Category Theory and Topos Theory, Spring 2018 Hand-In Exercises

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1 Exercises

Exercise 1 (To be handed in February 19) A *Forest* is a partially ordered set $(F, <)$ such that for any $x \in F$, the set $F_{<x} = \{y \in F \mid y < x\}$ is a finite linear order. The cardinality of $F_{<x}$ is called the *level* of x. A morphism of forests $F \to G$ is an order-preserving and level-preserving function. Clearly, we have a category of forests $\mathcal{F}$.

a) A *tree* is a forest which has exactly one element of level 0 (the *root* of the tree. Let $\mathcal{T}$ be the full subcategory (i.e. having the same morphisms) of $\mathcal{F}$ on the trees. Show that the categories $\mathcal{F}$ and $\mathcal{T}$ are equivalent. Are they isomorphic? Motivate your answer.

b) Show that the category $\mathcal{F}$ is isomorphic to a category of the form $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ for a suitable small category $\mathcal{C}$.

c) A forest F is called *well-founded* is there is no infinite sequence

$$x_0 < x_1 < x_2 < \cdots$$

in $\mathcal{F}$. Give a purely category-theoretic property which characterizes the well-founded forests in $\mathcal{F}$.

Exercise 2 (To be handed in March 5) a) Consider the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{e_0} & B & \overset{f_0}{\underset{g_0}{\rightrightarrows}} & C \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow[e_1]{} & B' & \overset{f_1}{\underset{g_1}{\rightrightarrows}} & C' \end{array}$$

where we assume that the horizontal rows are equalizer diagrams, that $cf_0 = f_1 b$, $cg_0 = g_1 b$, the left hand square commutes and the arrow c is monic. Prove that the left-hand square is a pullback.

b) Let $\mathcal{C}$ be a small category and C an object of $\mathcal{C}$; consider the functor $\mathcal{C}(C,-):\mathcal{C}\to \mathrm{Set}$. Prove that this functor preserves all limits which exist in $\mathcal{C}$.

Exercise 3 (To be handed in March 19) Throughout, we assume a regular category $\mathcal{C}$.

a) Show that an arrow $g:X\to Y$ is a regular epi precisely if the following condition holds: for every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{b} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[a]{} & B \end{array}$$

with m mono, there is a unique arrow $h:Y\to A$ such that $mh=a$ and $hg=b$.

b) Use part a) to show that for any composable pair $A\xrightarrow{g}B\xrightarrow{f}C$ of arrows of $\mathcal{C}$ we have: if fg is regular epi, then so is f.

c) For any arrow $f:X\to Y$ in $\mathcal{C}$ we define the *graph* of f as the subobject of $X\times Y$ represented by the mono $\langle \mathrm{id}_X, f\rangle : X\to X\times Y$.

Suppose X,Y,Z are objects of $\mathcal{C}$, $g:Z\to Y$ is a regular epi and $R\in \mathrm{Sub}(X\times Y)$. Let $S=(\mathrm{id}_X\times g)^*(R)\in \mathrm{Sub}(X\times Z)$. Assume that the following two sequents of regular logic are true, with the evident interpretation:

$$\vdash_x \exists z S(x,z)$$
$$S(x,z)\wedge S(x,z')\vdash_{x,z,z'} z=z'$$

Prove that there is an arrow $f:X\to Y$ such that R is the graph of f.

Exercise 4 (To be handed in April 9) Let $\mathcal{D}\underset{G}{\overset{F}{\leftrightarrows}}\mathcal{C}$ be an adjunction with $F\dashv G$ and G full and faithful. We denote the induced monad GF on $\mathcal{C}$ by T.

a) Prove that μ (the multiplication of the monad) is a natural isomorphism.

b) Is the functor G monadic? Justify your answer.

Exercise 5 (To be handed in April 23) Let X be a presheaf on a small category $\mathcal{C}$ and let Y be a subpresheaf of X. We see X as a structure for ther language which has just one unary relation symbol R, and $[\![R]\!]=Y$.

a) Prove that the following three conditions are equivalent:

  i) The sentence $\forall x\neg\neg R(x)$ is true in the structure X.

  ii) For every $C\in\mathcal{C}_0$, every $\xi\in X(C)$ and every arrow $g:C'\to C$ in $\mathcal{C}$, there is an arrow $h:C''\to C'$ such that $X(gh)(\xi)\in Y(C'')$.

iii) For every subpresheaf Z of X which is not the initial presheaf, the intersection $Y \cap Z$ is not the initial presheaf.

If these conditions hold then Y is said to be a *dense* subpresheaf of X.

b) Assume that $\mathcal{C}$ is a groupoid (all arrows are isomorphisms). Show that the only dense subpresheaf of X is X itself.

Exercise 6 (May be handed in digitally until May 9, midnight) In a category $\mathcal{C}$ with pullbacks, a *partial map classifier* for an object X is a monomorphism $\zeta_X : X \to \tilde{X}$ with the property that for any mono $m : A \to B$ and arrow $f : A \to X$ (this is regarded as a *partial map* from B to X) there is a unique arrow $\bar{f} : B \to \tilde{X}$ which makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \zeta_X} \\ B & \xrightarrow[\bar{f}]{} & \tilde{X} \end{array}$$

a pullback.

a) (4 points) Suppose for every object X of $\mathcal{C}$ there is a partial map classifier. Show that there is a functor $\widetilde{(\cdot)} : \mathcal{C} \to \mathcal{C}$ and a natural transformation $\zeta : \mathrm{id}_{\mathcal{C}} \Rightarrow \widetilde{(\cdot)}$ such that for every object X of $\mathcal{C}$, the arrow $\zeta_X : X \to \tilde{X}$ is a partial map classifier for X.

b) (6 points) Let X be a topological space; we consider the category $\mathrm{Sh}(X)$ of sheaves over X. Given such a sheaf F, we denote the action of F on inclusions $U \subseteq V$ (the morphisms in the category of open sets of X) by $\restriction$: for $x \in F(V)$ we write $x\restriction U$ for $F(U \subseteq V)(x)$. Now we define $\tilde{F}$ as follows:

$$\tilde{F}(V) \;=\; \{(U, x) \,|\, U \subseteq V, x \in F(U)\}$$

and for $V' \subseteq V$, we define $(U, x)\restriction V'$ to be $(U \cap V', x\restriction(U \cap V'))$. Show that there is a natural map $F \to \tilde{F}$ in $\mathrm{Sh}(X)$ which is a partial map classifier for F.

c) (2 bonus points) Can you generalize the construction in b) to toposes of the form $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$?

2 Solutions

Solution to Exercise 1.

a) Define functors $F : \mathcal{W} \to \mathcal{T}$ and $G : \mathcal{T} \to \mathcal{W}$ as follows: given a forest W, add a new bottom element to this poset, obtaining $F(W)$. For a morphism $f : W \to W'$ we have $F(f) : F(W) \to F(W')$ which is f when restricted to W, and sends the bottom element to the bottom element of $F(W')$. Note that the level of each element of W is 1 higher in $F(W)$ than in W. In the other direction, given a tree T, $G(T) = T - \{r\}$ where r is the root of T. Here the levels get 1 lower, when we pass from T to $G(T)$. The definition of G on arrows is left to you. It is not hard to prove that F and G are functors. Clearly, $G(F(W)) = W$, and $F(G(T))$ is isomorphic to T. The isomorphism is natural, because it is the identity except for the root.

The categories $\mathcal{W}$ and $\mathcal{T}$ can*not* be isomorphic: look at initial objects in both categories. In $\mathcal{T}$, every singleton set is initial; but in $\mathcal{W}$ there is exactly one initial object, the empty set. Since every isomorphism induces a bijection between the collections of initial objects, we cannot have an isomorphism.

b) Well. . . there was a difficulty in this exercise I wasn't fully aware of! The idea was: we take the poset $\mathbb{N}$ for $\mathcal{C}$. For a functor $X : \mathbb{N}^{\text{op}} \to \text{Set}$, we define the poset $G(X)$ as the set of pairs (n, x) satisfying $x \in X(n)$. We put $(n, x) \leq (m, y)$ iff $n \leq m$ and $X_{nm}(y) = x$ (where $X_{nm} : X(m) \to X(n)$ is the action of the functor X on the arrow $n \leq m$). It is easy to convince oneself that $G(X)$ is a forest. Conversely, given a forest W one has a functor $F(W) : \mathbb{N}^{\text{op}} \to \text{Set}$ by putting: $F(W)(m)$ is the set of elements of W of level m. If $n \leq m$ and $x \in F(W)(m)$, then there is a unique element of level n which is $\leq x$; we define the action of $F(W)$ on arrows accordingly. It is also straightforward that for a forest W, $F(G(W))$ is isomorphic to W and that for a functor X, $G(F(X))$ is isomorphic to X. So the pair F, G is an equivalence. However, it is not an isomorphism! Forests, being defined as posets, have the property that the level-sets (sets of elements of the same level) are pairwise disjoint. Functors $X : \mathbb{N}^{\text{op}} \to \text{Set}$ do not have the property that $X(n)$ is disjoint from $X(m)$ if $n \neq m$! In short, we need an isomorphism between the category $\text{Set}^{\mathbb{N}^{\text{op}}}$ and its full subcategory on the functors X for which the sets $X(n)$ are pairwise disjoint. There *is* a solution to this, but it seems to involve a bit of the foundations of category theory. . .

Consider $\mathbb{N}$-indexed sequences of cardinal numbers $\kappa = (\kappa_n)_{n\in\mathbb{N}}$. For each such κ, let A_κ be the class of $\mathbb{N}$-indexed families of sets $X = (X_n)_{n\in\mathbb{N}}$ which satisfy $|X_n| = \kappa_n$ for each n. Let B_κ be the subclass of A_κ consisting of those X which moreover satisfy $X_n \cap X_m = \emptyset$ for $n \neq m$. There is an injective operation from A_κ to B_κ, for example send X to the family $(\{(x, n) \mid x \in X_n\})_{n\in\mathbb{N}}$. By the Cantor-Schröder-Bernstein theorem (which also holds for classes), there is a bijection $F_\kappa : A_\kappa \to B_\kappa$ for each κ. Now we need a large axiom of choice (which is available if our category Set is "small" in some universe) to assign to any $\mathbb{N}$-indexed family X a sequence of bijections $f_n : X_n \to F_\kappa(X)_n$ (where $\kappa = (|X_n|)_{n\in\mathbb{N}}$).

Now, for an object X of $\text{Set}^{\mathbb{N}^{\text{op}}}$, we have its underlying $\mathbb{N}$-indexed family (also denoted X, or $(X_n)_{n\in\mathbb{N}}$), and the action on arrows $X_{nm} : X_m \to X_n$ for $n \leq m$. We define the structure of a functor $\mathbb{N}^{\text{op}} \to \text{Set}$ on $F_\kappa(X)$ by putting

$$F_\kappa(X)_{nm}(y) \;=\; f_n(X_{nm}(f_m^{-1}(y)))$$

and for an arrow $\mu : X \Rightarrow Y$ (where we have assigned $(f_n)_n : X_n \to F_\kappa(X)_n$ to X and $(g_n)_n : Y_n \to F_\lambda(Y)_n$ to Y), we define an arrow $G(\mu) : F_\kappa(X) \to F_\lambda(Y)$ by

$$G(\mu)_n(x) \;=\; g_n(\mu_n(f_n^{-1}(x)))$$

One has to check that $G(\mu)$ is indeed a natural transformation, and that the assignment G which sends every object X of $\mathrm{Set}^{\mathbb{N}^{\mathrm{op}}}$ to the functor $F_\kappa(X)$ defined above and every μ to $G(\mu)$, is indeed a functor; this is straightforward. We now have the desired isomorphism from $\mathrm{Set}^{\mathbb{N}^{\mathrm{op}}}$ to its full subcategory on the "pairwise disjoint" functors.

c) There is the tree $\mathbb{N}$, and it is clear that a forest F is well-founded if and only if there is no morphism of forests $\mathbb{N} \to F$. The forest $\mathbb{N}$ is the terminal object of $\mathcal{F}$; so a forest is well-founded if and only if it admits no arrow from the terminal object to itself.

Solution to Exercise 2.

a) Suppose that the diagram

$$\begin{array}{ccc} X & \xrightarrow{k} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[e_1]{} & B' \end{array}$$

commutes. Then $f_1bk = f_1e_1h = g_1e_1h = g_1bk$, so $cf_0k = f_1bk = g_1bk = cg_0k$. Since c is mono, we have $f_0k = g_0k$, and by the equalizer property of e_0 we find that k factors uniquely through e_0 by a map $n : X \to A$. Then $e_1an = be_0n = bk = e_1h$, so since e_1 is mono, we have $an = h$. We conclude that the left hand square in the exercise is a pullback.

b) Suppose $F : \mathcal{I} \to \mathcal{C}$ is a diagram and (D, μ) is a limiting cone for F in $\mathcal{C}$. Composition with $\mathcal{C}(C, -) : \mathcal{C} \to \mathrm{Set}$ gives a diagram $G(i) = \mathcal{C}(C, F(i))$ in Set, where, for $f : i \to j$ in $\mathcal{I}$, $G(f) : \mathcal{C}(C, F(i)) \to \mathcal{C}(C, F(j))$ is given by composition with $F(f)$.

If X is a set and $\nu : \Delta_X \Rightarrow G$ a natural transformation then for each $x \in X$ and $i \in \mathcal{I}_0$ we have $\nu_i(x) : C \to G(i)$ and for $f : i \to j$ the diagram

$$\begin{array}{ccc} C & \xrightarrow{\nu_i(x)} & G(i) = \mathcal{C}(C, F(i)) \\ & \searrow^{\nu_j(x)} & \downarrow{\scriptstyle G(f)} \\ & & G(j) = \mathcal{C}(C, F(j)) \end{array}$$

So for every $x \in X$ we have a cone $\rho(x)$ in $\mathcal{C}$ with vertex C. Since (D, μ) is limiting, we have a unique map of cones $\rho(x) \to (D, \mu)$; that is, for each $x \in X$ an arrow $C \to D$ in $\mathcal{C}$. We conclude that the cone $\mathcal{C}(C, D) \stackrel{\mathcal{C}(C,\mu)}{\longrightarrow} \mathcal{C}(C, F)$ is limiting in Set.

Solution to Exercise 3.

a) First, suppose g is regular epi. The uniqueness of the required arrow $h : Y \to A$ is immediate from the assumption that m is mono, so we prove that such h exists. For the arrows a and b, choose regular epi-mono factorizations $a = m_1 e_1$, $b = m_2 e_2$. Using Proposition 4.3ii), we have that both $m_1(e_1 g)$ and $(m m_2)e_2$ are regular epi-mono factorizations of the composition ag:

$$\begin{array}{ccccc} X & \xrightarrow{e_2} & Z_2 & \xrightarrow{m_2} & A \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[e_1]{} & Z_1 & \xrightarrow[m_1]{} & B \end{array}$$

By the essential uniqueness of the regular epi-mono factorization, there is an isomorphism $\sigma : Z_1 \to Z_2$ satisfying $e_2 = \sigma e_1 g$ and $m m_2 \sigma = m_1$. Then $m_2 \sigma e_1 : Y \to A$ is the required diagonal filler.

Conversely, suppose such a diagonal filler always exists for any diagram meeting the specifications of the exercise. We have to prove that g is regular epi. Let $X \stackrel{e}{\to} Z \stackrel{m}{\to} Y$ be the regular epi-mono factorization. Since the diagram

$$\begin{array}{ccc} X & \xrightarrow{e} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle m} \\ Y & \xrightarrow[\mathrm{id}]{} & Y \end{array}$$

commutes and m is mono, there is a unique $h : Y \to Z$ with $mh = \mathrm{id}_Y$ and $hg = e$. Now $mhm = \mathrm{id}_Y m = m\mathrm{id}_Z$ so since m is mono, $hm = \mathrm{id}_Z$. We see that h is an inverse for m, so g is regular epi.

b) Let $A \stackrel{g}{\to} B \stackrel{f}{\to} C$ be arrows such that fg is regular epi. To show: f is regular epi. We use the criterion of part a), so suppose we have a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{b} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[a]{} & B \end{array}$$

Compose this with g to obtain:

$$\begin{array}{ccc} A & \xrightarrow{bg} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow{b} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle m} \\ C & \xrightarrow[a]{} & B \end{array}$$

Since fg is regular epi we have a unique $h : C \to A$ such that $mh = a$ and $hfg = bg$. Then $mhf = af = mb$ whence, since m is mono, $hf = b$; this means that h is also a diagonal filler for the original diagram.

c) This part requires some more work. The first thing to notice is, that a subobject R of $X \times Y$ is tha graph of some $f : X \to Y$ if and only if the composition $R \to X \times Y \to X$ is an isomorphism. I leave this to you. We also use the fact that an arrow is an isomorphism if and only if it is both mono and regular epi.

We consider the subobjects $[\![S(x, z)]\!]$, $[\![S(x, z')]\!]$ and $[\![z = z']\!]$ of $X \times Z \times Z$. We have the projections $\pi_{12} : X \times Z \times Z \to X \times Z$ (projection on the first and second coordinate) and $\pi_{13} : X \times Z \times Z \to X \times Z$. We have: $[\![S(x, z)]\!] = \pi_{12}^*(S)$ and $[\![S(x, z')]\!] = \pi_{13}^*(S)$, and $[\![S(x, z) \wedge S(x, z')]\!] = \pi_{12}^*(S) \wedge \pi_{13}^*(S)$ (the second $\wedge$ means: the meet in $\mathrm{Sub}(X \times Z \times Z)$). The assumption that the sequent $S(x, z) \wedge S(x, z') \vdash_{x,z,z'} z = z'$ is true means that

$$\pi_{12}^*(S) \wedge \pi_{13}^*(S) \leq [\![z = z']\!] \text{ in } \mathrm{Sub}(X \times Z \times Z)$$

Here $[\![z = z']\!]$ is the subobject of $X \times Z \times Z$ represented by the map $\mathrm{id}_X \times \delta : X \times Z \to X \times Z \times Z$, where $\delta : Z \to Z \times Z$ is the diagonal.

Furthermore we notice that for $S \in \mathrm{Sub}(X \times Z)$ the sequent $\vdash_x \exists z S(x, z)$ is true if and only if $S \to X$ is regular epi. Indeed, this sequent is true if and only if $\exists_{\pi_X}(S)$ is the top element of $\mathrm{Sub}(X)$ (where $\pi_X : X \times Z \to X$ is the projection), that is: if and only if the composition $S \to X \times Z \to X$ is regular epi.

Now suppose $\langle i_X, i_Y \rangle : R \to X \times Y$ represents the subobject R and $\langle u, v \rangle : S \times X \to Z$ represents S. We wish to show that $i_X : R \to X$ is an isomorphism. Because the map $u : S \to X$ is regular epi and it factors through i_X, by part b) of the exercise we know that i_X is regular epi. Therefore we have to see that i_X is mono.

Let $V \overset{f}{\underset{h}{\rightrightarrows}} R$ be a parallel pair such that $i_X f = i_X h$. Consider the map

$$a = \langle i_X f, i_Y f, i_Y h \rangle = \langle i_X h, i_Y f, i_Y h \rangle : V \to X \times Y \times Y$$

and consider the pullback

$$\begin{array}{ccc} W & \xrightarrow{c} & V \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle a} \\ X \times Z \times Z & \xrightarrow[\mathrm{id}_X \times g \times g]{} & X \times Y \times Y \end{array}$$

Writing q_{12}, q_{13} for the projections $X \times Y \times Y \to X \times Y$, we see that the map a factors through $q_{12}^*(R) \wedge q_{13}^*(R)$, and therefore the map b factors through $\pi_{12}^*(S) \wedge \pi_{13}^*(S)$. It follows from what we have seen before, that b factors through the subobject $\mathrm{id}_X \times \delta : X \times Y \to X \times Y \times Y$, and this means that $\pi_{12}b = \pi_{13}b$. Now we get

$$q_{12}ac = \langle \mathrm{id}_X \times g \rangle \pi_{12} b = \langle \mathrm{id}_X \times g \rangle \pi_{13} b = q_{13}ac.$$

Because c is regular epi, $q_{12}a = q_{13}a$. But this means that $f = h$. This concludes the proof that i_X is mono, and the exercise.

Solution to Exercise 4.

a) The multiplication of the monad GF has components

$$\mu_C = G(\varepsilon_{F(C)}) : GFGF(C) \to GF(C).$$

So in order to prove that μ is a natural isomorphism, it suffices to show that ε is a natural isomorphism. We prove that ε is both epi and split mono.

Consider a diagram $FG(D) \xrightarrow{\varepsilon_D} D \overset{f}{\underset{g}{\rightrightarrows}} D'$ in $\mathcal{D}$ such that $f\varepsilon_D = g\varepsilon_D$. Then their transposes along $F \dashv G$ are equal, which means $G(f) = G(g)$. Since G is faithful, we have $f = g$. We conclude that ε_D is epi.

Now, we prove that ε is split mono. Since G is full, we have an arrow $\alpha : D \to FG(D)$ such that $G(\alpha) = \eta_{G(D)} : G(D) \to GFG(D)$. The composition $FG(D) \xrightarrow{\varepsilon_D} D \xrightarrow{\alpha} FG(D)$ transposes to $G(\alpha) = \eta_{G(D)}$, which is also the transpose of the identity on $FG(D)$. We conclude that $\alpha\varepsilon_D$ is the identity on $FG(D)$, so ε_D is split mono.

b) The answer is yes. Suppose $h : GF(D) \to D$ is a T-algebra. Then $h\eta_D = \mathrm{id}_D$. We consider

$$\eta_D h : GF(D) \to GF(D)$$

Since G is full, there is an arrow $\beta : F(D) \to F(D)$ such that $G(\beta) = \eta_D h$. The transpose of β is $G(\beta)\eta_D : D \to GF(D)$, which by choice of β is equal to $\eta_D h \eta_D = \eta_D$, which is also the transpose of $\mathrm{id}_{F(D)}$. We conclude that $\beta = \mathrm{id}_{F(D)}$, so

$$\eta_D h = G(\beta) = G(\mathrm{id}_{F(D)}) = \mathrm{id}_{GF(D)}.$$

We see that h is a 2-sided inverse of η_D. So there is at most one T-algebra structure on an object D of $\mathcal{D}$. I leave it to you to prove that there is at least one, too, and to conclude that $T-\mathrm{Alg}$ is equivalent to $\mathcal{D}$.

Solution to Exercise 5.

a) i)⇔ii): this is just working out the definition.

ii)⇒iii): suppose Z is a nonempty subpresheaf of X; suppose $\xi \in Z(C)$. Taking the identity on C for g in ii), we see that for some $h : C' \to C$ we have $X(h)(\xi) \in Y(C')$. Since Z is a subpresheaf of X, $X(h)(\xi) \in Y(C') \cap Z(C')$, therefore $Y \cap Z$ is nonempty.

iii)⇒ii): suppose $\xi \in X(C)$, $g : C' \to C$. Consider the subpresheaf Z of X generated by $X(g)(\xi)$: $Z(C'') = \{X(gh)(\xi) \,|\, h : C'' \to C'\}$. Then Z is nonermpty. By iii), $Z \cap Y$ is nonempty, so there is some $h : C'' \to C'$ such that $X(gh)(\xi) \in Y(C'')$; i.e., ii) holds.

b) Suppose $Y \subseteq X$ is dense, and $\xi \in X(C)$. For some $h : C' \to C$ we have $X(h)(\xi) \in Y(C')$. Now if h is an isomorphism, we find that $X(h^{-1})(X(h)(\xi)) \in Y(C)$, that is: $\xi \in Y(C)$. So $Y = X$.

Solution to Exercise 6.

a) Choosing for each object X of $\mathcal{C}$ a partial map classifier $\zeta_X : X \to \tilde{X}$, we have an assignment $\widetilde{(\cdot)}$ on objects. In order to see that $\widetilde{(\cdot)}$ can be extended to a functor, use the defining property of ζ_X on arrows $f : X \to Y$: let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the unique arrow making the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \zeta_X}\downarrow & & \downarrow{\scriptstyle \zeta_Y} \\ \tilde{X} & \xrightarrow[\tilde{f}]{} & \tilde{Y} \end{array}$$

a pullback (note, that f is a partial map from $\tilde{X}$ to Y).

If $f = \mathrm{id}_X$, then clearly $\tilde{f} = \mathrm{id}_{\tilde{X}}$ since this turns the relevant square into a pullback. Similarly, for $g : Y \to Z$ we have that the outer square of the composite diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle \zeta_X}\downarrow & & \downarrow{\scriptstyle \zeta_Y} & & \downarrow{\scriptstyle \zeta_Z} \\ \tilde{X} & \xrightarrow[\tilde{f}]{} & \tilde{Y} & \xrightarrow[\tilde{g}]{} & \tilde{Z} \end{array}$$

is a pullback; hence $\tilde{g}\tilde{f} = \widetilde{gf}$ by uniqueness. So $\widetilde{(\cdot)}$ is a functor, and $\zeta : \mathrm{id}_{\mathcal{C}} \Rightarrow \widetilde{(\cdot)}$ is a natural transformation.

b) First we need to see that $\tilde{F}$ as defined in part b) is indeed a sheaf on X. So, suppose we have a compatible family in $\tilde{F}$ at some open $U \subseteq X$, indexed by a covering sieve. That is, we have an open cover $(U_i)_{i\in I}$ of U and elements (V_i, x_i) of $\tilde{F}(U_i)$. Hence $V_i \subseteq U_i$ and $x_i \in F(V_i)$. That this is a compatible family in $\tilde{F}$ at U, means that for $i, j \in I$ we have $x_i{\restriction}V_i{\cap}V_j = x_j{\restriction}V_i{\cap}V_j$. We see that the family $(x_i)_{i\in I}$ is a compatible family in F at $\bigcup_{i\in I} V_i$. Since F is a sheaf, this family has a unique amalgamation $x \in F(V)$ where $V = \bigcup_{i\in I} V_i$. Now the pair $(V, x) \in \tilde{F}(U)$ is the uniqwue amalgamation of the original family; we conclude that $\tilde{F}$ is a sheaf.

Clearly, we have a natural transformation $\zeta_F : F \to \tilde{F}$, defined by

$$(\zeta_F)_U(x) = (U, x).$$

Now suppose G is a sheaf on the space X, $H \subseteq G$ a subsheaf and $\mu : H \to F$ a morphism of sheaves. We define $\bar{\mu} : G \to \tilde{F}$ as follows: for $x \in G(U)$ let $\bar{\mu}_U(x)$ be (V, y) where

$$V \ = \ \bigcup\{W \subseteq U \mid x{\restriction}W \in H(W)\}$$

and $y \in F(V)$ is $\mu_V(x{\restriction}V)$ (check that $x{\restriction}V \in H(V)$). This is the only option for $\bar{\mu}$, and the pullback property is left to you to check.

c) [Sketch.] Now let G be a sheaf on a site $(\mathcal{C}, J)$. For a subsheaf H of G, an object C of $\mathcal{C}$ and $x \in G(C)$, the sieve

$$R_x \ = \ \{f : C' \to C \mid G(f)(x) \in H(C')\}$$

is closed, since H is a sub*sheaf*. Therefore, if F is a sheaf and $\mu : H \to F$ a map of sheaves, for each $x \in G(C)$ we have a closed sieve R_x on C and an arrow $R_x \to F$.

So we define $\tilde{F}(C)$ to be the set of pairs (R, ξ) where R is a closed sieve on C and ξ a morphism $R \to F$ (i.e., a compatible family in F at C, indexed by the closed sieve R). The rest is analogous to the case in b) and left to you.