Fast computation of principal $\mathcal{A}$-determinants by means of dimer models; an introduction through three concrete examples.

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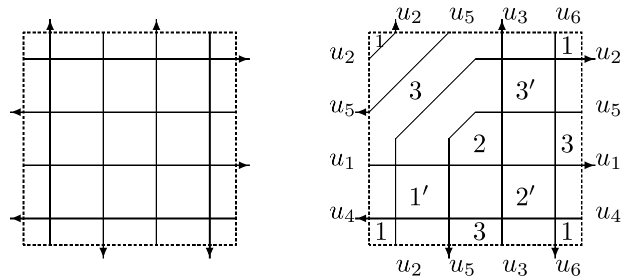
$\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ is a set of N vectors in $\mathbb{Z}^{N-2}$ which spans $\mathbb{Z}^{N-2}$ over $\mathbb{Z}$ and for which there exists a linear map $h : \mathbb{Z}^{N-2} \to \mathbb{Z}$ such that $h(\mathcal{A}) = \{1\}$. The *principal $\mathcal{A}$-determinant* was defined by Gelfand, Kapranov, Zelevinsky [1]. It describes the singularities of the associated GKZ $\mathcal{A}$-hypergeometric system of differential equations. It also gives the locus in $(u_1, \ldots, u_N)$-space where the zero-set of the $(N-2)$-variable Laurent polynomial $\sum_{j=1}^{N} u_j \mathbf{x}^{\mathbf{a}_j}$ has singularities. According to GKZ it is a linear combination of monomials in the variables $u_1, \ldots, u_N$ with exponents being the integer points in the so-called *secondary polygon.* GKZ gave the coefficients for the vertices of the secondary polygon. We want all coefficients. In [3] it was proven that the principal $\mathcal{A}$-determinant equals the determinant of (an appropriate form of) the *Kasteleyn matrix* of a *dimer model.* D. Gulotta describes in [2] an efficient algorithm for constructing this dimer model from $\mathcal{A}$, or rather from a $2 \times N$-matrix $B = (b_{ij})$ of which the rows form a basis for the lattice of relations $\{(\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N \mid \ell_1 \mathbf{a}_1 + \ldots + \ell_N \mathbf{a}_N = \mathbf{0}\}$. Let me now restrict to concrete examples.

Appell's hypergeometric function F_1:

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}.$$

We present Gulotta's algorithm for this example in two pictures. These should be viewed on a torus which is drawn here as a square with opposite sides identified:



Start in the left hand picture with $\sum_{j=1}^{N} |b_{1j}|$ horizontal lines alternatingly oriented $\to$ resp. $\leftarrow$ and $\sum_{j=1}^{N} |b_{2j}|$ vertical lines alternatingly oriented $\uparrow$ resp. $\downarrow$. The reader may try to guess how the other pictures are created from the starting picture, or look up the algorithm in [2]. The final situation shown in the right-hand picture consists of six oriented zigzag paths with winding numbers on the torus given by the columns of matrix B; $u_1, u_2, u_3, u_4, u_5, u_6$ are variables corresponding with the columns of B and with the zigzags. The connected components of the complement of this zigzag configuration are 2-cells of which the boundary is either oriented or unoriented. There are three cells with positive (resp. negative) oriented boundary which we label 1, 2, 3 (resp. 1′, 2′, 3′). The *dimer model* mentioned in the title "is" this configuration of 2-cells with oriented boundaries. As in [3] we collect the information on the zigzags, their intersection points and intersection numbers and the cells with oriented boundary in the following 3×3-matrix, which is a form of the *Kasteleyn matrix* of the dimer model.

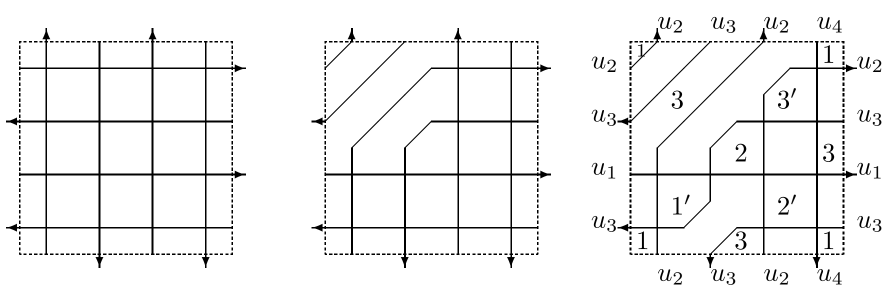
$$\mathbb{K}(u_1,u_2,u_3,u_4,u_5,u_6) \;=\; \begin{bmatrix} u_2u_4 & u_1u_5 & u_1u_2+u_4u_5 \\ u_4u_6 & u_1u_3 & u_1u_6+u_3u_4 \\ u_2u_6 & u_3u_5 & u_2u_3+u_5u_6 \end{bmatrix};$$

the matrix entries in this matrix correspond to the intersection points of the zigzags: an entry $p\,u_ku_m$ in position (i,j) means that it is an intersection point of zigzags u_k and u_m, that it is a vertex of the oriented 2-cells i' and j and that p is the number of intersection points of the zigzags u_k and u_m. According to [3] the sought for principal A-determinant for this example is

$$\begin{aligned}&\det(u_1u_2u_3u_4u_5u_6\,\mathbb{K}(u_1^{-1},u_2^{-1},u_3^{-1},u_4^{-1},u_5^{-1},u_6^{-1})) \;=\\ &\quad=\; u_1u_2u_3u_4u_5u_6(u_1u_2-u_4u_5)(u_2u_3-u_5u_6)(u_1u_6-u_3u_4)\,.\end{aligned}$$

This result is written in the GKZ formalism. In order to connect it with Appell's F_1 one must set $x=u_1u_2u_4^{-1}u_5^{-1}$ and $y=u_2u_3u_5^{-1}u_6^{-1}$ (cf. with the rows of B).

Horn's hypergeometric function G_3:
$\mathcal{A}=\begin{bmatrix}1&1&1&1\\0&2&1&3\end{bmatrix}$, $B=\begin{bmatrix}1&1&-2&0\\0&2&-1&-1\end{bmatrix}$. Gulotta's algorithm for Horn's G_3 in three pictures, the first two of which are the same as for Appell's F_1:



The final situation shown in the right-hand picture consists of four oriented zigzag paths with winding numbers on the torus given by the columns of matrix B; u_1,u_2,u_3,u_4 are variables corresponding with the columns of B and with the zigzags. In the complement of this zigzag configuration there are three cells with positive (resp. negative) oriented boundary which we label 1, 2, 3 (resp. 1′, 2′, 3′). As before we collect the information on the zigzags, their intersection points and intersection numbers and the cells with oriented boundary in the following

3×3-matrix, which is a form of the *Kasteleyn matrix* of the dimer model.

$$\mathbb{K}(u_1, u_2, u_3, u_4) \ = \ \begin{bmatrix} 3\,u_2u_3 & 1\,u_1u_3 & 2\,u_1u_2 \\ 2\,u_3u_4 & 2\,u_1u_2 & 1\,u_1u_4 + 3\,u_2u_3 \\ 1\,u_2u_4 & 3\,u_2u_3 & 2\,u_3u_4 \end{bmatrix}.$$

According to [3] the sought for principal A-determinant for this example is

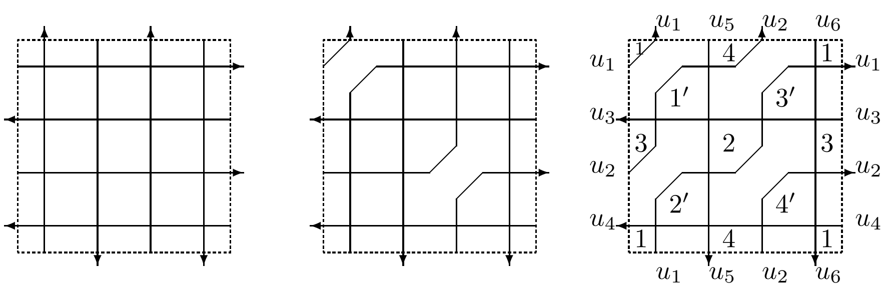
$$\det(u_1u_2u_3u_4\,\mathbb{K}(u_1^{-1}, u_2^{-1}, u_3^{-1}, u_4^{-1})) \ = \\ = \ 18\,u_1^2u_2u_3u_4^2 \ + \ u_1u_2^2u_3^2u_4 \ - \ 4\,u_1u_3^3u_4 \ - \ 4\,u_1^2u_2^3u_4 \ - \ 27\,u_1^3u_4^3\,.$$

This becomes 0 exactly when the polynomial $u_1 \, + \, u_3\,x \, + \, u_2\,x^2 \, + \, u_4\,x^3$ does not have three distinctinct non-zero roots.

Appell's hypergeometric function F_4:

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

Gulotta's algorithm for this example in three pictures:



The right-hand picture leads to the matrix

$$\mathbb{K}(u_1, u_2, u_3, u_4, u_5, u_6) \ = \ \begin{bmatrix} 0 & u_3u_5 & u_2u_3 & u_2u_5 \\ u_1u_4 & u_1u_5 & 0 & u_4u_5 \\ u_1u_6 & u_1u_3 & u_3u_6 & 0 \\ u_4u_6 & 0 & u_2u_6 & u_2u_4 \end{bmatrix}.$$

We leave the computation of the principal $\mathcal{A}$-determinant as an exercise.

References

[1] Gelfand, I.M., M.M. Kapranov, A.V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser Boston, 1994

[2] Gulotta, D., *Properly ordered dimers, R-charges and an efficient inverse algorithm*, arXiv:0807.3012

[3] Stienstra, J., *Chow Forms, Chow Quotients and Quivers with Superpotential*, arXiv:0803.3908