Hilbert 10 Seminar

Homework Set 15

Simon Docherty (Due Jan 21st)

In this homework we prove some properties of the Chebyshev polynomials:

$A_0(T) = 1$	$B_0(T) = 0$
$A_1(T) = T$	$B_1(T) = 1$
$A_{n+2}(T) = 2T \cdot A_{n+1}(T) - A_n(T)$	$B_{n+2}(T) = 2T \cdot B_{n+1}(T) - B_n(T)$

1. Show $B_n(1) = m$ and $deg(B_{n+1}) = n$ for all n.

2. Recall the definition of the Chebyshev polynomials given in Jetze's lecture:
   The n-th Chebyshev polynomials are given by $(T + \sqrt{T^2 - 1})^n = X_n(T) + Y_n(T)\sqrt{T^2 - 1}$
   Show the two definitions coincide.

3. In the lecture we showed the Chebyshev polynomials give solutions to the $\mathbb{Z}[T]$ Pell equation $X^2 - (T^2 - 1)Y^2 = 1$. In this exericse we prove the sequences exhaust all solutions to the equation.

   Let $U(T)$,$V(T) \in \mathbb{Z}[T]$ be such that $[U(T)]^2 - (T^2 - 1)[V(T)]^2 = 1$

   a) Show that we may assume there exists $N \in \mathbb{Z}^+$ such that for all $a \in \mathbb{Z}^+$ if $a > N$ then $U(a) = a_{f(a)}$ and $V(a) = a_{f(a)}$ for some function f.
   b) Show there exists $K \in \mathbb{N}$ such that for all $\mathbb{Z}^+ \ni a > N$, $f(a) < K$
   [Hint: Consider the characterisation of solutions of the integer Pell equation and use this to determine the behaviour of $f(a)$ as $a \to \infty$]
   c) Conclude that for some $n \in \mathbb{N}$: $U(T) = A_n(T)$, $V(T) = B_n(T)$