# Seminar on Hilbert's Tenth Problem
# Homework, due October 14

Jetze Zoethout

**Exercise 1.**
In this exercise, we will prove our second congruence property.

We can extend the sequence $\alpha_b(n)$ to all $n \in \mathbb{Z}$ by extending our recurrence relation to all of $\mathbb{Z}$. That is: we put $\alpha_b(n+2) = b\alpha_b(n+1) - \alpha_b(n)$ for all $n \in \mathbb{Z}$. Of course, this also gives us an extension of $A_b(n)$ to all $n \in \mathbb{Z}$.
**a)** Prove that $\alpha_b(-n) = -\alpha_b(n)$ for all $n \in \mathbb{N}$ and that $A_b(n) = B_b^n$ for all $n \in \mathbb{Z}$.
**b)** Using these results, find $A_b^{-1}(n)$.

Let $j, l, m, n$ be natural numbers such that $n = 2lm \pm j$ en define $v = \alpha_b(m+1) - \alpha_b(m-1)$.
**c)** Prove that $A_b(m) \equiv -A_b^{-1}(m) \mod v$.
**d)** Show that $A_b(n) \equiv \pm (A_b(j))^{\pm 1} \mod v$.
**e)** Prove the second congruence property, i.e. $\alpha_b(n) \equiv \pm\alpha_b(j) \mod v$.
For d) and e), please note that the $\pm$-sign(s) do(es) not necessarily correspond with the one in $n = 2lm \pm j$.

**Exercise 2.**
Let $b \geq 2$ and $x$ be natural numbers.
**a)** Prove that $x = \alpha_b(m)$ for some $m \in \mathbb{N}$ if and only if $4 + (b^2 - 4)x^2$ is a square. Hint: consider the characteristic equation.

Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$. We will now prove a nice property of these numbers.
**b)** Prove that $\alpha_3(n) = F_{2n}$ for all $n \in \mathbb{N}$. Conclude that $5x^2 + 4$ is a square if and only if $x$ is of the form $x = F_{2n}$.

Let $c \geq 2$ be a natural number. Consider the Pell equation $x^2 - (c^2 - 1)y^2 = 1$.
**c)** Prove that

$$\{y \in \mathbb{N} : \exists x \in \mathbb{N}(x^2 - (c^2 - 1)y^2 = 1)\} = \{\alpha_{2c}(n) : n \in \mathbb{N}\}.$$