Exam Topos Theory, June 15, 2023, 14:00–17:00
with solutions

Exercise 1 Given an object X of a topos $\mathcal{E}$, we consider a subobject $a : A \to X$ of X, classified by an arrow $\phi : X \to \Omega$. We also consider the least subobject of X, $0 \to X$, classified by $n : X \to \Omega$.

a) (5 pts) Let $E \xrightarrow{e} X \underset{n}{\overset{\phi}{\rightrightarrows}} \Omega$ be an equalizer diagram. Show that the subobject E of X has the following universal property: whenever C is a subobject of X such that $C \cap A = 0$, then $C \leq E$. In logical terms, E is the Heyting implication $A \Rightarrow 0$.

b) (5 pts) Suppose we now replace 0 by an arbitrary subobject $b : B \to X$ of X. We now obtain a binary operation on subobjects of X by considering the equalizer

$$E \xrightarrow{e} X \underset{\psi}{\overset{\phi}{\rightrightarrows}} \Omega$$

where $\psi : X \to \Omega$ classifies B. Does this give $A \Rightarrow B$? Determine what we obtain in the case $\mathcal{E} = \text{Set}$.

Solution a): For a subobject C of X, say $c : C \to X$ mono, the intersection $C \cap A$ is classified by ϕc, and $0 = C \cap 0$ is classified by nc. So $C \cap A = 0$ if and only if $\phi c = nc$, that is: iff c equalizes n and ϕ and hence factors through E, in other words $C \leq E$.

b): We have a binary operation on subobjects of X, sending a pair of subobjects $(a : A \to X, b : B \to X)$ the equalizer $e : E \to X$ of their two classifying maps. This does, of course, *not* give $A \Rightarrow B$, since our operation is symmetric whereas $\Rightarrow$ is not. In Set, we get

$$(A \cap B) \cup (X - (A \cup B))$$

Exercise 2 This exercise shows that every slice of a presheaf category is again a presheaf category: show that for a small category $\mathcal{C}$ and a presheaf F on $\mathcal{C}$, the slice category $\widehat{\mathcal{C}}/F$ is equivalent to a presheaf category. [Hint: consider the category of elements of F]

Solution: We define a functor $\mathcal{I} : \widehat{\mathcal{C}}/F \to \widehat{\text{Elts}(F)}$. For an object $\mu : G \to F$ of $\widehat{\mathcal{C}}/F$ we define $\mathcal{I}(\mu) : \text{Elts}(F)^{\text{op}} \to \text{Set}$ by: $\mathcal{I}(\mu)(x, C) = \mu_C^{-1}(x) \subseteq G(C)$.

For an arrow $\alpha : (y, D) \to (x, C)$ in $\mathrm{Elts}(F)$, i.e. an arrow $\alpha : D \to C$ in $\mathcal{C}$ for which $F(\alpha)(x) = y$, we see that $G(\alpha)$ sends $\mu_C^{-1}(x)$ to $\mu_D^{-1}(y)$, i.e. $\mathcal{I}(\mu)(\alpha)$ sends $\mathcal{I}(\mu)(x)$ to $\mathcal{I}(\mu)(y)$. Checking functoriality is left to you. We see that $\mathcal{I}(\mu)$ is a presheaf on $\mathrm{Elts}(F)$.

Now consider an arrow $f : \mu \to \nu$ in $\widehat{\mathcal{C}}/F$:

$$\begin{array}{ccc} G & \xrightarrow{\;f\;} & H \\ & {\scriptstyle\mu}\searrow \qquad \swarrow{\scriptstyle\nu} & \\ & F & \end{array}$$

We see that f_C sends $\mu_C^{-1}(x)$ to $\nu_C^{-1}(x)$, that is: we have $\mathcal{I}(f) : \mathcal{I}(\mu) \to \mathcal{I}(\nu)$ and a functor $\mathcal{I} : \widehat{\mathcal{C}}/F \to \widehat{\mathrm{Elts}}(F)$.

In the other direction we have the embedding from $\mathrm{Elts}(F) = y{\downarrow}F$ into $\widehat{\mathcal{C}}/F$ (sending an object (x, C) to the object $y_C \to F$ of $\widehat{\mathcal{C}}/F$ to which it corresponds) and take the left Kan extension of this.

If you had roughly this amount of detail, you got full credits for the exercise.

Exercise 3 Suppose $f : A \to B$ is monic, $g : A \to C$ arbitrary. Let $h : B \to \Omega^C$ be the transpose of the classifying map of the mono $\langle f, g \rangle : A \to B \times C$. So

$$\begin{array}{ccc} A & \longrightarrow & B \times C \\ \downarrow & & \downarrow{\scriptstyle \tilde{h}} \\ 1 & \xrightarrow[\;t\;]{} & \Omega \end{array}$$

is a pullback, and h is the transpose of $\tilde{h}$. Show that the square

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \{\cdot\}} \\ B & \xrightarrow[\;h\;]{} & \Omega^C \end{array}$$

is a pullback.

Solution: first we check that the diagram commutes. We consider the exponential transposes of both compositions (clockwise and counterclockwise).

Clockwise we get $A \times C \xrightarrow{g\times \mathrm{id}} C \times C \xrightarrow{\Delta} \Omega$ which, by considering the pullback

$$\begin{array}{ccccc} A\times C & \xrightarrow{g\times \mathrm{id}} & C\times C & \xrightarrow{\Delta} & \Omega \\ {\scriptstyle\langle \mathrm{id},g\rangle}\uparrow & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle t} \\ A & \xrightarrow[g]{\qquad} & C & \xrightarrow{\qquad} & 1 \end{array}$$

classifies the graph of g as subobject of $A \times C$.

Counterclockwise we get the top row of the following diagram of pullbacks:

$$\begin{array}{ccccc} A\times C & \xrightarrow{f\times \mathrm{id}} & B\times C & \xrightarrow{\tilde{h}} & \Omega \\ {\scriptstyle\langle \mathrm{id},g\rangle}\uparrow & & \uparrow{\scriptstyle\langle f,g\rangle} & & \uparrow{\scriptstyle t} \\ A & \xrightarrow[\mathrm{id}]{\qquad} & A & \xrightarrow{\qquad} & 1 \end{array}$$

which is seen to classify the same graph of g. The two compositions agree, and the square commutes.

To show that the diagram is a pullback, suppose we have arrows $v : V \to C$ and $w : V \to B$ satisfying $\{\cdot\} \circ v = hw$. Again transposing, we see that the maps $V \times C \xrightarrow{v\times \mathrm{id}} C \times C \xrightarrow{\Delta} \Omega$ and $V \times C \xrightarrow{w\times \mathrm{id}} B \times C \xrightarrow{\tilde{h}} \Omega$ agree. We have a commutative square

$$\begin{array}{ccc} V\times C & \xrightarrow{v\times \mathrm{id}} & C\times C \\ {\scriptstyle w\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle\Delta} \\ B\times C & \xrightarrow[\tilde{h}]{\qquad} & \Omega \end{array}$$

However, by definition of $\tilde{h}$ and Δ, the following diagram consists of pullbacks, hence its outer square is a pullback:

$$\begin{array}{ccccc} A\times C & \xrightarrow{\pi_C} & C & \xrightarrow{\delta} & C\times C \\ {\scriptstyle\pi_A}\downarrow & & \downarrow & & \\ A & \xrightarrow{\qquad} & 1 & & \downarrow{\scriptstyle\Delta} \\ {\scriptstyle\langle f,g\rangle}\downarrow & & & \searrow{\scriptstyle t} & \\ B\times C & & \xrightarrow[\tilde{h}]{\qquad\qquad} & & \Omega \end{array}$$

So the pair (v, w) factors uniquely through A, as desired.

Exercise 4 In this exercise we consider toposes $\mathcal{E}$ and $\mathcal{F}$ and a geometric morphism f from $\mathcal{E}$ to $\mathcal{F}$:

$$\mathcal{E} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{F}$$

with $f^* \dashv f_*$. Now suppose j is a Lawvere-Tierney topology on $\mathcal{E}$ and k is one on $\mathcal{F}$.

a) (5 pts) Show: the functor f_* sends j-sheaves to k-sheaves, if and only if the functor f^* sends k-dense monos to j-dense monos.

b) (5 pts) Show that if the equivalent conditions of part a) hold, then the geometric morphism restricts to a geometric morphism between the respective sheaf categories:

$$\begin{array}{ccc} \mathrm{Sh}_j(\mathcal{E}) & \longrightarrow & \mathrm{Sh}_k(\mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{E} & \underset{f}{\longrightarrow} & \mathcal{F} \end{array}$$

Solution a) it seems most expedient to prove a little lemma first. We say that an object X has the *right lifting property* (RLP) with respect to an arrow $m : M \to N$, and equivalently that m has then the *left lifting property* (LLP) w.r.t. X, if every diagram

$$\begin{array}{ccc} M & \xrightarrow{m} & N \\ \downarrow & & \\ X & & \end{array}$$

has a unique filler: an arrow $N \to X$ making the triangle commute.

Lemma 0.1 Let $\mathcal{S}$ be a topos, and l a Lawvere-Tierney topology. The following two statements are equivalent for a mono $M \overset{m}{\to} N$ in $\mathcal{S}$:

i) m has the LLP wit respect to every l-sheaf.

ii) m is l-dense.

Proof. The direction ii)⇒i) is immediate; this is the definition of an l-sheaf. In order to prove i)⇒ii), i.e. that m is dense provided it has the LLP w.r.t. all sheaves, we calculate the closure of m. Let i denote the embedding of the category of l-sheaves into $\mathcal{S}$, and L its left adjoint (the sheafification functor). Now the closure of m is the left hand vertical mono in the following pullback square:

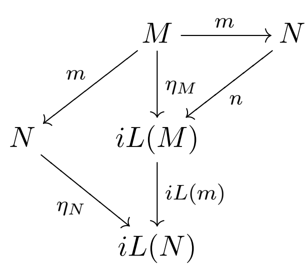
$$\begin{CD}
\overline{M} @>>> iL(M) \\
@VVV @VV{iL(m)}V \\
N @>>{\eta_N}> iL(N)
\end{CD}$$

where η denotes the unit of the adjunction $L \dashv i$.

By applying assumption i) to the diagram

$$\begin{CD}
M @>{m}>> N \\
@V{\eta_M}VV \\
iL(M)
\end{CD}$$

we obtain a filler $n : N \to iL(M)$ (satisfying $nm = \eta_M$). If we look at the diagram



we see that both η_N and $iL(m)n$ are fillers for the diagram involving M, N and $iL(N)$, so they must be equal. This means that the diagram

$$\begin{CD}
N @>{n}>> iL(M) \\
@V{\mathrm{id}}VV @VV{iL(m)}V \\
N @>>{\eta_N}> iL(N)
\end{CD}$$

commutes. So the identity on N (the maximal subpbject of N) factors through the closure of m. We conclude that this closure is N, which is to say that m is dense. ∎

Now we can succinctly answer part a): if f^* sends k-dense monos to j-dense monos and X is a j-sheaf in $\mathcal{E}$, then for any k-dense mono m in $\mathcal{F}$, X has the RLP w.r.t. $f^*(m)$ (since $f^*(m)$ is j-dense), hence f_*X has the RLP w.r.t. m. So f_*X is a k-sheaf.

Conversely, if f_* sends j-sheaves to k-sheaves, then f^* sends k-dense monos (which have the LLP w.r.t. all objects f_*X for j-sheaves X), to monos which have the LLP w.r.t. all j-sheaves X; that is, by the Lemma, to j-dense monos.

b) Since f_* sends j-sheaves to k-sheaves, we have a functor $\phi_* : \mathrm{Sh}_j(\mathcal{E}) \to \mathrm{Sh}_k(\mathcal{F})$. If we denote the sheafification on $\mathcal{E}$ by L_j and its right adjoint by i_1, and for $\mathcal{F}$ by L_k and i_2, then we see that that the composites $i_2\phi_*$ and f_*i_1 are naturally isomorphic. We need only to show that ϕ_* has a left adjoint ϕ^*, for then, by composition of adjoints, we will have a natural isomorphism between $L_j f^*$ and $\phi^* L_k$.

Define ϕ^* as $L_j f^* i_2 : \mathrm{Sh}_k(\mathcal{F}) \to \mathrm{Sh}_j(\mathcal{E})$. The adjunction is trivial, using the natural isomorphism between $i_2\phi_*$ and f_*i_1, the adjunctions between f_* and f^*, between the L's and the i's, and the fact that i_2 is full and faithful:

$$\begin{array}{rcll} \mathrm{Sh}_j(\mathcal{E})(\phi^*Y, X) & \simeq & \mathrm{Sh}_j(\mathcal{E})(L_j f^* i_2 Y, X) & \simeq \\ \mathcal{E}(f^* i_2 Y, i_1 X) & \simeq & \mathcal{F}(i_2 Y, f_* i_1 X) & \simeq \\ \mathcal{F}(i_2 Y, i_2 \phi_* X) & \simeq & \mathrm{Sh}_k(\mathcal{F})(Y, \phi_* X) & \end{array}$$