Hilbert 10 Seminar

Homework Set 7

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1. Consider the ring $\mathbb{Z}[\sqrt{a^2-1}] = \{x + y\sqrt{a^2-1} \mid x, y \in \mathbb{Z}\}$.
   Define for $z = x + y\sqrt{a^2-1}$:

$$\begin{aligned} \overline{z} &:= x - y\sqrt{a^2-1} \\ N(z) &:= z\overline{z} \end{aligned}$$

   a) Prove $G = \{z \in \mathbb{Z}[\sqrt{a^2-1}] \mid N(z) = 1\}$ forms an infinite abelian group under multiplication.
   b) Show there exists least such $z_0 > 1$ with $N(z_0) = 1$. Show G is generated by $\{z_0, -1\}$.
   c) Conclude for all $x, y \in \mathbb{N}$:

$$x^2 - (a^2-1)y^2 = 1 \iff \exists n \left[x + y(a^2-1)^{1/2} = (a + (a^2-1)^{1/2})^n\right]$$

2. Let R be a relation of roughly exponential growth order n; that is:

   i) $\forall u, v\, [Ruv \implies v < u * n]$
   ii) $\forall k \exists u, v \left[Ruv \wedge u^k \leq v\right]$

   where $u * 0 = 1$ and $u * (n+1) = u^{u*n}$

   a) Assume $\exists k \forall u, v \left[Ruv \implies v < u^{ku}\right]$
      Define $\mathcal{J}xy$ iff $\exists v \left[Rxv \wedge y^k \leq v\right]$
      Show $\mathcal{J}$ is a Robinson Relation.
   b) Assume $\neg \exists k \forall u, v \left[Ruv \implies v < u^{ku}\right]$
      Consider R_1 defined by R_1uv iff $\exists a\, [\psi(a, u) \wedge Rav]$ where

$$\psi(a, u) \iff \exists x, y \left[x^2 - (a^2-1)(a-1)^2y^2 = 1 \wedge x > 1 \wedge a > 1 \wedge u = ax\right]$$

      i) Show $\forall a > 1 \exists u\, [\psi(a, u) \wedge u < a^{2a}]$
      ii) Show R_1 is a relation of roughly exponential growth order $n - 1$.
   c) Conclude: Given a relation of roughly exponential growth R, a Robinson Relation $\mathcal{J}$ is existentially definable in terms of R.