Tame Topology and O-minimal Structures, Smoothness and triangulation, homework set

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For these questions, we assume an o-minimal structure $(R, <, S)$ extending an ordered field $(R, <, 0, 1, +, -, \cdot)$.

1 C^k functions

Let $f = (f_1, f_2, ..., f_n) : U \to R^n$ be a definable map on an open set $U \subset R^m$. We give an inductive definition for f to be C^k, where k is a positive integer:
f is C^1 if it satisfies the original definition.
f is C^{k+1} if f is C^1 and $df : U \to R^{nm}$ is C^k.

a) Show that for an open $U \subset R^m$, the inclusion map $U \to R^m$ is C^k for all $k > 0$.
b) Assume $f : U \to R^n$ ($U \subset R^m$ open and f is C^k) and $g : V \to R^p$ ($V \subset R^n$ open and g is C^k). Proof that $g \circ f : W \to R^p$ is C^k for any set $W \subset f^{-1}(V)$ open in R^m.
c) For $f : A \to R$ with $A \subset R$ and $k > 0$. Proof that there is decomposition of R partitioning A such that on all open cells C in the partition, $f|_C$ is C^k.

2 Good linear spaces

Let $A \subset R^m$ definable with $dim(A) \leq k < m$. Show that there is a linear space L in R^m of dimension $(m - k)$ such that for all $v \epsilon R^m$ we have that with $L_v := \{v + x : x \epsilon L\}$, the intersection $L_v \cap A$ is finite.
Hint: use the good directions lemma (7.4.2)

3 Open faces of complexes

Let K be a complex of R^m, in the ordered field $(R, <, 0, 1, +, -, \cdot)$.

a) Let $\sigma \epsilon K$ with $dim(\sigma) = dim(|K|)$. Show that σ is open in $|K|$.
Note: You may assume that the dimension of $|K|$ is the maximal dimension of its elements.
b) Give an example of a complex K with an element $\sigma \epsilon K$ for which σ is not open in K.