *not* a unique y with $(x,y) \in G$. If this set is empty, then clearly there is a unique g such that $\mathrm{dom}(g) = TC(R)(u)$ and g satisfies the condition in the definition of G; but then, clearly, $(u, F(\langle u, g|Ru\rangle)) \in G$ and $F(\langle u, g|Ru\rangle)$ is the unique such element.

If the set is nonempty, it has an R-minimal element w; however, then we have a unique g such that $\mathrm{dom}(g) = TC(R)w$ and g satisfies the condition in the definition of G; whence $(w, F(\langle w, g|Rw\rangle)) \in G$ and $F(\langle w, g|Rw\rangle)$ unique with this property, contradicting the assumption on w.

So G is a function, and the property follows at once.