Partial Combinatory Algebras – Variations on a Topos-theoretic Theme

Jaap van Oosten

Department of Mathematics
Utrecht University

Wisconsin Logic Seminar, November 17, 2020

Partial Combinatory Algebras (PCAs) PCAs are the partial variant of "models for Combinatory Logic".

Combinatory Logic was defined by Moses Schönfinkel in a paper from 1924: *Über die Bausteine der mathematischen Logik*.

Peter Johnstone therefore calls PCAs "Schönfinkel algebras".

Which prompts the following short biographical intermezzo:

Moses Ilyich (or is it *Isayevich*?) *Schönfinkel* is one of the more mysterious figures in the history of logic. He was born in 1889 (or was it 1887?) in Ukraina. He worked from 1914 (!) to 1924 under Hilbert in Göttingen, during which period one paper appeared: *Über die Bausteine der mathematischen Logik* in *Mathematische Annalen* 92, 1924. However, this paper appears to have been written by someone else, who took notes during lectures by Schönfinkel.

A second paper, coauthored by Bernays, appeared in 1927; by this time, however, Schönfinkel was already in a mental hospital in Moscow.

He died in 1942 in Moscow; his papers were used for firewood by his neighbours.

Combinatory Logic a calculus of terms in variables and constants k and s, and an operation which we write as juxtaposition, with axioms

$$(kx)y = x$$
$$((sx)y)z = (xz)(yz)$$

The letter k stands for "Konstante Funktion"; the letter s is mysteriously called "Verschmelzungsfunktion" (blending function). Aim: to provide an alternative foundation of mathematics in which not sets, but *functions* are the primitive notion.

The partial version

Partial version of CL: term do not necessarily denote (may be "undefined"). We write $t\downarrow$ to mean that t is defined.

A *Partial Combinatory Algebra* is a set A together with a partial map $A \times A \rightharpoonup A$ (the *application map*). Also here we write, for elements $a, b \in A$, $ab \downarrow$ to indicate that the pair (a, b) is in the domain of the application map.

Moreover, a PCA A should have elements k and s satisfying:

$$kx \downarrow$$
$$(ka)b = a$$
$$(sa)b \downarrow$$

and: if $(ac)(bc) \downarrow$ then $((sa)b)c \downarrow$ and

$$((sa)b)c = (ac)(bc)$$

.

We use the following conventions for brackets and other notations: a statement $t = s$ implies that t, s and all their subterms are defined.

We write $t \preceq s$ to mean: if $s \downarrow$ then $t = s$. We write $t \simeq s$ to mean $t \preceq s$ and $s \preceq t$.

The notion of PCA is due to Feferman (1975). He had the aim of giving a language and axioms for "explicit mathematics". This is a form of realizability (but within classical logic); still prominent in research of the Swiss school (Jaeger, Strahm).

Inge Bethke (thesis under Troelstra, 1988) studies notions like "extensionality" and "totality" of PCAs.

Examples of PCAs a. $\mathcal{K}_1$ ("Kleene's first model") is the set of natural numbers; $mn \simeq \varphi_m(n)$ with φ_m the m-th partial recursive function.

b. $\mathcal{K}_2$ ("Kleene's second model") is the set $\mathbb{N}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$. We assume a coding of sequences $\langle a_0, \ldots a_{n-1} \rangle$. For functions α, β, we let $\alpha\beta \downarrow$ if and only if for each natural number n there is some k such that

$$\alpha(\langle n, \beta(0), \ldots, \beta(k-1) \rangle) > 0$$

and we let $\alpha\beta(n) = \alpha(\langle n, \beta(0), \ldots, \beta(k-1) \rangle) - 1$ for the least such k.

Examples of PCAs (continued)
A *total* combinatory algebra (in fact, a λ-model) was defined by
Dana Scott: let $\mathcal{S}$ be the powerset of $\mathbb{N}$. We assume bijections:

$$\begin{aligned}
\langle \cdot, \cdot \rangle : & \quad \mathbb{N}^2 \to \mathbb{N} \\
e_- : & \quad \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathbb{N})
\end{aligned}$$

Let $AB = \{y \,|\, \text{for some } n, e_n \subseteq B \text{ and } \langle n, y \rangle \in A\}$
Both $\mathcal{K}_2$ and $\mathcal{S}$ can be restricted to (respectively) *recursive*
functions and *recursively enumerable* sets: $\mathcal{K}_2^{\mathrm{rec}}$, $\mathcal{S}^{\mathrm{r.e.}}$.

Some recursion theory inside a PCA

For any term t in variables $x_0, \ldots, x_n$ there is a term $\Lambda x_0 \cdots x_n.t$ with the following properties: for each tuple $a_0, \ldots, a_n$ from A we have

- $(\Lambda x_0 \cdots x_n.t)a_0 \cdots a_{n-1} \downarrow$
- $(\Lambda x_0 \cdots x_n.t)a_0 \cdots a_n \preceq t(a_0, \ldots, a_n)$

Let $p = \Lambda xyz.zxy$ so $pab = \Lambda z.zab$; let $p_0 = \Lambda v.vk$ and let $p_1 = \Lambda v.v(\Lambda wu.u)$. Then $p_0(pab) = a$ and $p_1(pab) = b$ so p is an *ordered pair* operator, with *unpairings* p_0 and p_1.

There are also *Booleans* t and f and a *definition by cases* term C satisfying $Ctab = a$ and $Cfab = b$.

Recursion theory in PCAs (continued) There is a copy of $\mathbb{N}$ in A:
$\{\bar{n} \mid n \in \mathbb{N}\}$, the *Curry numerals*.

For every k-ary partial recursive function ϕ there is an element a_ϕ
of A simulating ϕ: for all $n_1, \ldots, n_k \in \mathbb{N}$,

$$a_\phi \bar{n}_1 \cdots \bar{n}_k \preceq \overline{\phi(n_1, \ldots, n_k)}$$

We can manipulate finite sequences $\langle a_0, \ldots, a_{k-1} \rangle$ of elements of
A. For example we have for suitable $c, d \in A$:

$$c\bar{i}\langle a_0, \ldots, a_{k-1} \rangle = a_i$$
$$d\langle a_0, \ldots, a_{k-1} \rangle = \bar{k}$$

Recursion theory in PCAs (continued) We have a *recursion theorem* in every PCA A: there are elements y, z satisfying, for each $f \in A$:

  i) $yf \preceq f(yf)$

 ii) $zf \downarrow$

iii) $zfx \preceq f(zf)x$ for all $x \in A$.

In Andy Pitts' thesis (1981) and a paper by Hyland, Johnstone and Pitts (1980) it is explained how every PCA A gives rise to a topos, the *realizability topos* over A, $RT(A)$.

Hyland's paper "The effective topos" describes the topos $RT(\mathcal{K}_1)$ in great detail.

The starting point: given a PCA A we have a category $Ass(A)$ of *assemblies* over A.

An assembly over A is a pair (X, E) where X is a set and $E(x)$ is a *nonempty* subset of A, for each $x \in X$.

A *morphism of assemblies* $(X, E) \to (Y, F)$ is a function $f : X \to Y$ of sets, for which there is an element $a \in A$ such that for all $x \in X$ and all $b \in E(x)$, $ab \in F(f(x))$. One says that a *tracks* the function f.

The category Ass(A) is locally cartesian closed, regular, has a weak subobject classifier (is a quasi-topos). Moreover, Ass(A) comes with an adjunction

$$(\Gamma : \text{Ass}(A) \to \text{Set}) \dashv (\nabla : \text{Set} \to \text{Ass}(A))$$

$\Gamma(X, E) = X$; $\nabla(X) = (X, \lambda x.A)$.
The category Ass(A) also has a *natural numbers object*
$N = (\mathbb{N}, E)$ with $E(n) = \{\bar{n}\}$.

Structure of Ass(A):

Product $(X, E) \times (Y, F)$ is $(X \times Y, G)$ where
$G(x, y) = \{\mathsf{p}ab \,|\, a \in E(x), b \in F(y)\}$.

Exponent $(Y, F)^{(X,E)}$ is (Z, G) where Z is the set of morphisms $(X, E) \to (Y, F)$ in Ass(A), and $G(f)$ is the set of elements a which track f.

Example. Let us consider, in $\text{Ass}(\mathcal{K}_1)$, the finite type structure over the natural numbers object N. It is isomorphic to $(\mathbb{N}, E)$ where $E(n) = \{n\}$.

We have the basic type o and for types σ, τ the arrow type $\sigma \Rightarrow \tau$. In $\text{Ass}(\mathcal{K}_1)$ we form objects X_σ for each type σ, starting with $X_o = N$ and taking exponents for the arrow types.

We obtain the structure of "hereditarily effective operations" of Kreisel-Troelstra; one of the models of the system HA^ω of intuitionistic arithmetic in all finite types. This was Hyland's original motivation for developing the effective topos.

The realizability topos $RT(A)$ is the *exact completion* of the *regular category* $Ass(A)$. One formally adds quotients of equivalence relations. Details are skipped.

The category $Ass(A)$ is a full subcategory of $RT(A)$. Actually, the category Set is the category of $\neg\neg$-sheaves in $RT(A)$, and $Ass(A)$ is the category of $\neg\neg$-separated objects (the objects X for which the statement $\forall xy \in X(\neg\neg(x = y) \to x = y)$ holds).

We now wish to understand: how functorial is the construction $A \mapsto \mathrm{RT}(A)$?

It turns out that there is a very nice categorical structure on the class of PCAs, which was first explored by John Longley in his thesis (1995). It has the following features:

It ties up with the standard notion of morphism for toposes, namely: geometric morphisms (Johnstone 2013, Faber/vO 2014).

It ties up with standard notions of classical recursion theory (Longley 1995, vO 2006, Longley/Normann 2015, Faber/vO 2016).

Applicative morphisms of PCAs

Let A, B be PCAs. An *applicative morphism* $A \to B$ is a total relation γ (we think of γ as a function from A to the set of nonempty subsets of B, so (A, γ) is an assembly over B) for which there is an element $r \in B$ which satisfies:

For each pair a, a' of elements of A and $b \in \gamma(a), b' \in \gamma(a')$, if $aa' \downarrow$ in A then $rbb' \downarrow$ in B, and $rbb' \in \gamma(aa')$.

The element r *realizes* the morphism γ. Composition of morphisms is composition of total relations.

We think of γ as a *simulation* in B of computations in A; the element r is a machine that translates code for an A-program into code for a B-program.

Examples of applicative morphisms

$\delta_1 : \mathcal{K}_1 \to A$: $\delta_1(n) = \{\bar{n}\}$ is the essentially unique applicative morphism $\mathcal{K}_1 \to A$ (up to a suitable notion of isomorphism of applicative morphisms)

$\delta_2 : \mathcal{K}_2^{\mathrm{rec}} \to \mathcal{K}_1$: $\delta_2(\phi) = \{e \in \mathbb{N} \,|\, \phi = \varphi_e\}$. Think of what a realizer of this morphism does; how it simulates the action of $\mathcal{K}_2^{\mathrm{rec}}$ in $\mathcal{K}_1$!

There are interesting applicative morphisms between $\mathcal{K}_2$ and $\mathcal{S}$ in both directions.

Theorem (Longley, 1995): every applicative morpjism $A \xrightarrow{\gamma} B$ gives rise to a regular functor $\mathsf{Ass}(\gamma) : \mathsf{Ass}(A) \to \mathsf{Ass}(B)$ which makes the diagrams

$$
\begin{array}{ccc}
\mathsf{Ass}(A) \xrightarrow{\mathsf{Ass}(\gamma)} \mathsf{Ass}(B) & \quad & \mathsf{Ass}(A) \xrightarrow{\mathsf{Ass}(\gamma)} \mathsf{Ass}(B) \\
\searrow_{\Gamma} \quad \downarrow^{\Gamma} & & \nwarrow_{\nabla} \quad \uparrow^{\nabla} \\
\mathrm{Set} & & \mathrm{Set}
\end{array}
$$

commute. Conversely, every regular functor making the two diagrams commute, is of the form $\mathsf{Ass}(\gamma)$ for some applicative morphism $\gamma : A \to B$.

A *geometric morphism* of toposes $f : \mathcal{F} \to \mathcal{E}$ consists of an adjoint pair

$$(f^* : \mathcal{E} \to \mathcal{F}) \dashv (f_* : \mathcal{F} \to \mathcal{E})$$

such that the left adjoint f^* preserves finite limits.

Examples: 1. If $\mathcal{F}$ and $\mathcal{E}$ are categories of sheaves over sober spaces X and Y, respectively, then these correspond exactly to continuous maps $X \to Y$.

2. The adjunction $\Gamma \dashv \nabla$ between Set and Ass(A) extends to a geometric morphism $\mathrm{Set} \to \mathrm{RT}(A)$, which embeds Set as the category of $\neg\neg$-sheaves in $\mathrm{RT}(A)$

What do geometric morphisms between realizability toposes look like?

Fundamental observation by P.T. Johnstone: Every geometric morphism $RT(A) \to RT(B)$ restricts to an adjunction between the categories of assemblies.

The left adjoint of such a restriction is always a regular functor commuting with the Γ's and ∇'s, and therefore corresponds to an applicative morphism $B \xrightarrow{\gamma} A$. The question then is:

For which applicative morphisms $\gamma : B \to A$ does the regular functor $\mathsf{Ass}(\gamma) : \mathsf{Ass}(B) \to \mathsf{Ass}(A)$ have a right adjoint?

Answer: (Hofstra/vO 2003; Johnstone 2013) For an applicative morphism $\gamma : B \to A$ the functor $\mathrm{Ass}(\gamma)$ has a right adjoint if and only if γ satisfies the following condition:

There is an element $q \in A$ such that for each $a \in A$ there exists a $b \in B$ satisfying $q\gamma(b) = \{a\}$

Here $q\gamma(b) = \{a\}$ means: for all $a' \in \gamma(b)$, $qa' = a$.

Special case of geometric morphisms: inclusions

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called an *inclusion* if the right adjoint f_* is full and faithful. In the case of categories of sheaves over spaces, this corresponds to an *embedding* of topological spaces.

Here I wish to draw attention to some specific inclusions between realizability toposes.

Definition: Let A and B be PCAs; let us write t_A, f_A for the Booleans in A and ditto t_B, f_B for B.

An applicative morphism $\gamma : A \to B$ is *decidable* if there is an element $d \in B$ such that $d\gamma(t_A) = \{t_B\}$ and $d\gamma(f_A) = \{f_B\}$.

Equivalently, the functor $\mathrm{Ass}(\gamma)$ preserves finite sums.

Note, that if $\mathrm{Ass}(\gamma)$ has a right adjoint, γ is necessarily decidable.

Computations in PCAs with an oracle

Let $\gamma : A \to B$ be an applicative morphism. A partial function $f : A \rightharpoonup A$ is *representable* w.r.t. γ if there is an element $b \in B$ satisfying: for each $a \in A$, if $f(a) \downarrow$ then $b\gamma(a) \subseteq \gamma(f(a))$.

Theorem (vO 2006): Given PCA A and partial function f on A, there is a PCA $A[f]$ which is universal with the property that there is a decidable applicative morphism $\iota_f : A \to A[f]$ w.r.t which f is representable: if $\gamma : A \to B$ is decidable and f is representable w.r.t. γ, then γ factors uniquely through ι_f:

$$A \xrightarrow{\iota_f} A[f]$$
$$\gamma \searrow \quad \downarrow$$
$$B$$

Applying this construction to $\mathcal{K}_1$ gives us the PCA of "computations with oracle f".

Note, that this construction gives us a notion of "Turing reducibility in A": if f and g are partial functions on A, then $f \leq_T g$ if and only if f is representable w.r.t. $\iota_g : A \to A[g]$. Equivalently: for every decidable applicative morphism $A \xrightarrow{\gamma} B$ we have: if g is representable w.r.t. γ, then so is f.

An extension of the "oracle" result (Faber/vO 2016)

Given a PCA A, we can define what we call an "effective operation of type 2" in A, and we have, for any partial function $F : A^A \rightharpoonup A$ a similar universal solution for "forcing F to be an effective operation": a decidable applicative morphism $\iota_F : A \to a[F]$ with the expected universal property.

We have the following result (which should not come unexpected): For the Kleene functional E ($E(f) = 0$ if and only if $\exists n f(n) = 0$) we have: a function $\mathbb{N} \to \mathbb{N}$ is representable w.r.t. $\mathcal{K}_1[E]$ if and only if the function f is hyperarithmetical.

This opens up the possibility of "realizability with hyperarithmetical functions"; this is a sheaf subtopos of the effective topos in which there is a model of Peano Arithmetic (with classical logic!). Such a model cannot exist in the effective topos.